Artin presentations and fundamental groups of 3-manifolds

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ABSTRACT

We prove that the set of $n$-Artin presentations has a group structure. It is a known result but it seems that it does not appear explicitly in the literature. As an application we consider a special class of integral framed links $\hat{\beta}$ in $S^3$ such that $\beta = \prod_{i=1}^n \Delta^2 \sigma_1^{2e_i} \sigma_2^{2f_i}$ and we calculate the fundamental group of the 3-manifolds obtained by integral Dehn surgery on these links. In some cases we say when the groups obtained cannot be trivial.

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1. Introduction

By the fundamental theorem of surgery proved by Lickorish and Wallace [8,13], we know that any closed, connected and oriented 3-manifold can be obtained by integral Dehn surgery on a link in $S^3$. It follows from a second paper of Lickorish [9] that if $M^2$ is a closed, oriented 2-manifold of genus $g$ then every orientation preserving homeomorphism of $M^2$ is isotopic to a product of twist homeomorphisms along certain $3g-1$ curves. This can be used to prove that any closed, connected and oriented 3-manifold can be obtained by integral Dehn surgery on a closed pure $n$-braid [12]. In [2] we give an elementary proof of this fact, showing that any surgery presentation of a closed, connected and oriented 3-manifold can be transformed to a surgery presentation on a closed pure $n$-braid by performing some surgery moves.

Let $P_n$ be the group of pure $n$-braids and let $\beta \in P_n$. We denote the closure of $\beta$ as $\hat{\beta}$ which is obtained by gluing the ends of the braid without forming new crossings. In particular, if we have pure 1-braids and perform Dehn surgery we get lens spaces; if we consider closed pure 2-braids one obtains, in general, Seifert fiber spaces. In this paper we consider the problem of determining when the 3-manifolds obtained by integral Dehn surgery on a special class of links are not $S^3$. This class of links are integral framed, closed pure 3-braids $\hat{\beta}$, where $\beta = \prod_{i=1}^n \Delta^2 \sigma_1^{2e_i} \sigma_2^{2f_i}$ with $e_i, f_i \in \mathbb{Z}$ and $\Delta^2 = (\sigma_1 \sigma_2 \sigma_3)^2$. This is made using the theory of $n$-Artin Presentations of fundamental groups of 3-manifolds. First, we calculate the fundamental groups of the 3-manifolds obtained by integral Dehn surgery on $\hat{\beta}$ and then using small cancellation theory determine which of these groups have as quotient a non-trivial group.

We recall the definition of Dehn surgery on a link. Let $L = \bigcup_{i=1}^n k_i$ be an oriented link contained in $S^3$. Denote by $\eta(L) = \bigcup \eta(k_i)$ the disjoint union of regular neighborhoods of its components. Let $(m_i, l_i)$ be a pair meridian-longitude for $\partial \eta(k_i)$. Call $r_i \subset \partial \eta(k_i)$ a curve $r_i = p_i m_i + q_i l_i$ where $p_i/q_i \in (0,1)\cap \mathbb{Q}$ and $l_i = 1$. The result of $r$-surgery on $L$, where $r = (r_1, \ldots, r_n)$, is the manifold $cl(S^3 - \text{int} \eta(L)) \cup \left( \bigcup_{i=1}^n T_i \right)$, where $T_i$ a standard solid torus which is glued along its boundary to $\partial \eta(k_i)$, in such a way that its meridian $m_i'$ is glued along $r_i$.

The paper is organized as follows. In Section 2, we define $n$-Artin presentations and give an operation in this set. Also we give an explicit proof of the fact that with this operation it is possible to give a group structure to the set of $n$-Artin presentations. In Section 3, we use the operation defined in Section 2 to calculate the fundamental groups of the 3-manifolds obtained by integral Dehn surgery on closed pure 3-braids. In Section 4, using small cancellation theory we prove that certain of these groups cannot be trivial.
2. \textit{n}-Artin presentations

We start this section giving some definitions about \textit{n}-Artin presentations. Then we prove that it is possible to give a group structure to the set \(A_n\) of \textit{n}-Artin presentations for a fixed \(n\). This is a known fact but it seems that it does not appear explicitly in the literature [14,7,5].

\textbf{Definition 2.1.} Given a presentation of a group \(G\) in terms of generators and relations

\[ \langle x_1, x_2, \ldots, x_n : r_1, r_2, \ldots, r_n \rangle \]

we say that this is an \textit{n}-Artin presentation if it satisfies the following equation

\[ \prod_{i=1}^{n} r_i^{-1} x_i r_i = \prod_{i=1}^{n} x_i \]

in the free group \(F_n (:= F(x_1, x_2, \ldots, x_n))\).

If \(w = \prod_{i=1}^{m} x_i^{\epsilon_i}\) (all \(\epsilon_i\) are \(\pm 1\)), i.e., it is a word in the free group with \(n\) generators and \(r = (r_1, r_2, \ldots, r_n)\) is an \(n\)-tuple where each entry is a word in the \(x_1, x_2, \ldots, x_n\) generators then we define

\[ w' = \prod_{i=1}^{m} (x_i^{\epsilon_i})^{-r_i} \]

where \((x_i^{\epsilon_i})^{-r_i} = x_i^{-r_i} x_i^{r_i} r_i^{-1}\), it is to say, we conjugate each element \(x_i^{\epsilon_i}\) by the word \(r_i\). Observe that, if \(\sigma_i(w)\) is the sum of the exponents of \(x_i\) in \(w\), then \(\sigma_i(w') = \sigma_i(w)\).

So, we have the following definition which is central to give to the set of \textit{n}-Artin presentations a group structure.

\textbf{Definition 2.2.} If \(r' = (r'_1, \ldots, r'_n)\) and \(r = (r_1, \ldots, r_n)\) define \(r'' = (r'_1, \ldots, r'_n)\) and \(r' \circ r = r'' \circ r''\) where

\[ (r_1, r_2, \ldots, r_n) \cdot (s_1, \ldots, s_n) = (r_1 s_1, r_2 s_2, \ldots, r_n s_n) \]

Let \(P_n\) be the group of pure \(n\)-braids, which can be seen as a group of automorphisms of a free group \(F_n\) of rank \(n\).

\textbf{Proposition 2.3.} Let \(A_n\) be the set of \textit{n}-Artin presentations. Then \(A_n\) is a group with respect to \(\circ\) and in fact it is isomorphic to \(P_n \times Z^n\) where \(P_n\) is the group of automorphisms \(\alpha\) of \(F_n\) satisfying:

\begin{enumerate}
  \item \(\alpha(x_i)\) is conjugate to \(x_i\) (\(i = 1, 2, \ldots, n\))
  \item \(\alpha(x_1 x_2 \cdots x_n) = x_1 x_2 \cdots x_n\).
\end{enumerate}

\textbf{Proof.} Let

\[ \psi : P_n \times Z^n \rightarrow A_n \]

be defined by

\[ \psi (\alpha, (a_1, \ldots, a_n)) = (r_1, r_2, \ldots, r_n) \]

where \(\alpha(x_i) = x_i^{r_i}\) (\(i = 1, 2, \ldots, n\)) and \(\sigma_n(r_i) = a_i\) where \(\sigma_n(r_i)\) is the sum of exponents of \(x_i\) in \(r_i\). We have to show that \(\psi\) is well defined.

i) It is well defined. Suppose \(\alpha(x_i) = x_i^{r_i}\) and \(\alpha(x_i) = x_i^{s_i}\). Assume that \(r_i = x_i^{s_i} r_i'\) and \(s_i = x_i^{s_i} s_i'\), where the first letter in \(r_i'\) and \(s_i'\) is not \(x_i\) and \(r_i'\) and \(s_i'\) are reduced words. Then \(\alpha(x_i) = (r_i')^{-1} x_i r_i'\) and \(\alpha(x_i) = (s_i')^{-1} x_i s_i'\) are reduced words, so it follows that \(r_i' = s_i'\). Uniqueness of \(r_i\) now follows from the fact that there is a unique \(r_i\) for which \(\sigma_n(r_i) = a_i\).

ii) To show that

\[ \psi ((\alpha, a) \cdot (\beta, b)) = \psi (\alpha, a) \circ \psi (\beta, b) \]

we follow the definitions and calculate

\[ \psi ((\alpha, a) \cdot (\beta, b)) = \psi ((\alpha \beta, a + b)) \]
Suppose that \( \psi(\alpha, a) = (t_1, t_2, \ldots, t_n) \) and \( \psi(\beta, b) = (s_1, s_2, \ldots, s_n) \) then
\[
\alpha \beta(x_k) = \alpha(x_k^s) = \alpha(s_k^{-1} x_k s_k) = \alpha(s_k^{-1}) \alpha(x_k) \alpha(s_k)
\]
\[
\sigma_i(t_i s_i^t) = (t_1 s_1^{t_1}, \ldots, t_n s_n^{t_n})
\]
Hence
\[
\psi(\alpha, a) \cdot \psi(\beta, b) = (\alpha \beta, a + b) = (t_1 s_1^{t_1}, t_2 s_2^{t_2}, \ldots, t_n s_n^{t_n})
\]
because
\[
\sigma_i(t_i s_i^t) = \sigma_i(t_i) + \sigma_i(s_i) = a_i + b_i
\]
and
\[
\psi(\alpha, a) \circ \psi(\beta, b) = t \cdot s^t
\]
iii) It is clear that this homomorphism is an isomorphism. 

The group \( P_n \times \mathbb{Z}^n \) is also called the group of framed pure \( n \)-braids. The group \( P_n \) is canonically isomorphic to the group of isotopy classes of homeomorphisms of \( W \) which are the identity on \( \partial W \) where \( W \) is a disk with \( n \) holes. This group is generated by \( A_{ij} \) \((1 \leq i < j \leq n)\) and \( X_i \) \((1 \leq i \leq n)\) where \( A_{ij} \) is the positive Dehn twist on the curve shown in Fig. 1 for the case of closed pure 3-braids and \( X_i \) is the positive Dehn twist on the boundary of the \( i \)-th hole. One also considers the Dehn twist \( \Delta \) on the exterior boundary of \( W \).

It is not difficult to see from the definition of \( \circ \) that, for example
\[
r \circ s \circ t \circ u = r \cdot s^t \cdot (t^s)^t \cdot (u^t)^t
\]
There is no loss of generality in considering \( n \)-Artin presentations in studying the fundamental groups of closed, connected and orientable 3-manifolds since in 1974, González-Acuña showed the following [7].

**Theorem 2.4.** The fundamental group of any orientable, connected and closed 3-manifold has an \( n \)-Artin presentation. Furthermore, a group \( G \) is the fundamental group of an orientable, connected and closed 3-manifold if it has an \( n \)-Artin presentation.

This follows from the fact that any closed, connected and orientable 3-manifold has an open book decomposition with planar pages. The monodromy of the open book, which is a homeomorphism such that it is the identity on the boundary, can be represented by a disc with \( n \) holes and paths going from the exterior boundary to the interior boundaries in such way that the \( i \)-th path finishes in the \( i \)-th hole (see Fig. 2).

### 3. The fundamental group of 3-manifolds obtained by integral Dehn surgery on closed pure 3-braids \( \hat{\beta} \)

In this section we get a presentation of the fundamental group of the 3-manifolds obtained by integral Dehn surgery on closed pure 3-braids \( \hat{\beta} \). We do it by considering the operation defined in Section 2.

Let \( B_n \) be the group of \( n \)-braids [4]. Remember [11] that \( P_n \subset B_n \) is the group of pure \( n \)-braids which has generators \( A_{ij} \) where
\[
A_{ij} = \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^{-1}
\]
Fig. 2. Disk with 3 holes before the action of a homeomorphism.

Fig. 3. Presentation of closed pure 3-braids.

for $1 \leq i < j \leq n$, and relations

$$A_{rs}A_{ik}A_i^{-1} = A_{rk} \quad \text{if } s < i \text{ or } k < r$$
$$A_{ks}A_{ik}A_k^{-1} = A_i^{-1}A_{ik}A_is, \quad i < k < s$$
$$A_{rk}A_{ik}A_r^{-1} = A_{ir}^{-1}A_{ik}A_{ir}A_{ik}, \quad i < r < k$$
$$A_{rs}A_{ik}A_r^{-1} = A_is^{-1}A_{ir}^{-1}A_{is}A_{ir}A_{is}A_is, \quad i < r < k < s$$

In our particular case where $n = 3$ we have generators $A_{12}, A_{13}$ and $A_{23}$ and relations

$$r_1 = [A_{13}, A_{12}A_{13}A_{23}] = 1$$
$$r_2 = [A_{12}, A_{13}A_{23}] = 1$$

where $[ ]$ denotes the commutator.

Furthermore, the group of pure 3-braids can be seen as the direct product of two free groups $F_2 \times Z$ [6], where $F_2$ is generated by $A_{12}, A_{23}$ and $Z$ by $\Delta = (\sigma_1 \sigma_2 \sigma_1)^2$. This give us a general diagram representing a closed pure 3-braid, see Fig. 3.

To get the presentation of the fundamental group of the 3-manifold obtained by integral Dehn surgery on $\hat{\beta}$, with $\beta \in P_3$, it is enough to consider the action of the Dehn twists along the curves representing $A_{12}$ and $A_{23}$ on $x_i$ where $i = 1, 2, 3$, and the Dehn twists along the curves $\Delta, X_1, X_2, X_3$ (cf. [1]). In our case, we have a disc with 3 holes and paths $r_1, r_2, r_3$ (see Fig. 2).

From this diagram we can read the fundamental group of the 3-manifold since if we blow up the disk with 3 holes, we get a handlebody whose complement is also a handlebody. Furthermore, we have 3 simple closed curves, see Fig. 4. This give us a Heegaard splitting of the 3-manifold. So, reading the words described by these closed, simple curves we get
a 3-Artin presentation of the fundamental group of the 3-manifold obtained by integral Dehn surgery on $\hat{h}$. In Fig. 4, the 3-Artin presentation has relations $r_1 = x_1 x_2 x_3 x_1 x_2 x_1$, $r_2 = x_1 x_2 x_3 x_1 x_2^2$, $r_3 = x_1 x_2 x_3^2$.

Applying Dehn twists along $A_{12}$, $A_{23}$, $\Delta$, $X_1$, $X_2$, $X_3$ we can read the 3-Artin presentations

$A_{12} = ((x_1 x_2)^{e_1}, (x_1 x_2)^{e_1}, 1)$
$A_{23} = (1, (x_2 x_3)^{f_1}, (x_2 x_3)^{f_1})$
$\Delta = ((x_1 x_2 x_3)^{e}, (x_1 x_2 x_3)^{e}, (x_1 x_2 x_3)^{e})$
$X_1 = (x_1, 1, 1), \quad X_2 = (1, x_2, 1), \quad X_3 = (1, 1, x_3)$

and composing these presentations in the following order

$X_1^m \circ X_2^n \circ X_3^p \circ A_{12} \circ A_{23} \circ \Delta$

we get the next theorem.

**Theorem 3.1.** Let $\hat{h}$ be a link in $S^3$, where $\beta = \Delta^{e_2} (\sigma_1^{e_1} \sigma_2^{e_2})^{f_1}$ with an integral framing $(m, n_1, p)$, where $e, e_1, f_1 \in Z$. Then the 3-manifold obtained by integral Dehn surgery on $\hat{h}$ has the 34-Artin presentation given by the generators $x_1, x_2, x_3$ and relations

$r_1 = x_1^{m-e_1-e_2}(x_2 x_3)^{f_1} x_2 (x_2 x_3)^{-f_1} (x_1 x_2 x_3)^e$
$r_2 = x_2^{n-e_1-f_1} (x_2 x_3)^{f_1} (x_1 x_2 x_3)^f (x_2 x_3)^{-f_1} (x_1 x_2 x_3)^e$
$r_3 = x_3^{p-e_2-f_1} (x_2 x_3)^{f_1} (x_1 x_2 x_3)^e$

In a previous work [3], we have a solution of the problem of determining when these groups are trivial. This is made geometrically, using the observation that the links $\hat{h} \in S^3$ where $\beta = \Delta^{2e} (\sigma_1^{e_1} \sigma_2^{e_2})^{f_1}$ are strongly invertible. So, we take the quotient given by the action of the involution and from that quotient we get what we call an hexatangle, which is double branched covered by the exterior of $\hat{h}$. By analyzing this hexatangle it is possible to say when we have a trivial knot which implies a solution for the original problem, i.e. we can determine explicitly in which cases the groups given in Theorem 3.1 are trivial.

Let

$C_1 = (x_1 (x_2 x_3)^{f_2} x_2 (x_2 x_3)^{-f_2})^{e_2} x_1 (x_2 x_3)^{f_2} x_2 (x_2 x_3)^{-f_2}^{-e_2}$
$C_2 = (x_1 (x_2 x_3)^{f_2} x_2 (x_2 x_3)^{-f_2})^{e_2} (x_2 x_3)^{f_2} x_2 (x_2 x_3)^{-f_2} (x_1 (x_2 x_3)^{f_2} x_2 (x_2 x_3)^{-f_2})^{-e_2} (x_2 x_3)^{f_2} x_2 (x_2 x_3)^{-f_2}$
$C_3 = (x_1 (x_2 x_3)^{f_2} x_2 (x_2 x_3)^{-f_2})^{e_2} (x_2 x_3)^{f_2} x_2 (x_2 x_3)^{-f_2} (x_1 (x_2 x_3)^{f_2} x_2 (x_2 x_3)^{-f_2})^{-e_2} (x_2 x_3)^{f_2} x_2 (x_2 x_3)^{-f_2}$
$C_4 = (x_1 (x_2 x_3)^{f_2} x_2 (x_2 x_3)^{-f_2})^{e_2}$

**Theorem 3.2.** Let $\hat{h}$ be a link in $S^3$, where

$\beta = \Delta^{2e} (\sigma_1^{e_1} \sigma_2^{e_2})^{f_1} (\sigma_1^{e_1} \sigma_2^{e_2})^{f_2}$

with an integral framing $(m, n_1, p)$, where $e, e_1, f_1 \in Z$. Then the 3-manifold obtained by integral Dehn surgery on $\hat{h}$ has the 3-Artin presentation $G$ given by the generators $x_1, x_2, x_3$ and relations
Proof. Doing $x_i$ and $x_j$ simultaneously equal to 1 for $i \neq j$ and $i, j = 1, 2, 3$ the result follows. □

Furthermore, we can consider the general case. It is to say, we can calculate a 3-Artin presentation of the fundamental group of any 3-manifold obtained by integral Dehn surgery on a link $\hat{\beta} \in S^3$ where $\beta = \prod_{i=1}^n \Delta^2 (\sigma_1^2)^{e_i} (\sigma_2^2)^{f_i}$. Let

$$A_{23}^n \circ A_{12}^n \circ \cdots \circ A_3^n \circ A_{12}^1 =: A_{23}^n \cdot (A_{12}^1)^{A_{23}^n} \cdot ((A_{23}^1)^{A_{12}^n})^{A_{23}^n} \cdots ((A_{23}^1)^{A_{12}^n})^{A_{23}^n}$$

Let

$$B = A_{23}^n \cdot (A_{12}^1)^{A_{23}^n} \cdot ((A_{23}^1)^{A_{12}^n})^{A_{23}^n} \cdots ((A_{23}^1)^{A_{12}^n})^{A_{23}^n} \cdot (\cdots ((A_{12}^1)^{A_{23}^n})^{A_{12}^n})^{A_{23}^n}$$

Since in the third component we have that $A_{12}^1 = 1$ for all $j$, then in the last expression only appears the terms $A_{23}^1$ and we have

$$B_1 = A_{23}^n \cdot ((A_{23}^1)^{A_{12}^n})^{A_{23}^n} \cdots ((A_{23}^1)^{A_{12}^n})^{A_{23}^n}$$

And calculating for the first component where $A_{23}^1 = 1$ for all $j$, we have

$$B_2 = (A_{12}^1)^{A_{23}^n} \cdot ((A_{23}^1)^{A_{12}^n})^{A_{23}^n} \cdots ((A_{23}^1)^{A_{12}^n})^{A_{23}^n}$$

To calculate the 3-Artin presentation of the 3-manifold we compose in the next order $X_1^m \circ X_2^n \circ X_3^p \circ A_{23}^n \circ A_{12}^n \circ \cdots \circ A_{23}^1 \circ A_{12}^1 \circ \Delta^e$ and we have the following.

**Theorem 3.4.** Let $\hat{\beta} \in S^3$ where

$$\beta = \prod_{i=1}^n \Delta^2 (\sigma_1^2)^{e_i} (\sigma_2^2)^{f_i}$$

with an integral framing $(m, n_1, p)$, where $e, e_i, f_i \in \mathbb{Z}$. Then the 3-manifold obtained by integral Dehn surgery on $\hat{\beta}$ has the 3-Artin presentation given by generators $x_1, x_2, x_3$ and relations

$$r_1 = x_1^{m-\sum e_i - e} B_2 (x_1 x_2 x_3)^e$$
$$r_2 = x_2^{n_1-\sum e_i - \sum f_i} B (x_1 x_2 x_3)^e$$
$$r_3 = x_3^{p-\sum f_i - e} B_1 (x_1 x_2 x_3)^e$$

□

**Corollary 3.5.** There exists representations of $G$ onto the groups $Z(e, e+\sum e_i, m), Z(e, e+\sum f_i, p)$ and $Z(e+\sum e_i, e+\sum f_i, n_1)$. So, if some gcd is different of 1 the group $G$ is not trivial.

4. The non-triviality of some fundamental groups of 3-manifolds obtained by integral Dehn surgery on closed pure 3-braids

The corollaries in Section 3 are an estimation of when the fundamental groups are not trivial, but they are very general since in these cases we do not have, in fact, a homology sphere. So, if we suppose that the numbers $e, e_i, f_i, m, n_1$, and $p$ are such that we have a homology sphere, i.e. when the link matrices which are

$$\begin{pmatrix} m & e+\sum e_i & e \\ e+\sum e_i & n_1 & e+\sum f_i \\ e & e+\sum f_i & p \end{pmatrix}$$

have determinant $\pm 1$ then we have a result more efficient which is saying when this homology sphere cannot be $S^3$. 
A group $T$ is a generalized triangular group if it can be represented as

$$T = \langle a, b : a^m = b^n = 1 \rangle \quad (l, m, n > 1)$$

where $w = a^{r_1}b^{s_1} \cdots a^{r_k}b^{s_k}$ ($k \geq 1, 0 < r_1 < l, 0 < s_1 < m$). We denote it as $T_{(l, m, n)}$. Doing $x_1 = 1, x_2 = 1$ and $x_3 = 1$ in each case respectively, we have the following result.

**Proposition 4.1.** There exists representations $G$ of the groups

(i) $T_{(m, p, k)}$ if $\sum f_1 = 0$ and $e = 0$ where $r_1 = (\prod_{j=1}^{s} x_2^{-f_1} (x_2 x_3)^{-f_1})^k$ and $s \geq 2, k > 1$

(ii) $T_{(m, m, k)}$ if $e = 0$ and $r_2 = (\prod_{j=1}^{s} x_1^{-f_2} (x_2 x_3)^{-f_2})^k$ and $s \geq 1, k \geq 2$

(iii) $T_{(m, n, k)}$ if $\sum e_1 = 0$ and $e = 0$ and $r_3 = (\prod_{j=1}^{s} x_2^{-f_3} (x_1 x_2)^{-f_3})^k$ with $s \geq 2, k > 1$.

In the next results we use small cancellation theory over free products. For convenience of the reader we recall some basic definitions and results of this theory which come from [10].

If $F$ is a free product of non-trivial groups $X_j$, then each non-identity element $w$ of $F$ has a unique representation in normal form as $w = y_1 y_2 \cdots y_n$ where each of the letters $y_i$ is a non-trivial element of one of the factors $X_j$, and where no adjacent $y_i y_{i+1}$ come from the same factor. The integer $n$ is the length of $w$, written $|w|$.

If $w = y_1 y_2 \cdots y_k c_1 c_2 \cdots c_l$ and $v = c_1^{-1} \cdots c_l^{-1} d_1 d_2 \cdots d_l$ are in normal form where $d_1 \neq y_k^{-1}$, we say that the letters $c_1, \ldots, c_l$ are canceled in forming the product $uv$. If $y_k$ and $d_1$ are in different factors of $F$, then $w = uv$ has normal form $y_1 \cdots y_k d_1 \cdots d_l$. It is possible that $d_1$ and $y_k$ are in the same factor of $F$ with $d_1 \neq y_k^{-1}$. Let $a = y_k d_1$. Then $w = uv$ has normal form $y_1 \cdots y_{k-1} a d_2 \cdots d_l$. We say that $y_k$ and $d_1$ have been consolidated to give the single letter $a$ in the normal form of $uv$.

We say that a word $w$ has reduced form $uv$ if the normal form for $w$ is obtained by concatenating the normal forms for $u$ and $v$. Thus there is neither cancellation nor consolidation between $u$ and $v$. We say that $w$ has semi-reduced form $uv$ if $w = uv$ and there is no cancellation between $u$ and $v$. Consolidation is expressly allowed.

Recall that an element $w$ of $F$ with normal form $w = y_1 \cdots y_k$ is said to be cyclically reduced if $|w| \leq 1$ or $y_1$ and $y_n$ are in different factors of $F$. We say that $w$ is weakly cyclically reduced if $|w| \leq 1$ or $y_1 \neq y_n^{-1}$. Thus there is no cancellation between $y_n$ and $y_1$, although consolidation is allowed.

A subset $R$ of $F$ is called symmetrized if every $r \in R$ is weakly cyclically reduced and every weakly cyclically reduced conjugate of $r$ and $r^{-1}$ is also in $R$.

A word $b$ is called a piece if $R$ contains distinct elements $r_1$ and $r_2$ with semi-reduced forms $r_1 = b c_1$ and $r_2 = b c_2$. Note that the last letter of $b$ does not have to be a letter of the normal form of $r_1$ or $r_2$.

**Condition $C(\lambda)$:** If $r \in R, r = b c$ in semi-reduced form where $b$ is a piece, then $|b| < \lambda |r|$. To avoid pathological cases, we further require that if $r \in R$ then $|r| \geq 1/\lambda$.

We use the next result to prove that certain groups are not trivial.

**Corollary 4.2.** ([p. 278 of [10]]) Let $F = \ast X_i$ be a free product, and let $R$ be a symmetrized subset of $F$ which satisfies $C(1/6)$. Let $N$ be the normal closure of $R$ in $F$. Then the natural map $\gamma : F \to F/N$ embeds each factor $X_i$ of $F$.

By doing $x_1 = 1$ in the group presentation given in Theorem 3.1, we have a group $H$ with presentation given by the generators $x_2$ and $x_3$ and relations

$$r_1 = (x_2 x_3)^{f_i} x_2^{e_2} (x_2 x_3)^{f_{i-1}} x_2^{e_2} \cdots x_2^{e_1} (x_2 x_3)^{-\sum f_i + e}$$

$$r_2 = x_2^{n_i - e - \sum f_i} (x_2 x_3)^{\sum f_i + e}$$

$$r_3 = x_3^{-p - \sum f_i - e} (x_2 x_3)^{\sum f_i + e}$$

By using the next lemma we will prove that under certain hypothesis this group $H$ is not trivial. Observe that if $z$ is an element of the center of a group $G$ and $G/z = 1$ then $G$ is cyclic. In this case $z = (x_2 x_3) \sum f_i + e$.

We will prove that a quotient of $H$ has a presentation as $G$ of Lemma 4.3. Let

$$(x_2 x_3) \sum f_i + e = 1 = x_2^{-n_i + e + \sum f_i}$$

$$(x_2 x_3) \sum f_i + e - p = 1 = (x_2 x_3) \sum f_i + e$$

and call $g = x_3 x_3$. Then $g \sum f_i + e = 1$ and $x_3 = x_2^{-1} g$. Call $x_2^{-1} d$ and $\sum f_i + e = F$. Then this quotient of $H$ has the presentation

$$\langle d, g : d^{e-n_1} = g^F = (dg)^{p-e} = g^{f_{i-1} d^{e_{i-1}} d^{-e_{i-1}} \cdots g f_i d^{-e_i} 2^e} \rangle$$

$$= \langle (dg)^{p-e}, g^{2 e + f_{i-1} d^{e_{i-1}} d^{-e_{i-1}} \cdots g f_i d^{-e_i}} \rangle$$
Lemma 4.3. Let \( G \) be the group with generators \( d \) and \( g \) and relations \( d^a = g^b = 1, (dg)^r = 1, d^{s_1}d^{s_2}g^{f_1}d^{s_3}g^{f_2} \cdots d^{s_k}g^{f_k} = 1 \). Suppose

(i) \(-1, 1, -s_1, s_1, -s_2, s_2, \ldots, -s_k, s_k \) are different numbers modulo \( a \)
(ii) \(-1, 1, -t_1, t_1, -t_2, t_2, \ldots, -t_k, t_k \) are different numbers modulo \( b \)
(iii) \( 6 \leq r, 6 \leq k \).

Then \( G \) is not trivial (in fact, the order of \( d \) in \( G \) is \( a \) and the order of \( g \) in \( G \) is \( b \)) and \( a > 2k + 2, b > 2k + 2 \).

Proof. Using small cancellation theory it is not difficult to see it since there are no pieces of length greater than 2. \( \square \)

Here is another way of describing \( H \). Consider the framed sublink consisting of the second and third components and the 3-manifold \( M \) obtained by surgery on it. \( M \) is a Seifert space or, if \( \sum f_i + e = 0 \), a connected sum of two lens spaces. Then

\[
H = \frac{\pi_1 M}{\langle \langle \text{first component} \rangle \rangle}
\]

\( H \) is trivial if and only if \( n_1 p - (e + \sum f_i)^2 = \pm 1 \) and

\[
\frac{\pi_1(M)/Z(\pi_1(M))}{\langle \langle f_1 \rangle \rangle} = 1
\]

where \( r_1 \) is represented by the first component. Notice that \( \pi_1(M)/Z(\pi_1(M)) \), the quotient of \( \pi_1 M \) by its center is a triangular group or a free product of two cyclic groups. Thus one is lead to the problem of deciding when a quotient of a triangular group or a free product of two cyclic groups is trivial.

Joining the previous results it follows that

Theorem 4.4. Let \( \hat{\beta} \) be a link in \( S^3 \) where

\[
\beta = \prod_{i=1}^{n} \Delta^{2e} (\sigma_1^2)^{e_1} (\sigma_2^2)^{f_i}
\]

with an integral framing \((m, n_1, p)\), where \( e, e_i, f_i \in \mathbb{Z} \). Then the 3-manifold obtained by integral Dehn surgery on \( \hat{\beta} \) is not \( S^3 \) if

(i) \(-1, 1, -e_1, e_1, -e_2, e_2, \ldots, -e_n, e_n \) are different numbers modulo \( F - n_1 \)
(ii) \(-1, 1, -f_1, f_1, -f_2, f_2, \ldots, -f_n, f_n \) are different numbers modulo \( F \)
(iii) \( 6 \leq F - p, 6 \leq n \). \( \square \)

Proof. It is clear since the group of the 3-manifold obtained by integral Dehn surgery has as quotient a non-trivial group of the form of \( G \) as in Lemma 4.3. \( \square \)

Relating to this we have the following conjecture.

Conjecture 4.5. The fundamental group of the 3-manifold obtained by integral Dehn surgery on \( \hat{\beta} \) where

\[
\beta = \Delta^{2e} \prod_{i=1}^{n} (\sigma_1^2)^{e_i} (\sigma_2^2)^{f_i}
\]

is not trivial for \( n \geq 2 \) and \( e, e_i, f_i \in \mathbb{Z} \) sufficiently large.

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References