A Vanishing Finite Sum Associated with Jacobi’s Triple Product Identity

KENNETH B. STOLARSKY*

Institute for Advanced Study, Princeton, New Jersey 08540
Communicated by Freeman J. Dyson
Received March 5, 1968

ABSTRACT

Jacobi’s triple product identity is shown to be equivalent to the vanishing of a finite sum. The vanishing of the sum is then established independently of Jacobi’s identity, and the behavior of a related finite sum is discussed. Finally, the possibility of generalizing Jacobi’s identity to include variables with non-quadratic exponents is examined; no decisive result is obtained.

1. INTRODUCTION

In [1] Andrews deduces the triple product identity

\[ \prod_{n=0}^{\infty} \{(1 - x^{2n+2})(1 + x^{2n+1}z)(1 + x^{2n+1}z^{-1})\} = \sum_{n=-\infty}^{\infty} x^{n^2} z^n \]  

(1)

from formulae (E1) and (E2), the latter being equivalent to

\[ \prod_{k=0}^{\infty} (1 + x^{2k+1}z)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n z^n}{(x^2; x^2)_n} \]  

(2)

where \((a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\). Here, by means of (2), we shall expand the quotient of the infinite sum on the right of (1) into a Laurent series in \(z\). Denote the coefficient of \(z^b\) in this expansion by \(A_b(x)\). One naturally expects that \(A_0(x)\) will be the first product of (1), and that \(A_b(x) = 0\) for \(b \neq 0\). The main purpose of this paper is to prove the latter fact without using (1). Note that all series involved here are absolutely convergent, provided \(|x| < 1\) and \(|x| < |z| < |x|^{-1}\).

In Section 2 the coefficients \(A_b(x)\) are calculated, and in Section 3

---

* This work was supported in part by a WARF fellowship held by the author at the University of Wisconsin, and also by National Science Foundation grant GP-7952X.

392
their behavior is established without using (1) by showing that a certain finite sum vanishes. A related finite sum of more complex behavior is discussed in Section 4. Finally, the possibility of generalizing Jacobi's identity to include variables with non-quadratic exponents is examined in Section 5.

For some other recent proofs of the Jacobi identity see [4] and [5].

2. EXPANSION OF THE QUOTIENT

Professor G. E. Andrews has kindly provided the author with proofs of

\[ H_j(x) = \sum_{n=0}^{\infty} \frac{x^{2n+j}}{(x^2; x^2)_n(x^2; x^2)_{n+j}} = (x^2; x^2)_\infty \sum_{n=0}^{\infty} (-1)^n x^{(2n+1)j + n(n+1)} \]  

for non-negative integers \( j \); e.g., let \( x = 0, \tau = q = x^2, \gamma = x^{2j+2}, \) and \( \beta \to 0 \) in (11) of his paper [2]. Expanding the quotient by means of (2), we find that

\[ A_b(x) = \sum_{j=0}^{\infty} (-1)^j x^{(j+b)^2} H_j(x) + \sum_{j=0}^{\infty} (-1)^{j+1} x^{(j+1-b)^2} H_{j+1}(x) \]  

Thus, by means of (3), we see that the coefficient of \( x^N \) in \((x^2; x^2)_\infty A_b(x)\) is the finite sum

\[ \sum'(-1)^m + \sum(-1)^{j+m} + \sum'(-1)^{j+m} \]  

where the primes indicate that the sums are over \( m \geq 0, j \geq 1, \) with the constraints \( b^2 + m(m+1) = N, (j+b)^2 + (2m+1)j + m(m+1) = N, \) and \( (j-b)^2 + (2m+1)j + m(m+1) = N, \) respectively. \( A_0(x) = (x^2; x^2)_\infty \) is easily seen to follow from an identity of Jacobi [3, p. 285, theorem 357].

3. VANISHING OF THE FINITE SUM

Let \( u = m + j \) and define the \( u \)-parity of a lattice point \((u, j)\) in the \( uj \) plane to be \((-1)^u\). Then, for \( b \neq 0 \), the double sums in (5) give the sum of the \( u \)-parities of all lattice points lying both in the wedge \( W_1 \) defined by \( j \geq 1, u \geq j, \) and also on one of the parabolas \( P^+(u), P^-(u) \) defined by

\[ u(u+1) \pm 2bj = N - b^2. \]

Note that these parabolas intersect the positive \( u \)-axis at the same point \( p \). Let \( W_2 \) be the wedge defined by \( u \geq |j| \). Trivial considerations of
symmetry then allow us to write (5) as one-half the sum of the \(u\)-parities of all lattice points in the wedge \(W_2\) which lie on \(P^+\) or \(P^-\); if \(p\) is a lattice point it is counted twice. Thus, to show (5) vanishes, it suffices (again by considerations of symmetry) to show that the \(u\)-parity sum over \(P^+\) vanishes.

First, assume \(N \geq b^2\). Let \([u_1]\) denote the greatest integer in \(u_1\), but let \([u_1, u_2]\) denote the closed interval from \(u_1\) to \(u_2\). As \(u\) increases, the parabola \(P^+\) enters \(W_2\) at the point \((u_1, j_1)\), and exits at \((u_2, j_2)\), where

\[
u_1 = -b + \{-1 + \sqrt{(1 + 4(N + b))/2}\}
\]

and

\[
u_2 = b + \{-1 + \sqrt{(1 + 4(N - b))/2}\}.
\]

Thus we need to show that the set \(U\) of integers \(u \in [u_1, u_2]\) such that

\[u(u + 1) \equiv N - b^2 \mod 2b\]

contains an equal number of odd and even integers. It follows from (6) and \(N \geq b^2\) that \(u_2 - u_1 \geq 2b - 1\), so \([u_1, u_2]\) contains either \(2b - 1\) or \(2b\) integers. Clearly there is at most one integer \(i\) such that \(u_2 - b < i < u_1 + b\). If \(i\) does not exist, \([u_2] \geq u_1 + 2b - 1\), and \([u_1, u_2]\) contains \(2b\) integers. If \(i\) does exist, \([u_2] < u_1 + 2b - 1\), so \([u_1, u_2]\) contains \(2b - 1\) integers, \(i - b = [u_1]\) being the largest integer strictly to the left of this interval, and \(i + b = [u_2] + 1\) being the smallest to the right of it. But \(u_2 - b < i < u_1 + b\) implies

\[i^2 + i - b < N < i^2 + i + b,
\]

so neither \([u_1]\) nor \([u_2] + 1\) can be in \(U\). Hence in the definition of \(U\) we may replace \([u_1, u_2]\) by either \([u_1 - 1, u_2]\) or \([u_1, u_2 + 1]\), forcing \([u_1, u_2]\) to contain exactly \(2b\) integers. Next, \(u \equiv u' \mod 2b\) implies

\[u(u + 1) \equiv u'(u' + 1) \mod 2b,
\]

and the function \(\sigma\) defined by

\[\sigma(u) = 2b - 1 - u\]

maps the integers modulo \(2b\) onto themselves in such a way that parity is reversed. Now (7) holds if \(u'\) is replaced by \(\sigma(u)\), so \(U\) is mapped onto itself by \(\sigma\) and hence contains an equal number of odd and even integers. This proves the vanishing of (5) for \(N \geq b^2 > 0\).
For $N < b^2$ the points of entry and exit are $(u_1, j_1)$ and $(u_2, j_2)$, where

$$u_1 = b + \{-1 - \sqrt{1 + 4(N - b)}\}/2$$

and

$$2b - u_2 - 1 = u_1,$$  \hspace{1cm} (9)

so clearly $[u_1, u_2]$ contains fewer than $2b$ integers. However, by (9), as $N$ decreases from $b^2$ the interval $[u_1, u_2]$ successively loses the pairs of lattice points $0, 2b - 1; 1, 2b - 2; 2, 2b - 3; \ldots$. Since the elements of each pair are mapped onto each other by (8), the finite sum must also vanish in this case.

4. A RELATED FINITE SUM

The behavior of a sum $f_b(N)$ somewhat similar to (5) will now be examined; however, the author has not been able to connect it with a $q$-identity. Here $\equiv$ shall denote congruence modulo $2b$, and, if $I$ is a set of numbers, $r(I)$ shall denote the set of residue classes of the integers of $I$ modulo $2b$. The same letters are used to denote elements of $I$ and $r(I)$; no confusion results.

**Theorem.** For positive integers $b, N$, and a root of unity $\omega$, let

$$f_b(N; \omega) = f_b(N) = \sum_{W} \omega^{u}$$

where $W$ is the wedge defined by $j \geq 1, u \geq j$, and the prime indicates that $u$ and $j$ are subject to the constraint $u^2 \pm b(2j - 1) = N$. Then, for a fixed $b \geq 4$ and $N$ sufficiently large,

$$f_b(N + 2b) = f_b(N) \quad \text{if} \quad \omega^{2b} = 1,$$  \hspace{1cm} (10)

and

$$f_b(N + b) = -f_b(N) \quad \text{if} \quad \omega^{b} = -1 \quad \text{and} \quad b \text{ is odd.} \hspace{1cm} (11)$$

**Proof:** $f_b(N)$ is a sum over the lattice points of $W$ which lie on one of the parabolas $P^+(u), P^-(u)$ (using an obvious notation). For $N \geq b + 1$ define closed intervals $I_1(N), I_2(N)$ by

$$I_1(N) = [\sqrt{(N + b)}, b + \sqrt{(N + b^2 - b)}]$$

and

$$I_2(N) = [-b + \sqrt{(N + b^2 + b)}, \sqrt{(N - b)}].$$
Then $P^+(u), P^-(u)$ lies in $W$ if and only if $u \in I_1, I_2$, respectively. The sum defining $f_i(N)$ extends over those $u \in r(I_1 \cup I_2)$ such that

$$u^2 \equiv N - b.$$  \hspace{1cm} (12)

Call a $u$ satisfying (12) $N$-essential. In general, if $A_1(N) = A_1$ and $A_2(M) = A_2$ are sets of numbers, call a mapping $\sigma: r(A_1) \rightarrow r(A_2)$ trivially good if it is one-to-one and onto, and good if it can be made so by adjoining or removing elements from $r(A_1), r(A_2)$ which are not $N, M$-essential, respectively. Set

$$Z_i(N) = r(I_i(N)), \quad i = 1, 2, Z(N) = Z_1(N) \cup Z_2(N),$$

and define mappings $\sigma_1: Z(N) \rightarrow Z(N + 2b), \sigma_2: Z(N) \rightarrow Z(N + b)$ by

$$\sigma_i(x) \equiv x + (i - 1)b.$$  \hspace{1cm} (13)

Since $\sigma_1$ maps $N$-essential points to $N + 2b$-essential points, and for $b$ odd $\sigma_2$ maps $N$-essential points to $N + b$-essential points, (10) and (11) are equivalent, respectively, to "$\sigma_1$ is good" and "$\sigma_2$ is good" for $N$-sufficiently large.

For non-negative integers $h, k$ let $t = bh + k$ and define closed intervals $J_i(h, k; b) = J_i = [\alpha_i, \beta_i]$ and $K_j(h, k; b) = K_j$ for $i = 1, \ldots, 5$ and $j = 1, 2$ by

\begin{align*}
J_1 &= [t^2 - b, t^2 - b], \\
J_2 &= [t^2 - b + 1, t^2 + b - 1], \\
J_3 &= [t^2 + b, (t + 1)^2 - b^2 - b], \\
J_4 &= [(t + 1)^2 - b^2 - b + 1, (t + 1)^2 - b^2 + b - 1], \\
J_5 &= [(t + 1)^2 - b^2 + b, (t + 1)^2 - b - 1], \\
K_1 &= [t, t + b], \\
K_2 &= [t + 1 - b, t - 1];
\end{align*}

let $J_i$ be empty if $\alpha_i > \beta_i$. Since $J_4(h, b; b) = J_4(h + 1, 0; b)$, there is an $N_0(b)$ such that every $N > N_0(b)$ lies in precisely one of these intervals with $0 \leq k \leq b - 1$. Define $S_j(h, k; b) = S_j = r(K_j), j = 1, 2$; then for $N \in J_i, N > N_0(b)$, and $b \geq 2$, the following is easily verified:

\begin{align*}
i & \quad Z_i(N) & Z_0(N) \\
1 & \quad S_1 & S_2 \\
2 & \quad S_1 - \{t\} & " \\
3 & \quad " & S_2 \cup \{t\} \\
4 & \quad " & S_0(h, k + 1; b) \\
5 & \quad S_1(h, k + 1; b) & "
\end{align*}
The theorem is now proved by considering separately each of the cases $N \in J_i$, $i = 1, \ldots, 5$. The details are given here only for (10) with $N \in J_3$ and (11) with $N \in J_2$. The remaining cases are no more difficult.

If $N \in J_3$ and $N + 2b \in J_3$, $\sigma_1$ is trivially good. If $N + 2b \in J_4$, $Z(N + 2b) = Z(N) - \{u\}$ with $u \equiv t + 1 - b$. But

$$N = (t + 1)^2 - b^2 - 3b + c$$

for some integer $c$, $1 \leq c \leq 2b - 1$, so $u$ is not $N + 2b$-essential. $\sigma_1$ is good now follows by adjoining $u$ to $Z(N + 2b)$. Finally, if $N + 2b \in J_5$, $Z(N) - \{u\} = Z(N + 2b) - \{u_2\}$ with $u_1 \equiv u_2$, so $\sigma_1$ is trivially good.

If $N \in J_2$ and $N + b \in J_2$, $Z(N) = Z(N + b) = (S_1 \cup S_2) - \{t\}$. $\sigma_2$ can be made one-to-one and onto by removing the $u_1 \in Z(N)$ and $u_2 \in Z(N + b)$ such that $u_1 + b \equiv t$ and $u_2 - b \equiv t$, respectively. Write $N = t^2 - b + c$, $1 \leq c \leq 2b - 1$. Then it follows easily that $u_2$ is not $N + b$-essential, while $u_1$ is $N$-essential only if $b = c$. But $N + b \in J_2$, so $b \neq c$. Hence $\sigma_2$ is good. Finally, if $N + b \in J_3$, $Z(N) = Z(N + b) - \{t\}$. $t$ is not $N$-essential, so adjoining it to $Z(N)$ shows that $\sigma_2$ is good.

**Remark.** The proof is actually valid for $N > b^4/4$. $b \geq 4$ is used to show $N + b, N + 2b \notin J_2(h, k + 1; b)$ when $N \in J_4$.

5. **Nonquadratic Exponents**

The procedure of Section 2 will now be applied to the sum on the right of (15). Let $m(i) = (n + 1)^i - n^i$,

$$
\prod (y_1, y_2, y_3, y_4) = \prod_{n=0}^{\infty} \{1 + y_1^{n(1)}y_2^{n(2)}y_3^{n(3)}y_4^{n(4)}\},
$$

(14)

and define $Q$ by

$$Q(y_1, y_2, y_3, y_4) \prod (y_1, y_2, y_3, y_4) \prod (y_1^{-1}, y_2, y_3^{-1}, y_4)
= \sum_{n=-\infty}^{\infty} y_1^n y_2^2 y_3^2 y_4^4.
$$

(15)

Then a short calculation shows that

$$Q(y_1, y_2, y_3, y_4) = Q(y_1 y_2 y_3 y_4, y_2 y_3 y_4, y_3 y_4, y_4).
$$

(16)
A more tedious calculation shows that the expansion of $Q$ into a formal Laurent series is

$$Q = \sum_{n=-\infty}^{\infty} L_n(y_2, y_3, y_4)y_1^n$$

$$= \cdots + y_1^{-1}[y_2^3y_3^{-1}y_4^2(1 - y_3^{-6}y_4^{12}) + y_2^5y_3^{-1}y_4^5(1 - y_3^{-18}y_4^{60}) + \cdots]$$

$$+ [1 - y_2^2y_4^2 - y_2^4y_4^4 + y_2^6(y_4^{30} - y_4^6) + \cdots]$$

$$+ y_1[y_2^3y_3y_4^3(1 - y_3^{-6}y_4^{12}) + y_2^5y_3y_4^5(1 - y_3^{-18}y_4^{60}) + \cdots] + \cdots \quad (17)$$

and it follows that

$$L_n(y_2, y_3, y_4) = y_2^{2n}y_3^{3n}y_4^{4n}L_n(y_2, y_3^2y_4^6, y_2, y_4^4, y_4)$$

is false in general, although it can be obtained formally from (16) and (17) by equating coefficients of $y_1^n$. This is not a paradox; as the referee has kindly pointed out, the Laurent series for the left side of (16) can clearly be analytic only for

$$|y_2^{-1}y_4^4| < |y_4| < |y_2y_3y_4|^{-1}$$

while the series for the right-hand side can be analytic only for

$$|y_2y_3^2y_4^3| < |y_1^{-1}y_2^{-1}y_3y_4^4| < |y_2y_3^{-4}y_4^{11}|^{-1},$$

i.e.,

$$|y_2y_3y_4|^{-1} < |y_1| < |y_2^{-1}y_3y_4^{15}|^{-1}.$$