J. Math. Anal. Appl. 353 (2009) 553–565

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Multilayers in a modulated stochastic game ☆

Jewgeni H. Dshalalow *, Hao-Jan Ke

Department of Mathematical Sciences, College of Science, Florida Institute of Technology, 150 W. University Blvd, Melbourne, FL 32901, USA

ARTICLE INFO

Article history:
Received 24 March 2008
Available online 24 December 2008
Submitted by M. Quincampoix

Keywords:
Noncooperative games
Fluctuation theory
Marked point processes
Poisson process
Ruin time
Exit time
First passage time

ABSTRACT

We are concerned with an antagonistic stochastic game between two players A and B which finds applications in economics and warfare. The actions of the players are manifested by a series of strikes of random magnitudes at random times exerted by each player against his opponent. Each of the assaults inflicts a random damage to enemy's vital areas. In contrast with traditional games, in our setting, each player can endure multiple strikes before perishing. Predicting the ruin time (exit) of player A, along with the total amount of casualties to both players at the exit is a main objective of this work. In contrast to the time sensitive analysis (earlier developed to refine the information on the game) we insert auxiliary control levels, which both players will cross in due game before the ruin of A. This gives A (and also B) an additional opportunity to reevaluate his strategy and change the course of the game. We formalize such a game and also allow the real time information about the game to be randomly delayed. The delayed exit time, cumulative casualties to both players, and prior crossings are all obtained in a closed-form joint functional.

1. Introduction

This article deals with a stochastic game between two players A and B of a totally antagonistic nature. The actions of the players are manifested by a mutual series of strikes of random magnitudes at random times. Each of the assaults is intended to inflict damage to vital areas of defense, infrastructure, utilities, economics, and industry (warfare), as well as drive down prices, cause the outsourcing of labor, render hostile advertisement, unfavorable trading of competitor's stock, lobbying politicians (competition of enterprises), arrange terror acts that involve human casualties, poison utilities and spread infection through chemical or biological sabotage, hacking into banks, country defense network, and stock exchange to cripple the economy (terrorism and cyber-terrorism).

In contrast with strictly antagonistic games best known in the literature, where a game ends with one single successful hit, in our setting, each player can endure multiple strikes before perishing. Therefore, we assign to each player a (hypothetical) threshold of endurance that represents how much damage he can sustain before succumbing. Each player will try to defeat his adversary at his earliest opportunity, and the time when one of them collapses is referred to as the *ruin time*. The latter is also called the *exit from the game*.

Predicting the ruin time, along with the total amount of casualties to both players at the exit has been an objective of this and past work [14–17]. Actually, the defeat of one player, say of player A, is the focus of our investigation. There are various ideas on how to refine the game. One of them is to make the processes time dependent including their relationship with the exit time referred to as *time sensitive analysis* [14]. Another approach is to observe the time when player A crosses

☆ This research is supported by the US Army Grant No. W911NF-07-1-0121.

* Corresponding author.
E-mail addresses: eugene@fit.edu (J.H. Dshalalow), hke@fit.edu (H.-J. Ke).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.12.026

some smaller threshold M_1 $(< M)$, which we can vary, and that should take place at some earlier epoch. Both authors have done exactly that in [17] arriving at closed form functionals of the processes at the exit of the game and at the crossing of level M_1. The closed form enabled one to alter M_1 arbitrarily thereby refining the information on the status of player A w.r.t. various thresholds instead of refining it w.r.t time as in [14]. A further refinement is to lay out the cross-level behavior of the process associated with player B, in addition to that for player A. Thus we are looking into the time and value of the "B-process" upon its crossing some N_1 $(< N)$. The simultaneous analysis with auxiliary layers for both players turns out to be more complex than that for one player, but the results for one player was instrumental to get a closed form solution for the general model.

Since in true antagonistic systems, the real time information may not be accessible, we assume that the course of the game can be observed upon a sequence of random times $\mathcal{T} = \{\tau_0, \tau_1, \ldots\}$ causing some random delays to the entering information. The random times are not specified; but, dependent on what we want to model, they can be arbitrarily fine or crude. It may seem like the delayed observation can cause a difficulty in determining who of the two players is ruined first if their ruin times come close to each other or if the collected data is very crude. However, the probabilistic analysis we render will provide a prediction of one player perishing ahead of the other upon one of the observation epochs. "Overlooked" paths can be minimized by refining $\mathcal{T}$ and by introducing modulation. The latter can self control observation frequencies, dependent on the severity of the situation. Another improvement can be rendered by an earlier detection of troubles through the insertion of auxiliary thresholds.

The paper is organized as follows. In Sections 2 and 3 we introduce and describe a basic game. In Section 4 we discuss and formalize modulation. In Sections 5 and 6 we analyze the main functional of the game bringing it to a closed form.

The article carries a game-theoretical setting and modeling; however, not in the very traditional sense (optimal strategies, equilibria, differential equations, Markov decision processes, to name a few). As a chief tool, we apply and embellish fluctuation theory [9,10,18–20], which stemmed from random walks [4] and the behavior of sums of independent and dependent random variables (like Markov and semi-Markov sequences) about critical levels [43]. Then, the studies on fluctuation theory were extended to exit times from sets by Wiener processes and compensated Poisson processes [24,25,31]. The latter found applications to risk analysis. Fluctuation theory has also become a stand-alone area of stochastics, with wide spread applications to physics [23,26,27,37] economics [32–35], stock market [11,12], biology [26], and queueing theory [21,42]. The use of fluctuation theory in games has not been explored until recently [14–17].

From game-theoretic standpoint, our work concerns with purely antagonistic games [3,22,28,41] as oppose to cooperative or partially cooperative games. The latter often applied in economics [2,8,29]. Antagonistic games are also used in economics [32,39], warfare [1,5–7,27,28,38,40], and ecology [36].

An interesting article [30] by Kadankova is one of the recent papers on the theory of fluctuations somewhat (but not directly) related to ours. In this paper the author derives the joint distribution of the first exit time from an interval and the excess over a threshold at the exit time for a Poisson process with an exponentially distributed negative component and the supremum, infimum, and the number of upcrossings and downcrossings, the number of passages into an exponentially distributed interval of time and the excess over a boundary of an interval.

Finally, modulation has been widely used in finance, electrical and computer networks, and queueing. The first author has been investigating modulated processes in a number of articles. A most recent related work is [13].

2. Preliminaries

To layout the game one step at a time we begin with a more rudimentary model which will be embellished in the upcoming sections. Let $(\Omega, \mathcal{F}(\Omega), P)$ be a probability space and let $\mathcal{F}_A, \mathcal{F}_B, \mathcal{F}_\tau \subseteq \mathcal{F}(\Omega)$ be independent sub-σ-algebras. Suppose

$$\mathcal{A} := \sum_{j \geqslant 0} x_j \varepsilon_{s_j}, \qquad \mathcal{B} := \sum_{k \geqslant 0} y_k \varepsilon_{t_k}, \quad s_0 = 0,\ t_0 = 0,\ x_j, y_k \geqslant 0, \tag{2.1}$$

are $\mathcal{F}_A$-measurable and $\mathcal{F}_B$-measurable marked Poisson processes (ε_a is a point mass at a) with respective intensities λ_A and λ_B and position independent marking. They are specified by their transforms

$$Ee^{-u\mathcal{A}(\cdot)} = e^{\lambda_A|\cdot|[g(u)-1]}, \quad g(u) = Ee^{-ux_1},\ \mathrm{Re}(u) \geqslant 0, \tag{2.2}$$

$$Ee^{-u\mathcal{B}(\cdot)} = e^{\lambda_B|\cdot|[h(u)-1]}, \quad h(u) = Ee^{-uy_1},\ \mathrm{Re}(u) \geqslant 0, \tag{2.3}$$

$|\cdot|$ is the Borel–Lebesgue measure and x_j and y_k are nonnegative r.v.'s. Furthermore, let

$$\mathcal{T} := \sum_{i \geqslant 0} \varepsilon_{\tau_i}, \quad \tau_0 \geqslant 0,\ \text{a.s.} \tag{2.4}$$

be an $\mathcal{F}_\tau$-measurable delayed renewal process. If

$$\big(A(t), B(t)\big) := \mathcal{A} \otimes \mathcal{B}\big([0, t]\big), \quad t \geqslant 0, \tag{2.5}$$

then

$$(A_j, B_j) := \big(A(\tau_j), B(\tau_j)\big) = \mathcal{A} \otimes \mathcal{B}\big([0, \tau_j]\big), \quad j = 0, 1, \ldots, \tag{2.6}$$

forms the observation process upon $\mathcal{A} \otimes \mathcal{B}$ embedded over $\mathcal{T}$, with respective increments

$$(X_j, Y_j) = \mathcal{A} \otimes \mathcal{B}\big((\tau_{j-1}, \tau_j]\big), \quad j = 0, 1, \ldots, \ X_0 = A_0, \ Y_0 = B_0. \tag{2.7}$$

Obviously, the bivariate marked point process

$$\mathcal{A}_\tau \otimes \mathcal{B}_\tau := \sum_{j \geqslant 0} (X_j, Y_j) \varepsilon_{\tau_j}, \tag{2.8}$$

where

$$\mathcal{A}_\tau = \sum_{i \geqslant 0} X_i \varepsilon_{\tau_i} \quad \text{and} \quad \mathcal{B}_\tau = \sum_{i \geqslant 0} Y_i \varepsilon_{\tau_i} \tag{2.9}$$

is with position dependent marking and with X_j and Y_j being dependent. With the notation

$$\Delta_j := \tau_j - \tau_{j-1}, \quad j = 0, 1, \ldots, \quad \tau_{-1} := 0, \tag{2.10}$$

we can evaluate the functional

$$\gamma(x, y, \theta) = Ee^{-xX_j - yY_j - \theta \Delta_j}, \quad \mathrm{Re}(x) \geqslant 0, \ \mathrm{Re}(y) \geqslant 0, \ \mathrm{Re}(\theta) \geqslant 0, \tag{2.11}$$

using straightforward probabilistic arguments

$$\gamma(x, y, \theta) = \delta\big\{\theta + \lambda_A\big(1 - g(x)\big) + \lambda_B\big(1 - h(y)\big)\big\}, \quad j = 1, 2, \ldots, \tag{2.12}$$

where

$$\delta(\theta) = Ee^{-\theta \Delta_1}, \quad \mathrm{Re}(\theta) \geqslant 0 \tag{2.13}$$

is the marginal Laplace–Stieltjes transform of $\Delta_1, \Delta_2, \ldots$.

To allow the game to have some history prior to τ_0 we have the "delayed" functionals

$$\gamma_0(x, y, \theta) = Ee^{-xA_0 - yB_0 - \theta \tau_0} = \delta_0\big\{\theta + \lambda_A\big(1 - g(x)\big) + \lambda_B\big(1 - h(y)\big)\big\}, \tag{2.14}$$

where

$$\delta_0(\theta) = Ee^{-\theta \tau_0}. \tag{2.15}$$

Loosely speaking the *game* in this case is stochastic process $\mathcal{A}_\tau \otimes \mathcal{B}_\tau$ describing the evolution of a conflict between players A and B known to an observer upon process $\mathcal{T} = \{\tau_0, \tau_1, \ldots\}$.

3. The formalism of a rudimentary game

The game is over when on the kth observation epoch τ_k (for some k), the collateral damage A_k to player A or B_k to player B exceeds its respective threshold M or N, respectively. To further formalize the game we introduce the *exit indices*

$$\mu := \inf\{j \geqslant 0\colon \ A_j = A_0 + X_1 + \cdots + X_j > M\} \tag{3.1}$$

and

$$\nu := \inf\{k > M\colon \ B_k = B_0 + Y_1 + \cdots + Y_k > N\}. \tag{3.2}$$

Hence $\mu < \nu$ and hence, player A is defeated at τ_μ, which takes place earlier than player's B defeat at τ_ν. The *first passage time* τ_μ is the associated *exit time* from the game. The functional

$$\begin{aligned} &\Phi_{\mu\nu} = \Phi_{\mu\nu}(\alpha_0, \alpha_1, \beta_0, \beta_1, h_0, h_1) = E\big[e^{-\alpha_0 A_{\mu-1} - \alpha_1 A_\mu - \beta_0 B_{\mu-1} - \beta_1 B_\mu - h_0 \tau_{\mu-1} - h_1 \tau_\mu}\big], \\ &\quad \mathrm{Re}(\alpha_0) \geqslant 0, \ \mathrm{Re}(\alpha_1) \geqslant 0, \ \mathrm{Re}(\beta_0) \geqslant 0, \ \mathrm{Re}(\beta_1) \geqslant 0, \ \mathrm{Re}(h_0) \geqslant 0, \ \mathrm{Re}(h_1) \geqslant 0 \end{aligned} \tag{3.3}$$

of the game will represent the status of both players upon exit time τ_μ and *pre-exit time* $\tau_{\mu-1}$. The latter is of particular interest, because we would like to predict not only A's ruin time but also one observation prior to this.

Theorem 1 [17] below combined with Proposition 1 [17] establishes an explicit formula for $\Phi_{\mu\nu}$. With (2.12) and (2.15) we abbreviate

$$\gamma := \gamma(\alpha_0 + \alpha_1 + x, \beta_0 + \beta_1 + y, h_0 + h_1), \tag{3.4}$$

$$\gamma_0 := \gamma_0(\alpha_0 + \alpha_1 + x, \beta_0 + \beta_1 + y, h_0 + h_1), \tag{3.5}$$

$$\Gamma := \gamma(\alpha_1 + x, \beta_1 + y, h_1), \tag{3.6}$$

$$\Gamma_0 := \gamma_0(\alpha_1 + x, \beta_1 + y, h_1), \tag{3.7}$$

$$\Gamma^1 := \gamma(\alpha_1, \beta_1 + y, h_1), \tag{3.8}$$

$$\Gamma_0^1 := \gamma_0(\alpha_1, \beta_1 + y, h_1). \tag{3.9}$$

Furthermore, we introduce the Laplace–Carson transform

$$\mathcal{L}_{pq}(\cdot)(x,y) := xy \int_{p=0}^{\infty} \int_{q=0}^{\infty} e^{-xp-yq}(\cdot)\, d(p,q), \quad \mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0, \tag{3.10}$$

with the inverse

$$\mathcal{L}_{xy}^{-1}(\cdot)(p,q) = \mathfrak{L}^{-1}\left(\cdot\ \frac{1}{xy}\right), \tag{3.11}$$

where $\mathfrak{L}^{-1}$ is the inverse of the bivariate Laplace transform.

Theorem 1. *(See Dshalalow and Ke [17].) The functional $\Phi_{\mu\nu}$ satisfies the following formula*:

$$\Phi_{\mu\nu} = \mathcal{L}_{xy}^{-1}\left(\Gamma_0^1 - \Gamma_0 + \frac{\gamma_0}{1-\gamma}\left(\Gamma^1 - \Gamma\right)\right)(M,N), \tag{3.12}$$

where $\mathrm{Re}(h_0 + h_1) \geqslant 0$, $\mathrm{Re}(\alpha_0 + \alpha_1 + x) > 0$, $\mathrm{Re}(\beta_0 + \beta_1 + y) > 0$.

Remark 1. Because

$$\gamma(x,y,\theta) = \delta\{\theta + \lambda_A(1 - g(x)) + \lambda_B(1 - h(y))\} \tag{3.13}$$

(see Eq. (2.13)), the transform $\gamma = \gamma(\alpha_0 + \alpha_1 + x, \beta_0 + \beta_1 + y, h_0 + h_1)$ abbreviated so in (3.4) will precisely be

$$\gamma = \delta\{h_0 + h_1 + \lambda_A(1 - g(\alpha_0 + \alpha_1 + x)) + \lambda_B(1 - h(\beta_0 + \beta_1 + y))\}. \tag{3.14}$$

The other transforms, following (3.4), can be easily specified accordingly. By Proposition 1 below, the series $\sum_{j\geqslant 0}[\gamma(x,y,\theta)]^j$ (where $\gamma(x,y,\theta) = \delta\{\theta + \lambda_A(1-g(x)) + \lambda_B(1-h(y))\}$ as per (2.12)) related to the proof of Theorem 1, is convergent if $\mathrm{Re}(\theta) > 0$, $\mathrm{Re}(x) > 0$, $\mathrm{Re}(y) > 0$, of which any two of the inequalities can be relaxed to $\geqslant$.

Proposition 1 [17] below will be needed in the sequel.

Proposition 1. *The norm of δ, $\|\delta\{\theta + \lambda_A(1-g(x)) + \lambda_B(1-h(y))\}\|$, is strictly less than* 1 *if*

$$\mathrm{Re}(\theta) > 0, \qquad \mathrm{Re}(x) > 0, \qquad \mathrm{Re}(y) > 0. \tag{3.15}$$

Of the three inequalities in (3.15) *any two can be replaced with $\geqslant$.*

Remark 2. In (3.10) when we introduced the Laplace–Carson transform, we restricted x and y as to $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$. Proposition 1 states that, for the sake of convergence, Eq. (3.15) must hold and even be relaxed by any two inequalities replaced with $\geqslant$. Therefore, the natural requirements of (3.15) and Proposition 1 are met when we assume that merely $\mathrm{Re}(\theta) \geqslant 0$.

4. A modulated game

In this section we will analyze the game introduced in Section 3 by using auxiliary "layers". We will add two control levels $M_1 < M$ and $N_1 < N$ and incorporate it into the main functional $\Phi_{\mu\nu}$ of (3.3). The information associated with the so-called (M_1, N_1)-*layer* will become more conclusive if we can indicate on causes leading to the defeat of player A. We will define the corresponding *exit indices*

$$\mu_1 := \min\{j \geqslant 0\colon\ M_1 < A_j \leqslant M\}, \tag{4.1}$$

$$\nu_1 := \min\{k > \mu_1\colon\ N_1 < B_k \leqslant N\}, \tag{4.2}$$

$$\mu := \min\{m > \nu_1\colon\ A_m > M\}, \tag{4.3}$$

$$\nu := \min\{n > \mu\colon\ B_n > N\}, \tag{4.4}$$

in accordance with $\mu_1 < \nu_1 < \mu < \nu$ (to be referred to as Case 1), and then the functional

$$\begin{aligned}\Phi_{\mu_1<\nu_1\mu\nu} = E\big[&e^{-a_0 A_{\mu_1-1}-a_1 A_{\mu_1}-a_2 A_{\nu_1-1}-a_3 A_{\nu_1}-a_4 A_{\mu-1}-a_5 A_\mu} e^{-b_0 B_{\mu_1-1}-b_1 B\mu_1-b_2 B_{\nu_1-1}-b_3 B_{\nu_1}-b_4 B_{\mu-1}-b_5 B_\mu}\\ &\times e^{-h_0\tau_{\mu_1-1}-h_1\tau\mu_1-h_2\tau_{\nu_1-1}-h_3\tau_{\nu_1}-h_4\tau_{\mu-1}-h_5\tau_\mu}\big],\\ &\mathrm{Re}(\alpha_0)\geqslant 0,\ldots,\mathrm{Re}(h_5)\geqslant 0,\end{aligned} \tag{4.5}$$

which should give us a comprehensive information about the game (pretty much about both players) prior to the global exit. Notice that even though we defined the r.v. ν in (4.4), we did not include the information on τ_ν and the associated crossing value $\mathcal{B}_\tau$ at τ_ν, because the game will be over by τ_μ. In real-world situations though mutual actions can still go on beyond the truce possibly holding at τ_μ or at some later epoch. Nevertheless, we chose to drop them in order to simplify the final formulas. The extended information however can be readily revived.

By varying the layers M_1 and N_1 we can emulate the evolution of the game to some degree in light of the continuous time parameter processes. Furthermore, if we assume that this information will become available to both players, then the game can be calibrated upon their chief reference times τ_{μ_1}, τ_{ν_1}, τ_μ and τ_ν. In other words, the game can become modulated upon the main observation epochs changing their input parameters in accordance with the available data on casualties. The original assumptions on processes $\mathcal{A}$ and $\mathcal{B}$ will thus be altered to turn them from independent marked Poisson processes to modulated marked Poisson processes. Because of other technical challenges, we will keep the rigor of modulation to a minimum. For more details on modulation the reader is referred to the recent article [13] by the first author.

The *modulation* we are about to introduce will practically allow both players to elaborate the information about their own casualties and casualties of the enemy and change the pace of their strategies.

Firstly, the parameters of the modulated Poisson random measure in (2.1) may alter at any observation epoch from $\mathcal{T}$, but most significantly, upon crossings. We therefore assume that (2.2) and (2.3) will be modified as follows:

(1) If $\tau_0 < t \leqslant \tau_{\mu_1}$,

$$Ee^{-u\mathcal{A}((\tau_0,t])} = e^{\lambda_A^1(t-\tau_0)[g_1(u)-1]},\quad g_1(u) = Ee^{-ux_i},\ \mathrm{Re}(u)\geqslant 0, \tag{4.6}$$

$$Ee^{-u\mathcal{B}((\tau_0,t])} = e^{\lambda_B^1(t-\tau_0)[h_1(u)-1]},\quad h_1(u) = Ee^{-uy_i},\ \mathrm{Re}(u)\geqslant 0, \tag{4.7}$$

with i being such that corresponding casualties x_i, y_i take place between τ_0 and τ_{μ_1}.

(2) If $\tau_{\mu_1} < t \leqslant \tau_{\nu_1}$,

$$Ee^{-u\mathcal{A}((\tau_{\mu_1},t])} = e^{\lambda_A^2(t-\tau_{\mu_1})[g_2(u)-1]},\quad g_2(u) = Ee^{-ux_i},\ \mathrm{Re}(u)\geqslant 0, \tag{4.8}$$

$$Ee^{-u\mathcal{B}((\tau_0,t])} = e^{\lambda_B^1(t-\tau_0)[h_1(u)-1]},\quad h_2(u) = Ee^{-uy_i},\ \mathrm{Re}(u)\geqslant 0, \tag{4.9}$$

with i being such that corresponding casualties x_i, y_i take place between τ_{μ_1} and τ_{ν_1}.

(3) If $\tau_{\nu_1} < t \leqslant \tau_{\mu_2}$,

$$Ee^{-u\mathcal{A}((\tau_{\nu_1},t])} = e^{\lambda_A^3(t-\tau_{\nu_1})[g_3(u)-1]},\quad g_3(u) = Ee^{-ux_i},\ \mathrm{Re}(u)\geqslant 0, \tag{4.10}$$

$$Ee^{-u\mathcal{B}((\tau_{\nu_1},t])} = e^{\lambda_B^3(t-\tau_{\nu_1})[h_3(u)-1]},\quad h_3(u) = Ee^{-uy_i},\ \mathrm{Re}(u)\geqslant 0, \tag{4.11}$$

with i being such that corresponding casualties x_i, y_i take place between τ_{ν_1} and τ_{μ_2}.

(4) If $\tau_{\mu_2} < t$,

$$Ee^{-u\mathcal{A}((\tau_{\mu_2},t])} = e^{\lambda_A^4(t-\tau_{\mu_2})[g_4(u)-1]},\quad g_4(u) = Ee^{-ux_i},\ \mathrm{Re}(u)\geqslant 0, \tag{4.12}$$

$$Ee^{-u\mathcal{B}((\tau_{\mu_2},t])} = e^{\lambda_B^4(t-\tau_{\mu_2})[h_4(u)-1]},\quad h_4(u) = Ee^{-uy_i},\ \mathrm{Re}(u)\geqslant 0, \tag{4.13}$$

with i being such that corresponding casualties x_i, y_i take place after τ_{μ_2}.

The corresponding functionals of casualties accumulated over the periods between observations $\mathcal{T}$ are modified as follows:

$$\gamma_l(x,y,\theta) = Ee^{-xX_j-yY_j-\theta\Delta_j} = \delta_l\big\{\theta + \lambda_A^l\big(1-g_l(x)\big) + \lambda_B^l\big(1-h_l(y)\big)\big\},\quad j,l = 0,1,\ldots, \tag{4.14}$$

where

$$\delta_0(\theta) = Ee^{-\theta\Delta_0}, \qquad \delta_1(\theta) = Ee^{-\theta\Delta_i}, \quad i = 1, \ldots, \mu_1, \tag{4.15}$$

$$\delta_2(\theta) = Ee^{-\theta\Delta_i}, \quad i = \mu_1 + 1, \ldots, \nu_1, \tag{4.16}$$

$$\delta_3(\theta) = Ee^{-\theta\Delta_i}, \quad i = \nu_1 + 1, \ldots, \mu_2, \qquad \delta_4(\theta) = Ee^{-\theta\Delta_i}, \quad i > \mu_2, \tag{4.17}$$

formalizing the modulation of the observation process $\mathcal{T}$ which can also be impacted by the crossings data and thus yielding a feedback effect.

5. Multilayers models. The main theorem

We define the four-variate Laplace–Carson transform, also to be referred to as the $\mathcal{L}_{pqrs}$-*transform*,

$$\mathcal{L}_{pqrs}(\cdot)(u, v, x, y) := uvxy \int_{p=0}^{\infty} \int_{q=0}^{\infty} \int_{r=0}^{\infty} \int_{s=0}^{\infty} e^{-up-vq-xr-ys}(\cdot)\, d(p, q, r, s),$$
$$\operatorname{Re}(u) > 0,\ \operatorname{Re}(v) > 0,\ \operatorname{Re}(x) > 0,\ \operatorname{Re}(y) > 0. \tag{5.1}$$

Furthermore, introduce the following notation (having in mind (4.6)–(4.17)):

$$\gamma_0 = \gamma_0(a_0 + \cdots + a_5 + u + x, b_0 + \cdots + b_5 + v + y, h_0 + \cdots + h_5), \tag{5.2}$$

$$\gamma_1 = \gamma_1(a_0 + \cdots + a_5 + u + x, b_0 + \cdots + b_5 + v + y, h_0 + \cdots + h_5), \tag{5.3}$$

$$\Gamma_0 = \gamma_0(a_1 + \cdots + a_5 + u + x, b_1 + \cdots + b_5 + v + y, h_1 + \cdots + h_5), \tag{5.4}$$

$$\Gamma_0^1 = \gamma_0(a_1 + \cdots + a_5 + x, b_1 + \cdots + b_5 + v + y, h_1 + \cdots + h_5), \tag{5.5}$$

$$\Gamma = \gamma_1(a_1 + \cdots + a_5 + u + x, b_1 + \cdots + b_5 + v + y, h_1 + \cdots + h_5), \tag{5.6}$$

$$\Gamma^1 = \gamma_1(a_1 + \cdots + a_5 + x, b_1 + \cdots + b_5 + v + y, h_1 + \cdots + h_5), \tag{5.7}$$

$$G^1 = \gamma_2(a_2 + \cdots + a_5 + x, b_2 + \cdots + b_5 + v + y, h_2 + \cdots + h_5), \tag{5.8}$$

$$g^1 = \gamma_2(a_3 + a_4 + a_5 + x, b_3 + b_4 + b_5 + v + y, h_3 + \cdots + h_5), \tag{5.9}$$

$$g^{12} = \gamma_2(a_3 + a_4 + a_5 + x, b_3 + b_4 + b_5 + y, h_3 + h_4 + h_5), \tag{5.10}$$

$$F = \gamma_3(a_4 + a_5 + x, b_4 + b_5 + y, h_4 + h_5), \tag{5.11}$$

$$f = \gamma_3(a_5 + x, b_5 + y, h_5), \tag{5.12}$$

$$f^1 = \gamma_3(a_5, b_5 + y, h_5). \tag{5.13}$$

Theorem 2. *Under the condition that*

$$\operatorname{Re}(a_0 + \cdots + a_5 + u + x) > 0, \qquad \operatorname{Re}(b_0 + \cdots + b_5 + v + y) > 0, \tag{5.14}$$

and

$$\operatorname{Re}(h_0 + \cdots + h_5) \geqslant 0, \tag{5.15}$$

the $\mathcal{L}_{pqrs}$*-transform of functional* $\Phi_{\mu_1<\nu_1\mu\nu}$ *satisfies the following formula*:

$$\Phi^*_{\mu_1<\nu_1\mu\nu} := \Phi^*_{\mu_1<\nu_1\mu\nu}(u, v, x, y) := \mathcal{L}_{pqrs}\Phi_{\mu_1<\nu_1\mu\nu}(u, v, x, y)$$
$$= \left[\Gamma_0^1 - \Gamma_0 + \frac{\gamma_0}{1-\gamma_1}\big(\Gamma^1 - \Gamma\big)\right] \frac{g^{12} - g^1}{1 - G^1} \frac{f^1 - f}{1 - F}. \tag{5.16}$$

Proof. Introduce the auxiliary stochastic multiparametric families

$$\mu_1(p, r) := \min\{j \geqslant 0\colon\ p < A_j \leqslant r\}, \tag{5.17}$$

$$\nu_1(p, q, r) := \min\big\{k > \mu_1(p, r)\colon\ q < B_k \leqslant s\big\}, \tag{5.18}$$

$$\mu(p, q, r) := \min\big\{m > \nu_1(p, q, r)\colon\ A_m > r\big\}, \tag{5.19}$$

$$\nu(p, q, r, s) := \min\big\{n > \mu(p, q, r)\colon\ B_n > s\big\}. \tag{5.20}$$

In particular,

$$\mu_1 := \mu_1(M_1, M), \qquad \nu_1 := \nu_1(M_1, N_1, M), \qquad \mu := \mu(M_1, N_1, M), \qquad \nu := \nu(M_1, N_1, M, N).$$

The associated parametric family of random variables

$$\left\{\mathbf{1}_{\{\mu_1(p,r)=j,\ \nu_1(p,q,r)=k,\ \mu(p,q,r)=m,\ \nu(p,q,r,s)=n\}};\ p>0,\ q>0,\ r>0,\ s>0\right\} \tag{5.21}$$

will be incorporated into the functional $\Phi_{\mu_1<\nu_1\mu\nu}$ in due course. To continue with the proof of Theorem 2 we will need a lemma. The result of this lemma is quite unexpected. Unlike the game in Section 3, with independent thresholds for players A and B, now we have two of the four thresholds dependent and yet the lemma asserts that the application of operator $\mathcal{L}_{pqrs}$ to the family (5.21) produces the same result as for the case with no relationship between variable levels p and r and q and s. The reason why the invariance of the mentioned partial order is surprising, because upon crossing level p, process $\mathcal{A}$, if unrestricted as per (5.17), may also exceed level r as well. The latter is undesired, because under the current game settings, the r-crossing by $\mathcal{A}$ should occur only at a later time. So, by instructing these events to follow chronologically, we would naturally expect their dependence in some form, as oppose to what we will see in (5.23) (see also Section 2 of [17]). Yet the result of Lemma 1 seems to relax condition (5.17) that upon the p-crossing, the r-crossing does not occur. A special case of this lemma was proved in [17].

Lemma 1. *Let* $w(p,q,r,s,j,k,m,n)$

$$= \mathbf{1}_{\{\mu_1(p,r)=j,\ \nu_1(p,q,r)=k,\ \mu(p,q,r)=m,\ \nu(p,q,r,s)=n\}}\mathbf{1}_{(p<r)}\mathbf{1}_{(q<s)}\mathbf{1}_{(j<k<m<n)}. \tag{5.22}$$

Then, the following holds true:

$$\begin{aligned}&\mathcal{L}_{pqrs}\big(w(p,q,r,s,j,k,m,n)\big)(u,v,x,y)\\ &\quad= \big(e^{-uA_{j-1}}-e^{-uA_j}\big)\big(e^{-vB_{k-1}}-e^{-vB_k}\big)\big(e^{-xA_{m-1}}-e^{-xA_m}\big)\big(e^{-yB_{n-1}}-e^{-yB_n}\big)\mathbf{1}_{(j<k<m<n)}.\end{aligned} \tag{5.23}$$

Proof. From (5.17)–(5.20) we deduce that

$$\begin{aligned} w_1(p,q,r,s,j,k,m,n) &= \mathbf{1}_{\{A_{j-1}\leqslant p<A_j\}}\mathbf{1}_{\{p<r\}}\big(\mathbf{1}_{\{r<A_m\}}-\mathbf{1}_{\{r<A_{m-1}\}}\big)\\ &\quad\times \mathbf{1}_{\{B_{k-1}\leqslant q<B_k\}}\mathbf{1}_{\{q<s\}}\big(\mathbf{1}_{\{s<B_n\}}-\mathbf{1}_{\{s<B_{n-1}\}}\big)\mathbf{1}_{(j<k<m<n)}\end{aligned} \tag{5.24}$$

after dropping $1_{\{A_j\leqslant r\}}$ as redundancy under $1_{\{A_{m-1}\leqslant r\}}$ and $1_{(j<m)}$ and so doing it with $1_{\{B_k\leqslant s\}}$ using similar arguments. Applying operator (5.1) to (5.24) we have

$$\begin{aligned}&\mathcal{L}_{pqrs}\big(w_1(p,q,r,s,j,k,m,n)\big)(u,v,x,y)\\ &\quad= uvxy,\int_{p=A_{j-1}}^{A_j} e^{-up}\left\{\int_{r=p}^{A_m} e^{-xr}\,dr-\int_{r=p}^{A_{m-1}} e^{-xr}\,dr\right\}dp\int_{q=B_{k-1}}^{B_k} e^{-vq}\left\{\int_{s=q}^{B_n} e^{-ys}\,ds-\int_{s=q}^{B_{n-1}} e^{-ys}\,ds\right\}dq\,\mathbf{1}_{(j<k<m<n)}.\end{aligned}$$

The latter readily reduces to (5.23) thereby completing the proof. □

Now, we return to the proof of Theorem 2. For the sequel we will need the following factorization of (5.23), which after some algebra yields

$$\begin{aligned}&\mathcal{L}_{pqrs}\big(w_1(p,q,r,s,j,k,m,n)\big)(u,v,x,y)\\ &\quad= e^{-(u+x)A_{j-1}-(v+y)B_{j-1}}\big(e^{-xX_j}-e^{-(u+x)X_j}\big)e^{-(v+y)Y_j}e^{-\sum_{i=j+1}^{k-1}\{xX_i+(v+y)Y_i\}}\big(e^{-yY_k}-e^{-(v+y)Y_k}\big)e^{-xX_k}e^{-\sum_{i=k+1}^{m-1}\{xX_i+yY_i\}}\\ &\qquad\times\big(1-e^{-xX_m}\big)e^{-yY_m}e^{-\sum_{i=m+1}^{n-1}\{yY_i\}}\big(1-e^{-yY_n}\big)\mathbf{1}_{(j<k<m<n)}.\end{aligned} \tag{5.25}$$

The auxiliary process

$$\begin{aligned}&\mathfrak{X}(p,q,r,s;j,k,m,n)\\ &\quad= e^{-a_0A_{\mu_1(p,r)-1}-a_1A_{\mu_1(p,r)}-a_2A_{\nu_1(p,q,r)-1}-a_3A_{\nu_1(p,q,r)}-a_4A_{\mu(p,q,r)-1}-a_5A_{\mu(p,q,r)}}\\ &\qquad\times e^{-b_0B_{\mu_1(p,r)-1}-b_1B_{\mu_1(p,r)}-b_2B_{\nu_1(p,q,r)-1}-b_3B_{\nu_1(p,q,r)}-b_4B_{\mu(p,q,r)-1}-b_5B_{\mu(p,q,r)}}\\ &\qquad\times e^{-h_0\tau_{\mu_1(p,r)-1}-h_1\tau_{\mu_1(p,r)}-h_2\tau_{\nu_1(p,q,r)-1}-h_3\tau_{\nu_1(p,q,r)}-h_4\tau_{\mu(p,q,r)-1}-h_5\tau_{\mu(p,q,r)}}\\ &\qquad\times \mathbf{1}_{\{\mu_1(p,r)=j,\nu_1(p,q,r)=k,\mu(p,q,r)=m,\nu(p,q,r,s)=n\}}\mathbf{1}_{(p<r)}\mathbf{1}_{(q<s)}\mathbf{1}_{(j<k<m<n)}\end{aligned}$$

can be factorized as follows:

$$
\begin{aligned}
&\mathfrak{X}(p,q,r,s;\,j,k,m,n)\\
&\quad= e^{-(a_0+\cdots+a_5)A_{j-1}-(b_0+\cdots+b_5)B_{j-1}-(h_0+\cdots+h_5)\tau_{j-1}} e^{-(a_1+\cdots+a_5)X_j-(b_1+\cdots+b_5)Y_j-(h_1+\cdots+h_5)\Delta_j}\\
&\qquad\times e^{-\sum_{i=j+1}^{k-1}\{(a_2+\cdots+a_5)X_i+(b_2+\cdots+b_5)Y_i+(h_2+\cdots+h_5)\Delta_i\}} e^{-(a_3+\cdots+a_5)X_k-(b_3+\cdots+b_5)Y_k-(h_3+\cdots+h_5)\Delta_k}\\
&\qquad\times e^{-\sum_{i=k+1}^{m-1}\{(a_4+a_5)X_i+(b_4+b_5)Y_i+(h_4+h_5)\Delta_i\}} e^{-aX_m-b_5Y_m-h_5\Delta_m}\\
&\qquad\times \mathbf{1}_{\{\mu_1(p,r)=j,\nu_1(p,q,r)=k,\mu(p,q,r)=m,\nu(p,q,r,s)=n\}}\mathbf{1}_{(p<r)}\mathbf{1}_{(q<s)}\mathbf{1}_{(j<k<m<n)}.
\end{aligned}
\tag{5.26}
$$

Multiplying (5.25) by (5.26) and regrouping the factors in accordance with the previous factorization pattern we have

$$
\begin{aligned}
&\mathfrak{X}(p,q,r,s;\,j,k,m,n)\mathcal{L}_{pqrs}\big(w_1(p,q,r,s,j,k,m,n)\big)(u,v,x,y)\\
&\quad= e^{-(a_0+\cdots+a_5+u+x)A_{j-1}-(b_0+\cdots+b_5+v+y)B_{j-1}-(h_0+\cdots+h_5)\tau_{j-1}}\\
&\qquad\times \big(e^{-(a_1+\cdots+a_5+x)X_j}-e^{-(a_1+\cdots+a_5+u+x)X_j}\big)e^{-(b_1+\cdots+b_5+v+y)Y_j-(h_1+\cdots+h_5)\Delta_j}\\
&\qquad\times e^{-\sum_{i=j+1}^{k-1}\{(a_2+\cdots+a_5+x)X_i+(b_2+\cdots+b_5+v+y)Y_i+(h_2+\cdots+h_5)\Delta_i\}}\\
&\qquad\times \big(e^{-yY_k}-e^{-(v+y)Y_k}\big)e^{-(a_3+\cdots+a_5+x)X_k-(b_3+\cdots+b_5)Y_k-(h_3+\cdots+h_5)\Delta_k}\\
&\qquad\times e^{-\sum_{i=k+1}^{m-1}\{(a_4+a_5+x)X_i+(b_4+b_5+y)Y_i+(h_4+h_5)\Delta_i\}}\big(1-e^{-xX_m}\big)e^{-a_5X_m-(b_5+y)Y_m-h_5\Delta_m}\\
&\qquad\times e^{-\sum_{i=m+1}^{n-1}\{yY_i\}}\big(1-e^{-yY_n}\big)\mathbf{1}_{(j<k<m<n)}.
\end{aligned}
\tag{5.27}
$$

Then, returning to the main functional, by Fubini's theorem and independent increments property, we have

$$
\begin{aligned}
\Phi^*_{\mu_1<\nu_1\mu\nu}(u,v,x,y) &:= \mathcal{L}_{pqrs}\Phi_{\mu_1<\nu_1\mu\nu}(u,v,x,y)\\
&= \mathcal{L}_{pqrs}\bigg(\sum_j\sum_k\sum_m\sum_n E\big[\mathfrak{X}(p,q,r,s;\,j,k,m,n)w_1(p,q,r,s,j,k,m,n)\big]\bigg)(u,v,x,y)\\
&= \sum_{j\geqslant 0} E\big[e^{-(a_0+\cdots+a_5+u+x)A_{j-1}-(b_0+\cdots+b_5+v+y)B_{j-1}-(h_0+\cdots+h_5)\tau_{j-1}}\big]\\
&\quad\times E\big[e^{-(a_1+\cdots+a_5)X_j}\big(1-e^{-uX_j}\big)e^{-xX_j}e^{-(b_1+\cdots+b_5+v+y)Y_j-(h_1+\cdots+h_5)\Delta_j}\big]\\
&\quad\times \sum_{k>j} Ee^{-\sum_{i=j+1}^{k-1}\{(a_2+\cdots+a_5+x)X_i+(b_2+\cdots+b_5+v+y)Y_i+(h_2+\cdots+h_5)\Delta_i\}}\\
&\quad\times E\big[e^{-(a_3+\cdots+a_5+x)X_k}\big(1-e^{-vY_k}\big)e^{-(b_3+\cdots+b_5+y)Y_k-(h_3+\cdots+h_5)\Delta_k}\big]\\
&\quad\times \sum_{m>k} Ee^{-\sum_{i=k+1}^{m-1}\{(a_4+a_5+x)X_i+(b_4+b_5+y)Y_i+(h_4+h_5)\Delta_i\}}\\
&\quad\times E\big[e^{-a_5X_m}\big(1-e^{-xX_m}\big)e^{-(b_5+y)Y_m-h_5\Delta_m}\big]\sum_{n>m} Ee^{-\sum_{i=m+1}^{n-1} yY_i}E\big[1-e^{-yY_n}\big].
\end{aligned}
\tag{5.28}
$$

Considering $A_{-1}=B_{-1}=\Delta_{-1}=\sum_{i=s}^{s-1}=0$ and in light of (4.6)–(4.17) and (5.2)–(5.13), after the formal summation, we will arrive at formula (5.16). Notice that the summation in the last factor of (5.28) gives 1. The convergence of the series $\sum_{j\geqslant 0}\gamma_1^j$ (and other similar series) is due to Proposition 1 under a minor modification as in the formulation of Theorem 2. Also notice that all related functionals in formula (5.16) are of type δ_0, δ_1, δ_2, δ_3, δ_4. □

Remark 3. For the upcoming needs we introduce a tensor-like index operator $\mathcal{J}$ which will act on functionals like in (5.2)–(5.13) as follows:

$$
\mathcal{J}^i\Gamma = \Gamma_0^i - \Gamma_0 + \frac{\gamma_0}{1-\gamma_1}\big(\Gamma^i-\Gamma\big), \quad i=1,2,\ldots. \tag{5.29}
$$

Using (5.29), thus we have formula (5.16) of Theorem 2 in a more compact form:

$$
\Phi^*_{\mu_1<\nu_1\mu\nu} := \Phi^*_{\mu_1<\nu_1\mu\nu}(u,v,x,y) := \mathcal{L}_{pqrs}\Phi_{\mu_1<\nu_1\mu\nu}(u,v,x,y) = \bigg(\frac{g^{12}-g^1}{1-G^1}\,\frac{f^1-f}{1-F}\bigg)\mathcal{J}^1\Gamma. \tag{5.30}
$$

Remark 4. Note that the factor $\mathcal{J}^1\Gamma = \Gamma_0^1 - \Gamma_0 + \frac{\gamma_0}{1-\gamma}(\Gamma^1-\Gamma)$ in (5.16) and (5.30) is the same functional for the game of two players A and B with no layers except for variables u and v being "purged".

6. Other cases of layers

In this section we incorporate layers in slightly different chronology. Because the techniques will be very similar, we will cut them short. We begin with

Case 2: $\nu_1 < \mu_1 < \mu < \nu$

The corresponding control indices and the main functional are then defined as

$$\nu_1 := \min\{j \geqslant 0: \ N_1 < B_j \leqslant N\}, \tag{6.1}$$

$$\mu_1 := \min\{k \geqslant \nu_1: \ M_1 < A_j \leqslant M\}, \tag{6.2}$$

$$\mu := \min\{m > \mu_1: \ A_m > M\}, \tag{6.3}$$

$$\nu := \min\{n > \mu: \ B_n > N\}, \tag{6.4}$$

$$\begin{aligned}\Phi_{\nu_1<\mu_1\mu\nu} := E\big[&e^{-a_0 A_{\mu_1-1}-a_1 A_{\mu_1}-a_2 A_{\nu_1-1}-a_3 A_{\nu_1}-a_4 A_{\mu-1}-a_5 A_\mu} e^{-b_0 B_{\mu_1-1}-b_1 B_{u_1}-b_2 B_{\nu_1-1}-b_3 B_{\nu_1}-b_4 B_{\mu-1}-b_5 B_\mu} \\ &\times e^{-h_0\tau_{\mu_1-1}-h_1\tau_{u_1}-h_2\tau_{\nu_1-1}-h_3\tau_{\nu_1}-h_4\tau_{\mu-1}-h_5\tau_\mu}\big].\end{aligned} \tag{6.5}$$

Introducing auxiliary stochastic multiparametric families and proceeding as in Case 1 (including a similar utility of Lemma 1) we have

$$\begin{aligned}\Phi^*_{\nu_1<\mu_1\mu\nu} &:= \Phi^*_{\nu_1<\mu_1\mu\nu}(u,v,x,y) := \mathcal{L}_{pqrs}\Phi_{\nu_1<\mu_1\mu\nu}(u,v,x,y) \\ &= \sum_{j\geqslant 0} E\big[e^{-(a_0+\cdots+a_5+u+x)A_{j-1}-(b_0+\cdots+b_5+v+y)B_{j-1}-(h_0+\cdots+h_5)\tau_{j-1}}\big] \\ &\quad\times E\big[e^{-(a_1+\cdots+a_5+u+x)X_j}e^{-(b_1+\cdots+b_5)Y_j}\big(1-e^{-vY_j}\big)e^{-yY_j-(h_1+\cdots+h_5)\Delta_j}\big] \\ &\quad\times \sum_{k>j} E\big[e^{-\sum_{i=j+1}^{k-1}\{(a_2+\cdots+a_5+u+x)X_i+(b_2+\cdots+b_5+y)Y_i+(h_2+\cdots+h_5)\Delta_i\}}\big] \\ &\quad\times E\big[e^{-(a_3+\cdots+a_5+x)X_k}\big(1-e^{-uX_k}\big)e^{-(b_3+\cdots+b_5+y)Y_k-(h_3+\cdots+h_5)\Delta_k}\big] \\ &\quad\times \sum_{m>k} E\big[e^{-\sum_{i=k+1}^{m-1}\{(a_4+a_5+x)X_i+(b_4+b_5+y)Y_i+(h_4+h_5)\Delta_i\}}\big] \\ &\quad\times E\big[e^{-a_5X_m}\big(1-e^{-xX_m}\big)e^{-(b_5+y)Y_m-h_5\Delta_m}\big]\sum_{n>m}E\big[e^{-\sum_{i=m+1}^{n-1}\{yY_i\}}\big]E\big[1-e^{-yY_n}\big].\end{aligned}$$

As far as the modulation, without loss of generality, we just swap μ_1 and ν_1 to have the conditions similar to (4.6)–(4.17):

(1) If $\tau_0 < t \leqslant \tau_{\nu_1}$,

$$Ee^{-u\mathcal{A}((\tau_0,t])} = e^{\lambda_A^1(t-\tau_0)[g_1(u)-1]}, \quad g_1(u) = Ee^{-ux_i}, \ \operatorname{Re}(u) \geqslant 0, \tag{6.6}$$

$$Ee^{-u\mathcal{B}((\tau_0,t])} = e^{\lambda_B^1(t-\tau_0)[h_1(u)-1]}, \quad h_1(u) = Ee^{-uy_i}, \ \operatorname{Re}(u) \geqslant 0, \tag{6.7}$$

with i being such that corresponding casualties x_i, y_i take place between τ_0 and τ_{ν_1}.

(2) If $\tau_{\nu_1} < t \leqslant \tau_{\mu_1}$,

$$Ee^{-u\mathcal{A}((\tau_{\mu_1},t])} = e^{\lambda_A^2(t-\tau_{\mu_1})[g_2(u)-1]}, \quad g_2(u) = Ee^{-ux_i}, \ \operatorname{Re}(u) \geqslant 0, \tag{6.8}$$

$$Ee^{-u\mathcal{B}((\tau_0,t])} = e^{\lambda_B^1(t-\tau_0)[h_1(u)-1]}, \quad h_2(u) = Ee^{-uy_i}, \ \operatorname{Re}(u) \geqslant 0, \tag{6.9}$$

with i being such that corresponding casualties x_i, y_i take place between τ_{ν_1} and τ_{μ_1}.

(3) If $\tau_{\mu_1} < t \leqslant \tau_{\mu_2}$,

$$Ee^{-u\mathcal{A}((\tau_{\nu_1},t])} = e^{\lambda_A^3(t-\tau_{\nu_1})[g_3(u)-1]}, \quad g_3(u) = Ee^{-ux_i}, \ \operatorname{Re}(u) \geqslant 0, \tag{6.10}$$

$$Ee^{-u\mathcal{B}((\tau_{\nu_1},t])} = e^{\lambda_B^3(t-\tau_{\nu_1})[h_3(u)-1]}, \quad h_3(u) = Ee^{-uy_i}, \ \operatorname{Re}(u) \geqslant 0, \tag{6.11}$$

with i being such that corresponding casualties x_i, y_i take place between τ_{μ_1} and τ_{μ_2}.

(4) If $\tau_{\mu_2} < t$ (same as in (4.12)–(4.13)),

$$Ee^{-u\mathcal{A}((\tau_{\mu_2},t])} = e^{\lambda_A^4(t-\tau_{\mu_2})[g_4(u)-1]}, \quad g_4(u) = Ee^{-ux_i}, \ \operatorname{Re}(u) \geqslant 0, \tag{6.12}$$

$$Ee^{-u\mathcal{B}((\tau_{\mu_2},t])} = e^{\lambda_B^4(t-\tau_{\mu_2})[h_4(u)-1]}, \quad h_4(u) = Ee^{-uy_i}, \ \operatorname{Re}(u) \geqslant 0, \tag{6.13}$$

with i being such that corresponding casualties x_i, y_i take place after τ_{μ_2}.

The corresponding functionals of casualties accumulated over the periods between observations $\mathcal{T}$ are modified as follows:

$$\gamma_l(x,y,\theta) = Ee^{-xX_j - yY_j - \theta\Delta_j} = \delta_l\{\theta + \lambda_A^l(1 - g_l(x)) + \lambda_B^l(1 - h_l(y))\}, \quad j,l = 0,1,\ldots, \tag{6.14}$$

where

$$\delta_0(\theta) = Ee^{-\theta\Delta_0}, \qquad \delta_1(\theta) = Ee^{-\theta\Delta_i}, \quad i = 1,\ldots,\nu_1, \tag{6.15}$$

$$\delta_2(\theta) = Ee^{-\theta\Delta_i}, \quad i = \nu_1 + 1,\ldots,\mu_1, \tag{6.16}$$

$$\delta_3(\theta) = Ee^{-\theta\Delta_i}, \quad i = \mu_1 + 1,\ldots,\mu_2, \qquad \delta_4(\theta) = Ee^{-\theta\Delta_i}, \quad i > \mu_2, \tag{6.17}$$

formalizing the modulation of the observation process $\mathcal{T}$ which can also be impacted by the crossings data and thus yielding a feedback effect.

With the notation (5.2)–(5.13) and additionally,

$$\Gamma_0^2 = \gamma_0(a_1 + \cdots + a_5 + u + x, b_1 + \cdots + b_5 + y, h_1 + \cdots + h_5), \tag{6.18}$$

$$\Gamma^2 = \gamma_1(a_1 + \cdots + a_5 + u + x, b_1 + \cdots + b_5 + y, h_1 + \cdots + h_5), \tag{6.19}$$

$$G^2 = \gamma_2(a_2 + \cdots + a_5 + u + x, b_2 + \cdots + b_5 + y, h_2 + \cdots + h_5), \tag{6.20}$$

$$g^2 = \gamma_2(a_3 + \cdots + a_5 + u + x, b_3 + \cdots + b_5 + y, h_3 + \cdots + h_5), \tag{6.21}$$

we arrive at

$$\Phi^*_{\nu_1<\mu_1\mu\nu} = \left[\Gamma_0^2 - \Gamma_0 + \frac{\gamma_0}{1-\gamma_1}\left(\Gamma^2 - \Gamma\right)\right]\frac{g^{12} - g^2}{1 - G^2}\,\frac{f^1 - f}{1 - F} = \mathcal{J}^2\Gamma\,\frac{g^{12} - g^2}{1 - G^2}\,\frac{f^1 - f}{1 - F}. \tag{6.22}$$

The above result can be summarized as

Theorem 3. *Under the condition that*

$$\mathrm{Re}(a_0 + \cdots + a_5 + u + x) > 0, \qquad \mathrm{Re}(b_0 + \cdots + b_5 + v + y) > 0, \tag{6.23}$$

and

$$\mathrm{Re}(h_0 + \cdots + h_5) \geqslant 0, \tag{6.24}$$

the $\mathcal{L}_{pqrs}$-transform of $\Phi_{\nu_1<\mu_1\mu\nu}$ satisfies formula (6.22).

Case 3: $\mu_1 = \nu_1 < \mu < \nu$.

The corresponding control indices and the main functional are then defined as

$$\mu_1 := \min\{j \geqslant 0:\ M_1 < A_j \leqslant M\}, \tag{6.25}$$

$$\nu_1 := \min\{k \geqslant 0:\ N_1 < B_k \leqslant N\}, \tag{6.26}$$

$$\mu := \min\{m > \nu_1:\ A_m > M\}, \tag{6.27}$$

$$\nu := \min\{n > \mu:\ B_n > N\} \tag{6.28}$$

and

$$\begin{aligned}\Phi_{\mu_1=\nu_1\mu\nu} = E\big[&e^{-(a_0+a_2)A_{\mu_1-1} - (a_1+a_3)A_{\mu_1} - a_4A_{\mu-1} - a_5A_\mu}e^{-(b_0+b_2)B_{\mu_1-1} - (b_1+b_3)B\mu_1 - b_4B_{\mu-1} - b_5B_\mu}\\ &\times e^{-(h_0+h_2)\tau_{\mu_1-1} - (h_1+h_3)\tau\mu_1 - h_4\tau_{\mu-1} - h_5\tau_\mu}\mathbf{1}_{\{\mu_1=\nu_1\}}\big].\end{aligned} \tag{6.29}$$

Again proceeding as in Case 1 we have

$$\begin{aligned}\Phi^*_{\nu_1=\mu_1\mu\nu}(u,v,x,y) &= \mathcal{L}_{pqrs}\Phi_{\mu_1=\nu_1\mu\nu}(u,v,x,y)\\ &= \sum_{j\geqslant 0} E\big[e^{-(a_0+\cdots+a_5+u+x)A_{j-1} - (b_0+\cdots+b_5+v+y)B_{j-1} - (h_0+\cdots+h_5)\tau_{j-1}}\big]\\ &\quad\times E\big[e^{-(a_1+a_3+a_4+a_5+x)X_j}\left(1 - e^{-uX_j}\right)e^{-(b_1+b_3+b_4+b_5)Y_j}\left(1 - e^{-vY_j}\right)e^{-yY_j - (h_1+h_3+h_4+h_5)\Delta_j}\big]\\ &\quad\times \sum_{m>j} E\big[e^{-\sum_{i=j+1}^{m-1}\{(a_4+a_5+x)X_i + (b_4+b_5+y)Y_i + (h_4+h_5)\Delta_i\}}\big]\\ &\quad\times E\big[e^{-a_5X_m}\left(1 - e^{-xX_m}\right)e^{-(b_5+y)Y_m - h_5\Delta_m}\big].\end{aligned} \tag{6.30}$$

Below are our conditions for the modulation:

(1) If $\tau_0 < t \leqslant \tau_{\mu_1} = \tau_{\nu_1}$,

$$Ee^{-u\mathcal{A}((\tau_0,t])} = e^{\lambda_A^1(t-\tau_0)[g_1(u)-1]}, \quad g_1(u) = Ee^{-ux_i}, \ \mathrm{Re}(u) \geqslant 0, \tag{6.31}$$

$$Ee^{-u\mathcal{B}((\tau_0,t])} = e^{\lambda_B^1(t-\tau_0)[h_1(u)-1]}, \quad h_1(u) = Ee^{-uy_i}, \ \mathrm{Re}(u) \geqslant 0, \tag{6.32}$$

with i being such that corresponding casualties x_i, y_i take place between τ_0 and $\tau_{\mu_1} = \tau_{\nu_1}$.

(2) If $\tau_{\mu_1} = \tau_{\mu_1} < t \leqslant \tau_{\mu_2}$,

$$Ee^{-u\mathcal{A}((\tau_{\nu_1},t])} = e^{\lambda_A^3(t-\tau_{\nu_1})[g_3(u)-1]}, \quad g_3(u) = Ee^{-ux_i}, \ \mathrm{Re}(u) \geqslant 0, \tag{6.33}$$

$$Ee^{-u\mathcal{B}((\tau_{\nu_1},t])} = e^{\lambda_B^3(t-\tau_{\nu_1})[h_3(u)-1]}, \quad h_3(u) = Ee^{-uy_i}, \ \mathrm{Re}(u) \geqslant 0, \tag{6.34}$$

with i being such that corresponding casualties x_i, y_i take place between τ_{μ_1} and τ_{μ_2}.

(3) If $\tau_{\mu_2} < t$

$$Ee^{-u\mathcal{A}((\tau_{\mu_2},t])} = e^{\lambda_A^4(t-\tau_{\mu_2})[g_4(u)-1]}, \quad g_4(u) = Ee^{-ux_i}, \ \mathrm{Re}(u) \geqslant 0, \tag{6.35}$$

$$Ee^{-u\mathcal{B}((\tau_{\mu_2},t])} = e^{\lambda_B^4(t-\tau_{\mu_2})[h_4(u)-1]}, \quad h_4(u) = Ee^{-uy_i}, \ \mathrm{Re}(u) \geqslant 0, \tag{6.36}$$

with i being such that corresponding casualties x_i, y_i take place after τ_{μ_2}.

The corresponding functionals of casualties accumulated over the periods between observations $\mathcal{T}$ are modified as follows:

$$\gamma_l(x,y,\theta) = Ee^{-xX_j - yY_j - \theta\Delta_j} = \delta_l\{\theta + \lambda_A^l(1-g_l(x)) + \lambda_B^l(1-h_l(y))\}, \quad j,l = 0,1,\ldots, \tag{6.37}$$

where

$$\delta_0(\theta) = Ee^{-\theta\Delta_0}, \qquad \delta_1(\theta) = Ee^{-\theta\Delta_i}, \quad i = 1,\ldots,\mu_1 = \nu_1, \tag{6.38}$$

$$\delta_3(\theta) = Ee^{-\theta\Delta_i}, \quad i = \mu_1+1,\ldots,\mu_2, \qquad \delta_4(\theta) = Ee^{-\theta\Delta_i}, \quad i > \mu_2, \tag{6.39}$$

formalizing the modulation of the observation process $\mathcal{T}$ which can also be impacted by the crossings data and thus yielding a feedback effect.

Since $(1-e^{-uX_j})(1-e^{-vY_j}) = 1 - e^{-vY_j} - e^{-uX_j} + e^{-uX_j}e^{-vY_j}$, we have the second factor

$$\begin{aligned} &E\big[e^{-(a_1+\cdots+a_5+x)X_j}(1-e^{-uX_j})e^{-(b_1+\cdots+b_5)Y_j}(1-e^{-vY_j})e^{-yY_j-(h_1+\cdots+h_5)\Delta_j}\big] \\ &\quad = \begin{cases} \Gamma_0^{12} - \Gamma_0^1 - \Gamma_0^2 + \Gamma_0, & j = 0, \\ \Gamma^{12} - \Gamma^1 - \Gamma^2 + \Gamma, & j > 0, \end{cases} \end{aligned} \tag{6.40}$$

where

$$\Gamma = \gamma_1(a_1+a_3+a_4+a_5+u+x, b_1+b_3+b_4+b_5+v+y, h_1+h_3+h_4+h_5), \tag{6.41}$$

$$\Gamma^1 = \gamma_1(a_1+a_3+a_4+a_5+x, b_1+b_3+b_4+b_5+v+y, h_1+h_3+h_4+h_5), \tag{6.42}$$

$$\Gamma^2 = \gamma_1(a_1+a_3+a_4+a_5+u+x, b_1+b_3+b_4+b_5+y, h_1+h_3+h_4+h_5), \tag{6.43}$$

$$\Gamma^{12} = \gamma_1(a_1+a_3+a_4+a_5+x, b_1+b_3+b_4+b_5+y, h_1+h_3+h_4+h_5), \tag{6.44}$$

$$\Gamma_0 = \gamma_0(a_1+a_3+a_4+a_5+u+x, b_1+b_3+b_4+b_5+v+y, h_1+h_3+h_4+h_5), \tag{6.45}$$

$$\Gamma_0^1 = \gamma_0(a_1+a_3+a_4+a_5+x, b_1+b_3+b_4+b_5+v+y, h_1+h_3+h_4+h_5), \tag{6.46}$$

$$\Gamma_0^2 = \gamma_0(a_1+a_3+a_4+a_5+u+x, b_1+b_3+b_4+b_5+y, h_1+h_3+h_4+h_5), \tag{6.47}$$

$$\Gamma_0^{12} = \gamma_0(a_1+a_3+a_4+a_5+x, b_1+b_3+b_4+b_5+y, h_1+h_3+h_4+h_5). \tag{6.48}$$

Hence,

$$\begin{aligned} \Phi^*_{\nu_1=\mu_1\mu\nu} &= \left[\Gamma_0^{12} - \Gamma_0^1 - \Gamma_0^2 + \Gamma_0 + \frac{\gamma_0}{1-\gamma_1}\big(\Gamma^{12} - \Gamma^1 - \Gamma^2 + \Gamma\big)\right]\frac{f^1 - f}{1-F} \\ &= \left[\Gamma_0^{12} - \Gamma_0^1 + \frac{\gamma_0}{1-\gamma_1}\big(\Gamma^{12} - \Gamma^1\big)\right]\frac{f^1-f}{1-F} - \left[\Gamma_0^2 - \Gamma_0 + \frac{\gamma_0}{1-\gamma_1}\big(\Gamma^1 - \Gamma\big)\right]\frac{f^1-f}{1-F} \\ &= \left(\frac{f^1-f}{1-F}\right)\big(\mathcal{J}^{12} - \mathcal{J}^2\big)\Gamma. \end{aligned} \tag{6.49}$$

The above result can be summarized as

Theorem 4. *Under the condition that*

$$\mathrm{Re}(a_0 + \cdots + a_5 + u + x) > 0, \qquad \mathrm{Re}(b_0 + \cdots + b_5 + v + y) > 0, \tag{6.50}$$

and

$$\mathrm{Re}(h_0 + \cdots + h_5) \geqslant 0, \tag{6.51}$$

the $\mathcal{L}_{pqrs}$-transform of $\Phi^*_{\nu_1=\mu_1\mu\nu}$ satisfies formula (6.49).

Remark 5. The inverse of the $\mathcal{L}_{pqrs}$-transform is $\mathcal{L}^{-1}_{uvxy}(\cdot)(p,q,r,s) = \mathfrak{L}^{-1}(\cdot\ \frac{1}{uvxy})$, where $\mathfrak{L}^{-1}$ is the four-dimensional inverse Laplace transform. When applied to the transformed functionals Φ^* in Theorems 2–4, it will restore the original functionals Φ of the game.

Summing up all three transformations from Theorems 2–4 gives

$$\Phi^*_{\mu_1\nu_1\mu\nu} := \Phi^*_{\mu_1<\nu_1\mu\nu} + \Phi^*_{\nu_1<\mu_1\mu\nu} + \Phi^*_{\nu_1=\mu_1\mu\nu} = \frac{f^1 - f}{1-F}\left\{\frac{g^{12}-g^2}{1-G^2}\big(\mathcal{J}^1 + \mathcal{J}^2\big)\Gamma + \big(\mathcal{J}^{12} - \mathcal{J}^2\big)\Gamma\right\}. \tag{6.52}$$

(6.52) gives the transformed functional of the game lost by player A and with two variable layers. The total functional of the game with two variable auxiliary layers M_1 and N_1 (without any relation between the two) can be obtained by the application of the inverse $\mathcal{L}^{-1}_{uvxy}$ to (6.52):

$$\Phi_{\mu_1\nu_1\mu\nu} = \mathcal{L}^{-1}_{uvxy}\left(\frac{f^1 - f}{1-F}\left\{\frac{g^{12}-g^2}{1-G^2}\big(\mathcal{J}^1 + \mathcal{J}^2\big)\Gamma + \big(\mathcal{J}^{12} - \mathcal{J}^2\big)\Gamma\right\}\right)(M_1, N_1, M, N). \tag{6.53}$$

Acknowledgments

The authors are most thankful to the referees whose comments were very constructive and right to the point. We were happy to follow all of them, which greatly improved the paper.

References

[1] A. Ardema, M. Heymann, N. Rajan, Analysis of a combat problem: The turret game, J. Optim. Theory Appl. 54 (1) (1987) 23–42.
[2] A. Bagchi, Stackelberg Differential Games in Economics Models, Springer-Verlag, 1984.
[3] T.S. Basar, G.J. Olsder, Dynamic Noncooperative Game Theory, Academic Press, Orlando, 1982.
[4] N.H. Bingham, Random walk and fluctuation theory, in: D.N. Shanbhag, C.R. Rao (Eds.), Handbook of Statistics, vol. 19, Elsevier Science, 2001, pp. 171–213.
[5] M.H. Breitner, H.J. Pesch, W. Grimm, Complex differential games of pursuit–evasion type with state constraints, part 1: Necessary conditions for optimal open-loop strategies, J. Optim. Theory Appl. 78 (3) (1993) 419–441.
[6] M.H. Breitner, H.J. Pesch, W. Grimm, Complex differential games of pursuit–evasion type with state constraints, part 2: Numerical computation of optimal open-loop strategies, J. Optim. Theory Appl. 78 (3) (1993) 443–463.
[7] A. Davidovitz, J. Shinar, Two-target game model of an air combat with fire-and-forget all-aspect missiles, J. Optim. Theory Appl. 63 (2) (1989) 133–165.
[8] E. Dockner, S. Jørgensen, N.V. Long, G. Sorger (Eds.), Differential Games in Economics and Management Science, Cambridge University Press, Cambridge, 2000.
[9] J.H. Dshalalow, On termination time processes, in: J. Galambos, J. Gani (Eds.), Studies in Applied Probability; Essays in Honour of Lajos Takács, Applied Probability Trust, Sheffield, UK, J. Appl. Probab. 31A (1994) 325–336.
[10] J.H. Dshalalow, On the level crossing of multi-dimensional delayed renewal processes, J. Appl. Math. Stoch. Anal. 10 (4) (1997) 355–361.
[11] J.H. Dshalalow, Fluctuations of recurrent processes and their application to the stock market, Stoch. Anal. Appl. 22 (1) (2004) 67–79.
[12] J.H. Dshalalow, On exit times of a multivariate random walk with some applications to finance, Nonlinear Anal. 63 (2005) 569–577.
[13] J.H. Dshalalow, Characterization of modulated Cox measures on topological spaces, Int. J. Appl. Math. Stat. 11 (7) (2007) 21–37, a special volume in commemoration of Leonard Euler's tricentennial.
[14] J.H. Dshalalow, Random walk analysis in antagonistic stochastic games, Stoch. Anal. Appl. 26 (2008) 738–783.
[15] J.H. Dshalalow, W. Huang, On noncooperative hybrid stochastic games, Nonlinear Anal. Hybrid Syst. 2 (3) (2008) 803–811, Special Issue Section: Analysis and Design of Hybrid Systems.
[16] J.H. Dshalalow, W. Huang, A stochastic game with a two-phase conflict, in: Adv. Nonlinear Anal., vol. 18, Cambridge Academic Publishers, 2008, pp. 201–209.
[17] J.H. Dshalalow, H.-J. Ke, Layers of noncooperative games, Nonlinear Anal. Ser. A, doi:10.1016/j.na.2008.10.072, in press.
[18] J.H. Dshalalow, A. Liew, On exit time of a multivariate random walk and its embedding in a quasi-Poisson process, Stoch. Anal. Appl. 24 (2006) 451–474.
[19] J.H. Dshalalow, A. Liew, On fluctuations of a multivariate random walk with some applications to stock options trading and hedging, Math. Model. Comput. 44 (10) (2006) 931–944.
[20] J.H. Dshalalow, A. Liew, On level crossings of an oscillating marked random walk, Comput. Math. Appl. 52 (2006) 917–932.
[21] J.H. Dshalalow, L. Tadj, On applications of first excess random processes to queueing systems with random server capacity and capacity dependent service time, Stoch. Stochastic Reports 45 (1993) 45–60.
[22] P.C. Fishburn, Non-cooperative stochastic dominance games, Internat. J. Game Theory 7 (1) (1978) 51–61.
[23] L. Garrido (Ed.), Fluctuations and Stochastic Phenomena in Condensed Matter, Proceedings of the Sitges Conference on Statistical Mechanics, Sitges, Barcelona, Spain, Ma, 1986, Lecture Notes in Phys., vol. 268, Springer-Verlag, 1987.
[24] P.S. Griffin, T.R. McConnell, On the position of a random walk at the time of first exit from a sphere, Ann. Probab. 20 (2) (1992) 825–854.
[25] A. Grigor'yan, M. Kelbert, Range of fluctuations of Brownian motion on a complete Riemannian manifold, Ann. Probab. 26 (1) (1998) 78–111.

[26] T. Hida (Ed.), Mathematical Approach to Fluctuations: Astronomy, Biology and Quantum Dynamics, Proceedings of the Iias Workshop, Kyoto, Japan, May 18-21, 1992, World Scientific Publishers, 1995.
[27] W. Horsthemke, D. Kondepudi (Eds.), Fluctuations and Sensitivity in Nonequilibrium Systems, Proceedings of an International Conference, University of Texas, Austin, Springer-Verlag, New York, 1984.
[28] R. Isaacs, Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization, Dover, 1999.
[29] S. Jørgensen, G. Zaccour, Differential Games in Marketing, Internat. Ser. Quant. Mark., vol. 15, Springer-Verlag, 2004.
[30] T.V. Kadankova, Exit, passage, and crossing times and overshoots for a Poisson compound process with an exponential component, Theory Probab. Math. Statist. 75 (2007) 23–29.
[31] M. Kelbert, Y. Suhov, Three-indexed processes: A high level crossing analysis, J. Appl. Math. Stoch. Anal. 16 (2) (2003) 127–139.
[32] R.V. Konstantinov, E.S. Polovinkin, Mathematical simulation of a dynamic game in the enterprise competition problem, Cybernet. Systems Anal. 40 (5) (2004) 720–725.
[33] A.E. Kyprianou, M.R. Pistorius, Perpetual options and Canadization through fluctuation theory, Ann. Appl. Probab. 13 (3) (2003) 1077–1098.
[34] E. Mellander, A. Vredin, A. Warne, Stochastic trends and economic fluctuations in a small open economy, J. Appl. Econometrics 7 (4) (1992) 369–394.
[35] J. Muzy, J. Delour, E. Bacry, Modelling fluctuations of financial time series: From cascade process to stochastic volatility model, Eur. Phys. J. B 17 (2000) 537–548.
[36] J.C. Perry, B.D. Roitberg, Games among cannibals: Competition to cannibalize and parent–offspring conflict lead to increased sibling cannibalism, J. Evol. Biology 18 (6) (2005) 1523–1533.
[37] S. Redner, A Guide to First-Passage Processes, Cambridge University Press, Cambridge, 2001.
[38] A. Segal, T. Miloh, A new 3-D pursuit–evasion differential game between two bank-to-turn airborne vehicles, Optimal Control Appl. Methods 20 (5) (1999) 223–234.
[39] V.N. Shashikin, Antagonistic game with interval payoff functions, Cybernet. Systems Anal. 40 (4) (2004) 556–564.
[40] T. Shima, Capture conditions in a pursuit–evasion game between players with biproper dynamics, J. Optim. Theory Appl. 126 (3) (2005) 503–528.
[41] M. Sobel, Noncooperative stochastic games, Ann. Math. Stat. 42 (1971) 1930–1935.
[42] L. Takács, On fluctuations problems in the theory of queues, Adv. in Appl. Probab. 8 (3) (1976) 548–583.
[43] L. Takács, On fluctuations of sums of random variables, in: G.-C. Rota (Ed.), Studies in Probability and Ergodic Theory, in: Adv. Math. Suppl. Studies, vol. 2, 1978, pp. 45–93.