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Edge-choosability of multicircuits

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Abstract

A *multicircuit* is a multigraph whose underlying simple graph is a circuit (a connected 2-regular graph). The *List-Colouring Conjecture* (*LCC*) is that every multigraph G has edge-choosability (list chromatic index) $\mathrm{ch}'(G)$ equal to its chromatic index $\chi'(G)$. In this paper the LCC is proved first for multicircuits, and then, building on results of Peterson and Woodall, for any multigraph G in which every block is bipartite or a multicircuit or has at most four vertices or has underlying simple graph of the form $K_{1,1,p}$. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

Let $G=(V,E)$ be a multigraph with vertex-set $V(G)=V$ and edge-set $E(G)=E$. Let $f:E\to\mathbb{N}$ be a function into the positive integers. We say that G is *edge-f-choosable* if, whenever we are given sets ('lists') A_e of 'colours' with $|A_e|=f(e)$ for each $e\in E$, we can choose a colour $c(e)\in A_e$ for each edge e so that no two adjacent edges have the same colour; in this case, we say loosely that G can be *edge-coloured from its lists*. The *edge choosability*, *list edge chromatic number* or *list chromatic index* $\mathrm{ch}'(G)$ of G is the smallest integer k such that G is edge-f-choosable when $f(e)=k$ for each e. The ordinary edge chromatic number, or chromatic index, of G is denoted by $\chi'(G)$; clearly $\mathrm{ch}'(G)\geqslant\chi'(G)$. The following conjecture was made independently by Vizing, by Gupta, by Albertson and Collins, and by Bollobás and Harris (see [4,5]).

The List-Colouring Conjecture (LCC). For every multigraph G, $\mathrm{ch}'(G)=\chi'(G)$.

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Galvin [3] proved the LCC for bipartite multigraphs. Specifically, if G is a bipartite multigraph then $\mathrm{ch}'(G)=\chi'(G)=\Delta(G)$, the maximum degree of G. In [1] we strengthened this result by proving that a bipartite multigraph G is edge-f-choosable, where $f(e)=\max\{d(u),d(w)\}$ for each edge $e=uw$ of G. (Here $d=d_G$ denotes degree in G.)

These results were extended to line-perfect multigraphs in [7]. Let $\omega'(G)$ denote the size of a largest edge-clique (set of mutually adjacent edges) in G, and let $\omega'_G(e)$ denote the size of a largest edge-clique containing edge e. A multigraph G is *line-perfect* if its line graph $L(G)$ is perfect, that is, if $\chi'(H)=\omega'(H)$ for every submultigraph H of G (since induced subgraphs of $L(G)$ are the line graphs of arbitrary submultigraphs of G). It follows from a result of Maffray [6, Theorem 2] that a multigraph is line-perfect if and only if each of its blocks is bipartite, or has at most four vertices, or has underlying simple graph of the form $K_{1,1,p}$ $(p\geqslant 3)$. In [7] we proved that if G is a line-perfect multigraph then G is edge-ω'_G-choosable and $\mathrm{ch}'(G)=\chi'(G)=\omega'(G)$. This proves the LCC for line-perfect multigraphs, and it generalizes the results of the previous paragraph for bipartite multigraphs, since if G is bipartite then it is easy to see that $\omega'(G)=\Delta(G)$ and $\omega'_G(e)=\max\{d(u),d(w)\}$ for each edge $e=uw$ of G.

Now let a *multicircuit* be a multigraph whose underlying simple graph is a circuit (a connected 2-regular graph), and let $\gamma_G(e)$ denote the maximum value of $\lceil |E(C)|/\lfloor \frac{1}{2}|V(C)|\rfloor\rceil$ over all multicircuits C of odd order such that $e\in C\subseteq G$ (interpreted as 0 if no such C exists). Let

$$\psi'_G(e):=\max\{\omega'_G(e),\gamma_G(e)\} \quad \text{and} \quad \psi'(G):=\max\{\psi'_G(e)\colon e\in E(G)\}. \tag{1}$$

Clearly $\chi'(G)\geqslant\psi'(G)$, since at most $\lfloor\frac{1}{2}|V(C)|\rfloor$ edges of C can be given the same colour. In this paper we first prove the LCC for multicircuits (Theorem 1). We then go on to prove the LCC for multigraphs in which every block is line-perfect or a multicircuit. Specifically, we prove (Theorem 3) that if G is such a multigraph then G is edge-ψ'_G-choosable and $\mathrm{ch}'(G)=\chi'(G)=\psi'(G)$. This generalizes the results of [7] mentioned above for line-perfect multigraphs, since if G is line-perfect then G contains no multicircuits of odd order greater than 3 and so $\psi'(G)=\omega'(G)$ and $\psi'_G(e)=\omega'_G(e)$ for each edge e.

The underlying simple graphs of nonbipartite multicircuits are circuits of odd order, and they share with $K_{1,1,p}$ the form 'bipartite plus one edge'. Thus, it would generalize and unify some of the above results if one could prove the LCC for every multigraph whose underlying simple graph has this form. But even this comparatively modest extension appears to be hard.

The rest of this paper is devoted to the proofs of the theorems mentioned above.

2. Proofs

We assume throughout that $C=(V,E)$ is a multigraph with $V=\{v_0,\ldots,v_{n-1}\}$ and $E=X_0\cup\cdots\cup X_{n-1}$, where $n\geqslant 3$ and $X_i=\{e_{i,0},\ldots,e_{i,\mu_i-1}\}$ is the set of edges between

v_i and v_{i+1} (where subscripts to v, X and μ should be interpreted modulo n). Let C have $m=\sum_{i=0}^{n-1}\mu_i$ edges and maximum degree Δ, and let the degree of v_i be $d(v_i)=\mu_{i-1}+\mu_i$. It is convenient to allow $\mu_i=0$ for some values of i, even though C is not then a multicircuit. If each edge of C is given a list of colours, we say that a colour is *present on* X_i if it belongs to the list of at least one edge in X_i. The following result is somewhat stronger than is needed in order to prove Theorem 1, and it may perhaps be of some interest in its own right.

Lemma 1.1. *Let $l_0,\ldots,l_{n-1}$ be integers such that $l_i \geqslant d(v_i)$ for each i and, if n is odd, say $n=2k+1$, then*

$$\sum_{i=1}^{k} l_{2i} \geqslant m. \tag{2}$$

Suppose that each edge $e_{i,j}$ is given a list of at least $l_i - j$ colours. Then C can be edge-coloured from its lists.

Proof. We prove the result by induction on m, noting that it clearly holds if $m=1$; so suppose $m\geqslant 2$. Choose a colour c that is present in the list of at least one edge. There are two cases to consider.

Case 1: There is some i such that c is present on X_i but not on X_{i-1} (mod n). Then choose j maximal such that c is present in the list of $e_{i,j}$ and colour $e_{i,j}$ with c. Let $C^* := C - e_{i,j}$ with c deleted from the list of every edge in $X_i \cup X_{i+1}$ that contains it. All parameters are the same in C^* as in C except for the following (in an obvious terminology): $m^*=m-1$, $\mu_i^*=\mu_i-1$, $d^*(v_i)=d(v_i)-1$, $d^*(v_{i+1})=d(v_{i+1})-1$, and $e^*_{i,j'-1}=e_{i,j'}$ if $j<j'<\mu_i$. Let us define $l_i^* := l_i - 1$, $l_{i+1}^* := l_{i+1}-1$ and $l_{i'}^* := l_{i'}$ for all $i' \notin \{i, i+1\}$. Then the hypotheses of the theorem are satisfied for C^*, and we may suppose inductively that C^* can be edge-coloured from its lists. Restoring edge $e_{i,j}$ with colour c gives the required edge-colouring of C.

Case 2: Colour c is present on every X_i. Suppose first that $n=2k+1$. Note that $m=\mu_0+\sum_{i=1}^{k} d(v_{2i})$, and so (2) implies $\sum_{i=1}^{k}(l_{2i}-d(v_{2i}))\geqslant \mu_0>0$. Choose an $h\neq 0$ such that $l_{2h}>d(v_{2h})$. Let $I:=\{0,2,\ldots,2h-2,2h+1,2h+3,\ldots,n-2\}$. For each $i\in I$, choose $j(i)$ maximal such that c is present in the list of $e_{i,j(i)}$, colour $e_{i,j(i)}$ with c, and delete $e_{i,j(i)}$ from C. Delete c from the list of every other edge of C that contains c. If the resulting graph is C^*, then $m^*=m-|I|=m-k$, $d^*(v_{2h})=d(v_{2h})$, $d^*(v_i)=d(v_i)-1$ for every $i\neq 2h$, and $e^*_{i,j'-1}=e_{i,j'}$ if $i\in I$ and $j(i)<j'<\mu_i$. Let us define $l_i^* := l_i - 1$ for every i. Then the hypotheses of the theorem are satisfied for C^*, and we may suppose inductively that C^* can be edge-coloured from its lists. Restoring the edges $e_{i,j(i)}$ $(i\in I)$ with colour c gives the required edge-colouring of C.

Suppose finally that n is even. Then we can proceed in an exactly similar way with $I:=\{0,2,\ldots,n-2\}$, except that now there is no exceptional vertex corresponding to v_{2h} in the above argument. $\square$

We are now in a position to prove the LCC for multicircuits:

Theorem 1. *If C is a multicircuit with n vertices, m edges and maximum degree Δ then*

$$\mathrm{ch}'(C)=\chi'(C)=\psi'(C)=\begin{cases}\Delta & \text{if } n \text{ is even,}\\ \max\{\Delta,\lceil m/k\rceil\} & \text{if } n=2k+1,\end{cases}$$

and so the LCC holds for C.

Proof. It is clear from (1) that $\mathrm{ch}'(C)\geqslant\chi'(C)\geqslant\psi'(C)$ and that $\psi'(C)$ has the value stated above. Thus, it suffices to prove that $\mathrm{ch}'(C)\leqslant\psi'(C)$. This follows from Lemma 1.1, because if each edge of C is given a list of $\psi'(C)$ colours and we define $l_i:=\psi'(C)$ for each i, then the hypotheses of Lemma 1.1 are satisfied. □

Of course, if n is even then C is bipartite and the result of Theorem 1 follows from [3]. For a similar reason, in Theorem 2 we assume n is odd; but we continue to allow even values of n in Lemma 2.1. Theorem 2 is a nonuniform strengthening of Theorem 1, different from that in Lemma 1.1, which we will use in Theorem 3 to deal with graphs that have multicircuits as blocks.

Recall that a *kernel* of a digraph D is a set K of nonadjacent vertices such that every vertex in $V(D)\backslash K$ is joined by an arc to at least one vertex in K. A digraph D is *normal* if every induced complete subdigraph of D has a kernel, which necessarily consists of a single vertex. (A digraph is *complete* if each two vertices are adjacent in *at least* one direction.) We shall need the following lemma, in which we continue to use X_i and μ_i as defined at the start of this section.

Lemma 2.1. *Suppose that $X_i=X_i^{(1)}\cup X_i^{(2)}$ where $X_i^{(1)}\cap X_i^{(2)}=\emptyset$, for each $i\in\{0,\dots,n-1\}$, and let $\mu_i^{(j)}:=|X_i^{(j)}|$ $(j\in\{1,2\})$ so that $\mu_i^{(1)}+\mu_i^{(2)}=\mu_i$. Suppose that, for each i, each element $x\in X_i$ is given a list of at least*

$$\begin{array}{ll}\mu_{i-1}+\mu_i=d(v_i) & \textit{colours if } x\in X_i^{(1)},\\ \mu_{i-1}^{(1)}+\mu_i+\mu_{i+1}^{(2)}=d(v_i)+\mu_{i+1}^{(2)}-\mu_{i-1}^{(2)} & \textit{colours if } x\in X_i^{(2)}.\end{array} \tag{3}$$

Suppose moreover that, if n is odd, then no colour is present on every X_i. Then C can be edge-coloured from its lists.

Proof. Form a digraph D with $V(D)=E(C)$ by joining $x\in X_i$ to $x'\neq x$ by an arc if

$$x'\in\begin{cases}X_{i-1}\cup X_i & \text{if } x\in X_i^{(1)},\\ X_{i-1}^{(1)}\cup X_i\cup X_{i+1}^{(2)} & \text{if } x\in X_i^{(2)}.\end{cases} \tag{4}$$

Then D is an orientation of the *line multigraph* of C; this differs from the (simple) line graph $L(C)$ in having two edges between x and x' whenever x and x' are parallel edges in C. We illustrate D diagrammatically in Fig. 1. Note that D is normal if $n\geqslant 4$, since if D' is an induced complete subdigraph of D then $V(D')\subseteq X_{i-1}\cup X_i$ for some i, and then any vertex in the first nonempty set in the list

$$V(D')\cap X_{i-1}^{(1)},\ V(D')\cap X_i^{(2)},\ V(D')\cap X_{i-1}^{(2)},\ V(D')\cap X_i^{(1)}$$

forms a kernel of D'.

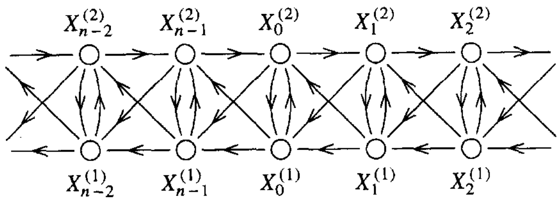

Fig. 1. A diagrammatic representation of D.

By comparing (3) and (4) we see that the lemma can be restated as follows: if each vertex x of D is given a list of more than $d^+(x)$ colours, where d^+ denotes outdegree in D, and if no colour is present on every X_i when n is odd, then D can be (vertex-)coloured from its lists. We prove this by induction on the number $|V(D)|$ of vertices in D, noting that it is certainly true if $|V(D)|=1$; so suppose $|V(D)|\geqslant 2$. Choose a colour c that is present in the list of at least one vertex of D, and let D_c be the subdigraph of D that is induced by all the vertices whose lists contain colour c. The hypotheses ensure that D_c is an orientation of the line multigraph of a *bipartite* submultigraph of C, and D_c is normal (even if $n=3$). Now, Maffray [6] proved that every normal orientation of the line multigraph of a bipartite multigraph has a kernel, and Galvin [3] gave a simpler proof of this (see also [2]). So let K be a kernel of D_c and colour all vertices in K with c. Let $D^* := D - K$ with c deleted from every list. Then the hypotheses of the lemma (as restated at the start of this paragraph) hold for D^*, and so we may suppose inductively that D^* can be (vertex-)coloured from its lists. Restoring the vertices of K with colour c gives the required colouring of D. □

Theorem 2. *Let C be a multicircuit with $n=2k+1$ vertices and m edges, and let $v\in V(C)$. Suppose that every edge incident with v is given a list of at least $d(v)$ colours, and every other edge uw is given a list of at least* $\max\{d(u),d(w),\lceil m/k\rceil\}$ *colours. Then C can be edge-coloured from its lists.*

Proof. We prove the result by induction on m. It is not difficult to see that it holds if $\mu_i=0$ for some i (and this also follows from Theorem 3.3 of [7]); so suppose $\mu_i\geqslant 1$ for every i. Label C so that $v_0=v$.

Suppose first that some colour c is present on every X_i. Since $m=\mu_0+\sum_{i=1}^{k} d(v_{2i})$, there exists an $h\neq 0$ such that $d(v_{2h})<\lceil m/k\rceil$. Proceed exactly as in Case 2 of Lemma 1.1. The graph C^* constructed there will satisfy the hypotheses of Theorem 2, and so we may assume inductively that it can be edge-coloured from its lists. It follows that we can construct the required edge-colouring of C.

So we may assume that no colour is present on every X_i. Note that the result follows immediately from Lemma 1.1 if $d(v_0)\geqslant d(v_{n-1})$, and so we may suppose $d(v_{n-1})=d(v_{2k})>d(v_0)$, that is, $\mu_{n-2}=\mu_{2k-1}>\mu_0$. Let $l_0:=l_{2k}:=d(v_0)$, and for $i\in\{1,\ldots,2k-1\}$ let $l_i:=\max\{d(v_i),d(v_{i+1}),\lceil m/k\rceil\}$, so that

every edge in X_i has a list of at least l_i colours. (5)

In order to apply Lemma 2.1, we define $\mu_{2k}^{(2)} := \mu_{2k}$ and $\mu_{2k-1}^{(2)} := \mu_{2k-1} - \mu_0$; then for $i = 2k-1, 2k-2, \ldots, 1$ in turn we define

$$\mu_{i-1}^{(2)} := \begin{cases} 0 & \text{if } \mu_i^{(2)} = 0, \\ \max\{0, \mu_{i+1}^{(2)} + d(v_i) - l_i\} & \text{otherwise.} \end{cases} \tag{6}$$

Note that

$$\mu_{i+1}^{(2)} + d(v_i) - l_i \leqslant \mu_{i+1} + d(v_i) - d(v_{i+1}) = \mu_{i-1},$$

and so $0 \leqslant \mu_{i-1}^{(2)} \leqslant \mu_{i-1}$ always. So, for each i, let $X_i^{(2)}$ be an arbitrary subset of X_i with cardinality $\mu_i^{(2)}$, and let $X_i^{(1)} := X_i \backslash X_i^{(2)}$ and $\mu_i^{(1)} := \mu_i - \mu_i^{(2)} = |X_i^{(1)}|$. Note that $\mu_{2k}^{(1)} = 0$.

Now, $m = \mu_{2k} + \sum_{i=1}^{k} d(v_{2i-1})$, so that $\sum_{i=1}^{k} (l_{2i-1} - d(v_{2i-1})) \geqslant m - m + \mu_{2k} = \mu_{2k}^{(2)}$. It follows from this and (6) (for all odd i) that $\mu_0^{(2)} = 0$. So if $\mu_i^{(2)} > 0$ then either $1 \leqslant i \leqslant 2k-1$, in which case $l_i \geqslant d(v_i) + \mu_{i+1}^{(2)} - \mu_{i-1}^{(2)}$ from (6), or else $i = 2k$, in which case

$$d(v_{2k}) + \mu_0^{(2)} - \mu_{2k-1}^{(2)} = d(v_{2k}) + 0 - \mu_{2k-1} + \mu_0 = d(v_0) = l_{2k}.$$

And if $\mu_i^{(1)} > 0$ then either $1 \leqslant i \leqslant 2k-1$, in which case $l_i \geqslant d(v_i)$ by definition, or $i = 0$, in which case $l_0 = d(v_0)$. In all cases the hypotheses of Lemma 2.1 are satisfied, by (5), and we deduce that C can be edge-coloured from its lists. □

Recall the definitions of $\psi'_G(e)$ and $\psi'(G)$ from (1), and for a fixed $v \in V(G)$ define $\psi'_{G,v} : E(G) \to \mathbb{N}$ by

$$\psi'_{G,v}(e) := \begin{cases} d_G(v) & \text{if } e \text{ is incident with } v, \\ \psi'_G(e) & \text{otherwise.} \end{cases}$$

We can now prove our final and most general result.

Theorem 3. *If $G = (V, E)$ is a multigraph in which every block is line-perfect or a multicircuit, and $v \in V$, then G is edge-$\psi'_{G,v}$-choosable. It follows that G is edge-ψ'_G-choosable and* $\mathrm{ch}'(G) = \chi'(G) = \psi'(G)$, *so that the LCC holds for G.*

Proof. The second sentence follows from the first since $\psi'(G) \geqslant \psi'_G(e) \geqslant \psi'_{G,v}(e)$ for each $e \in E$ and $v \in V$, and clearly $\mathrm{ch}'(G) \geqslant \chi'(G) \geqslant \psi'(G)$.

We prove the first sentence by induction on the number of blocks in G. There is no loss of generality in supposing that G is connected. If G has only one block then either G is line-perfect and the result follows from Theorem 2.3 of [7], or else G is a multicircuit of odd order and the result follows from Theorem 2. So suppose that G has more than one block. Then it has at least two endblocks (where an *endblock* is a block containing exactly one cutvertex). Let G_2 be an endblock of G not containing v except possibly as its cutvertex w, and let G_1 be the union of all blocks other than G_2. Then $v \in G_1$, and G_1 has one block fewer than G, and so we may suppose by induction that G_1 is edge-$\psi'_{G_1,v}$-choosable, and similarly that G_2 is edge-$\psi'_{G_2,w}$-choosable. Suppose that each edge e of G is given a list of at least $\psi'_{G,v}(e)$ colours. Since $\psi'_{G,v}(e) \geqslant \psi'_{G_1,v}(e)$ for each $e \in E(G_1)$, we can edge-colour G_1 from these lists. Now a total of $d_{G_1}(w)$ colours are used on edges of G_1 at w. Remove these colours from the lists on all edges of G_2 at w. If e is such an edge, then the number of colours remaining in its list is at least

$$\psi'_{G,v}(e) - d_{G_1}(w) \geqslant d_G(w) - d_{G_1}(w) = d_{G_2}(w) = \psi'_{G_2,w}(e)$$

(regardless of whether $v = w$ or not); and each edge e of $G_2 - w$ still has a list of at least $\psi'_{G,v}(e) \geqslant \psi'_{G_2,w}(e)$ colours. Hence we can complete the colouring from these lists. □

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