Available online at www.sciencedirect.com
SCIENCE DIRECT
Advances in Applied Mathematics 32 (2004) 380–390
ADVANCES IN Applied Mathematics
www.elsevier.com/locate/yaama

Problems related to type-A and type-B matrices of chromatic joins

Frank Schmidt

Department of Mathematics, The George Washington University, Washington, DC 20052, USA

Received 10 July 2002; accepted 12 July 2002

Abstract

We outline problems that Rodica Simion was investigating that concern factorizations of determinants of matrices whose entries are defined by combinatorial statistics.

1. Introduction

In this paper, we sketch as much as we can of one of the programs that Rodica Simion was very actively in the process of developing before her untimely passing in early 2000. This program concerns factorizations of determinants of matrices that are defined by combinatorial statistics. We also provide some of the tantalizing computational evidence that she produced that suggests that this area is likely to have considerable depth.

Rodica's interest in such problems began with Tutte's paper [28], and was further reinforced by learning about meanders and their Gram determinants [6,7]. To provide more complete motivation, we start with the earlier work that motivated [28], which we mention in Section 3, after first sketching a few of the combinatorial preliminaries that enter into the discussion. Section 3 also contains the key results of [4,28] that so interested Rodica. Some problems and directions for future work that were part of Rodica's program of research are outlined in Section 4.

E-mail address: fenris@gwu.edu.

0196-8858/$ – see front matter
doi:10.1016/S0196-8858(03)00090-3

2. Noncrossing partitions and type-*B* counterparts

We assume the reader is familiar with the set partition lattice Π_n. This lattice has a number of important relatives, the first of which we will encounter below is the lattice of noncrossing partitions. (For an extensive survey of noncrossing partitions, see [24].) A partition $\pi = \{X_1, X_2, \dots, X_k\}$ of $\{1, 2, \dots, n\}$ is *noncrossing* if whenever $a < b < c < d$ and a and c are in a block X_i of π and b and d are in a block X_j of π, then $X_i = X_j$. Under the ordering (refinement) induced by Π_n, the noncrossing partitions also form a lattice [16], which is denoted by NC_n. It is well known that the number of elements in NC_n is the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. The meet operations are the same in Π_n and NC_n, but the join operations do not always agree; for instance, the join of $1/24/3$ and $13/2/4$ in Π_4 is $13/24$, which is crossing, so the join of these elements in NC_4 is 1234. The rank function of NC_n is the restriction of that of Π_n; in both cases, $\mathrm{rk}(\pi) = n - \mathrm{bk}(\pi)$, where $\mathrm{bk}(\pi)$ is the number of blocks of π.

The partition lattice can be generalized to Dowling lattices [10]; however, we will focus on the particular Dowling lattice of interest, namely, the lattice of type-B partitions. A type-B partition of the set

$$[\pm n] := \{+1, +2, \dots, +n, -1, -2, \dots, -n\}$$

is a partition π of $[\pm n]$ that satisfies two properties:

(i) for each block X of π, the set $-X := \{-x \mid x \in X\}$ is also a block of π, and
(ii) there is at most one block X of π for which $X \cap (-X) \neq \emptyset$.

The block in condition (ii), when present, is called the *zero block* of π. Note that for the zero block X, we have $X = -X$. By condition (i), the blocks of π other than the zero block occur in pairs $X, -X$; the number of such pairs is the nonzero block statistic $\mathrm{nzbk}(\pi)$. Thus, $\pi = 1, -1/2, -4/-2, 4/3/-3$ has zero block $X = \{1, -1\}$ and $\mathrm{nzbk}(\pi) = 2$. As with ordinary (type-A) partitions, type-B partitions are ordered by refinement, that is, for such partitions π and ρ, we have $\pi \leqslant \rho$ if and only if each block of π is contained in a block of ρ. Under refinement, the type-B partitions of $[\pm n]$ form a geometric lattice, denoted by Π_n^B.

To get noncrossing partitions of type B, place $+1, +2, \dots, +n, -1, -2, \dots, -n$ in this order clockwise around a circle. For $\pi \in \Pi_n^B$ and each pair i, j of distinct elements of $[\pm n]$, draw a chord inside the circle between elements i and j if i and j are in the same block X of π and at least one of the two arcs from i to j contains no other element of X. We say that π is noncrossing if all such chords can be drawn without crossings. The example of π in the previous paragraph is crossing. Under refinement, the noncrossing partitions of $[\pm n]$ form a lattice, denoted NC_n^B. The meet operations are the same in Π_n^B and NC_n^B, but the join operations do not always agree. The number of elements in NC_n^B is the middle binomial coefficient $\binom{2n}{n}$. The rank function of NC_n^B is a restriction of that of Π_n^B, which is given by $\mathrm{rk}(\pi) = n - \mathrm{nzbk}(\pi)$. For more on NC_n^B, see [21].

3. Historical roots

In his quest for a proof of what was then the Four Color Conjecture, G.D. Birkhoff introduced the chromatic polynomial (or chromial). Later, Birkhoff and Lewis [1] defined two families of polynomials called constrained chromials and free chromials. They focused on a particular class of maps in the plane, namely those in which all bounded faces are triangles; the unbounded face may have any n-gon (or n-ring) as its boundary. The Birkhoff–Lewis equations express each free chromial of an n-ring as a linear combination of constrained chromials. The goal was to invert these relations, thus expressing each constrained chromial as a linear combination of free chromials.

Tutte [27] generalized these equations to planar maps (not requiring a triangulation inside the n-ring) and redefined the free chromials in terms of partitions of the set of vertices that lie on the n-ring. Now solving for the constrained chromials in terms of the free ones is a matter of Möbius inversion in the lattice of set partitions. Tutte also showed that these new chromials can be expressed as linear combinations of free chromials associated with noncrossing partitions, and now the question is reduced to finding the coefficients in these expressions. This requires inverting the matrix of chromatic joins. The matrix of chromatic joins is, up to similarity, the matrix

$$T_n(q) := \left[q^{\mathrm{bk}(\alpha \vee_{\Pi_n} \beta)}\right]_{\alpha,\beta \in NC_n}, \tag{1}$$

whose rows and columns are indexed by the elements of NC_n, using the same ordering of these partitions for the rows as for the columns, in which the entry in the row indexed by the partition α and the column indexed by the partition β is $q^{\mathrm{bk}(\alpha \vee_{\Pi_n} \beta)}$, where this join is computed in Π_n rather than in NC_n. For instance, using the ordering 1/2/3, 12/3, 13/2, 1/23, 123 of the elements of Π_3, we have

$$T_3(q) = \begin{bmatrix} q^3 & q^2 & q^2 & q^2 & q \\ q^2 & q^2 & q & q & q \\ q^2 & q & q^2 & q & q \\ q^2 & q & q & q^2 & q \\ q & q & q & q & q \end{bmatrix},$$

which has as its determinant $q^5(q-1)^4(q-2)$. Inverting the matrix A of chromatic joins raises the question of finding the determinant $\det(T_n(q))$ of this matrix.

Tutte [28] and Dahab [4] derived elegant expressions for the determinant of the matrix of chromatic joins. Their formulas involve the *Beraha polynomials*, which are defined as follows:

$$p_0(q) = 0,$$
$$p_n(q) = \sum_{i=0}^{[n/2]} (-1)^i \binom{n-i-1}{i} q^{[n/2]-i}, \quad \text{for } n \geqslant 1.$$

Thus, $p_1(q) = 1$, $p_2(q) = q$, $p_3(q) = q-1$, $p_4(q) = q^2 - 2q$, $p_5(q) = q^2 - 3q + 1$.

The following formula for $\det(T_n(q))$ is a reformulation of what Tutte showed in [28]:

$$\det\bigl(T_n(q)\bigr) = q^{\binom{2n-1}{n}} \prod_{m=1}^{n-1} \left(\frac{p_{m+2}(q)}{q\, p_m(q)}\right)^{\frac{m+1}{n}\binom{2n}{n-1-m}}. \tag{2}$$

It is not immediately evident that the right side of this equation is a polynomial in q, as it must be. The alternative formula derived in [4], given in Eq. (3) below, is clearly a polynomial formula. It turns out that for $n \geqslant 2$, each Beraha polynomial $p_n(q)$ has one irreducible factor, $f_n(q)$, called the nth *Beraha factor*, which does not divide any of the polynomials with lower index. The following four Beraha factors will play a role in the computational evidence presented in Section 4:

$$f_3(q) = q - 1,$$
$$f_6(q) = q - 3,$$
$$f_9(q) = q^3 - 6q^2 + 9q - 1,$$
$$f_{12}(q) = q^2 - 4q + 1.$$

Dahab [4] obtained the following formula:

$$\det\bigl(T_n(q)\bigr) = \prod_{i=1}^{n} f_{i+1}(q)^{\delta(n,i)}, \tag{3}$$

where the multiplicity $\delta(n,i)$ of the $(i+1)$st Beraha factor is described in terms of a continued fraction: the value of $C_n - \delta(n,i)$ is the coefficient of x^n in the power series $G_{i-1}(x)$ defined by the continued fraction $G_r(x) = 1/(1 - xG_{r-1}(x))$ for $r \geqslant 1$, with $G_0(x) = 1$. The multiplicities of the irreducible factors have the following combinatorial description: $\delta(n,i)$ is the number of Dyck paths from the origin to the point $(2n, 0)$, whose maximum y-coordinate does not exceed $n - i - 1$ (see, e.g., [11]).

It is natural to generalize the matrix $T_n(q)$ in Eq. (1) in the following way. Let L be a ranked lattice, let L' be an induced subposet of L (see, e.g., [26]). Let $\mathrm{co}(x)$ denote the corank of an element x of L, and let $\vee_L$ and $\wedge_L$ denote the join and meet operations of L. Let

$$M(L', \vee_L, q) := \bigl[q^{\mathrm{co}(\alpha \vee_L \beta)}\bigr]_{\alpha,\beta \in L'},$$

the matrix, defined up to similarity, whose rows and columns are indexed by the elements of L', using the same ordering of these elements for the rows as for the columns, in which the entry in the row indexed by the element α and the column indexed by the element β is $q^{\mathrm{co}(\alpha \vee_L \beta)}$. The matrix $M(L', \wedge_L, q)$ is defined in the same manner with $\wedge_L$ replacing $\vee_L$. Since $\mathrm{co}(\alpha) = \mathrm{bk}(\alpha) - 1$ for $\alpha \in \Pi_n$, the matrix of chromatic joins $T_n(q)$ is obtained by dividing each entry of $M(NC_n, \vee_{\Pi_n}, q)$ by q.

Lindstrom [18] gives an elegant factorization of the determinants of $M(L', \vee_L, q)$ and $M(L', \wedge_L, q)$ in the case that L equals L'. However, "hybrid" cases such as $M(NC_n, \vee_{\Pi_n}, q)$ and $M(NC_n^B, \vee_{\Pi_n^B}, q)$ are considerably more difficult to treat.

4. Directions and open problems

In this section, we mention various open problems and directions for research related to the matrices $M(NC_n, \vee_{\Pi_n}, q)$ and $M(NC_n^B, \vee_{\Pi_n^B}, q)$ that Rodica was developing. The reader should bear in mind that Rodica did not have sufficient time to fully investigate these topics; indeed, some were considered only very briefly. Thus, there may be easy proofs or counterexamples for some of the problems mentioned below. However, the computational evidence presented below suggests that the general thrust of this line of research is likely to be both very challenging and fertile.

One of the problems of central interest to Rodica was the following.

Problem 1. Develop a formula for the determinant of the type-B matrix of chromatic joins, $M(NC_n^B, \vee_{\Pi_n^B}, q)$.

In the first four cases, these matrices have dimensions 2×2, 6×6, 20×20, and 70×70. Rodica obtained the determinant of $M(NC_n^B, \vee_{\Pi_n^B}, q)$ for $n = 1, 2, 3, 4$, using circular symmetry to reduce the amount of computation required for the 70×70 case. These determinants are

$$q - 1, \tag{4}$$

$$(q-1)^5(q-3), \tag{5}$$

$$(q-1)^{21}(q-3)^6\bigl(q^3 - 6q^2 + 9q - 1\bigr), \tag{6}$$

and

$$(q-1)^{85}(q-3)^{29}\bigl(q^3 - 6q^2 + 9q - 1\bigr)^8\bigl(q^2 - 4q + 1\bigr), \tag{7}$$

that is,

$$f_3(q),$$

$$\bigl(f_3(q)\bigr)^5 f_6(q),$$

$$\bigl(f_3(q)\bigr)^{21}\bigl(f_6(q)\bigr)^6 f_9(q),$$

and

$$\bigl(f_3(q)\bigr)^{85}\bigl(f_6(q)\bigr)^{29}\bigl(f_9(q)\bigr)^8 f_{12}(q).$$

The unusual nature of these factors, namely that they are powers of every third Beraha factor, is part of what sustained Rodica's interest in this problem.

Problem 2. Are all factors of the determinant of $M(NC_n^B, \vee_{\Pi_n^B}, q)$ of the form $f_{3k}(q)$ for some k?

The matching polynomials of paths, i.e., Chebyshev polynomials of the second kind, play an essential role in the type-A case. Rodica noted a connection between the polynomials $f_{3k}(q)$ and the matching polynomials of cycles. Let $m(C_n; x)$ be the matching polynomial of the n-cycle C_n. (See [12, Chapter 1] for matching polynomials.) The polynomials $m(C_n; \sqrt{x})^2 - 1$ arise in computations by Dabkowski and Przytycki [3] of two-variable annular skein determinants. Using standard results about Chebyshev polynomials, one can show that the irreducible factors of $m(C_n; \sqrt{x})^2 - 1$ are all of the form $f_{3k}(x)$ for some k dividing n. This makes it seem even more likely that the answer to Problem 2 is affirmative; indeed, it seems that a wide variety of problems involve the same factors, $f_{3k}(x)$, and it may be that these are all special cases of a more fundamental problem.

We note that the sequences

$$\det\bigl(M(NC_1, \vee_{\Pi_1}, q)\bigr), \qquad \det\bigl(M(NC_2, \vee_{\Pi_2}, q)\bigr), \qquad \ldots,$$

and

$$\det\bigl(M(NC_1^B, \vee_{\Pi_1^B}, q)\bigr), \qquad \det\bigl(M(NC_2^B, \vee_{\Pi_2^B}, q)\bigr), \qquad \ldots,$$

are divisibility sequences, that is, each term divides the next term in its sequence. To see this, we first focus on the case of Π_n and NC_n. Note that NC_{n-1} is isomorphic to the sublattice NC_n' of NC_n that consists of the elements in which $\{n\}$ is a singleton block. In forming the matrix $M(NC_n, \vee_{\Pi_n}, q)$, list the C_{n-1} elements in NC_n' first. For α in $NC_n - NC_n'$, let α' be $\alpha \wedge \sigma_n$ where σ_n has just two blocks, $\{1, 2, \ldots, n-1\}$ and $\{n\}$; that is, α' is formed from α by taking n out of its block and making $\{n\}$ a singleton block. Thus, α' is in NC_n'. Note that for any α in $NC_n - NC_n'$ and β in NC_n', we have $\mathrm{bk}(\alpha \vee_{\Pi_n} \beta) = \mathrm{bk}(\alpha' \vee_{\Pi_n} \beta) - 1$ since $\alpha \vee_{\Pi_n} \beta$ and $\alpha' \vee_{\Pi_n} \beta$ differ only in that $\{n\}$ is a singleton block of the latter. Thus, the first C_{n-1} entries in column α of $M(NC_n, \vee_{\Pi_n}, q)$ are q times the corresponding entries in column α'. It follows that $M(NC_n, \vee_{\Pi_n}, q)$ can be reduced to a matrix of the form

$$\begin{pmatrix} A & \mathbf{0} \\ B & C \end{pmatrix},$$

where A is $q \cdot M(NC_{n-1}, \vee_{\Pi_{n-1}}, q)$, the entries in B and C are polynomials in q, and $\mathbf{0}$ is the $C_{n-1} \times (C_n - C_{n-1})$ matrix of zeros. The divisibility assertion follows immediately by taking the determinant. In the case of Π_n^B and NC_n^B, note that NC_{n-1}^B is isomorphic to the sublattice $NC_n'^B$ of NC_n^B that consists of the elements in which $\{n\}$ and $\{-n\}$ are singleton blocks. For α in $NC_n^B - NC_n'^B$, let α' be $\alpha \wedge \sigma_n$ where σ_n has just one pair of nonzero blocks, $\{-n\}$ and $\{n\}$; thus, in α', the elements n and $-n$ have been separated from the blocks in which they occur in α, whether this is the zero block, or two other blocks. Note that for any α in $NC_n^B - NC_n'^B$ and β in $NC_n'^B$, we have $\mathrm{nzbk}(\alpha \vee_{\Pi_n} \beta) = \mathrm{nzbk}(\alpha' \vee_{\Pi_n} \beta) - 1$. The divisibility assertion now follows as in the earlier case.

The factorizations of determinants in [18] arise from a factorization of the matrices of interest. This suggests the next problem.

Problem 3. Do the matrices $M(NC_n, \vee_{\Pi_n}, q)$ and $M(NC_n^B, \vee_{\Pi_n^B}, q)$ have factorizations that yield the desired factorizations of the determinants?

One can show that if L_n is the subsemilattice of modular elements of Π_n, then the determinant of $M(L_n, \vee_{\Pi_n}, q)$ can be reduced to computing the determinant of $M(L_n, \wedge_{L_n}, q)$, to which the results of [18] apply. More generally, one can ask the following question.

Problem 4. To what extent can matrices of the types considered here be altered and still yield interesting factorizations for the associated determinants?

In the example considered above using the modular elements of Π_n, the underlying lattice was altered, leading to a subdeterminant; one could also alter the entries in the original matrix to see how robust the factorizations are.

The next two questions seek structural reasons for the factorizations in Eqs. (2) and (3).

Problem 5. What are the essential structural features of Π_n and NC_n that account for the factorizations in Eqs. (2) and (3)?

Problem 6. More generally, find conditions on a pair of lattices L and L' such that the determinant of the matrix $M(L', \vee_L, q)$ has a nice factorization.

In Problem 6, we assume that L' is also an induced subposet of L. In relation to Problem 5, we note that the lattices NC_n and Π_n are connected by a number of interesting properties. For instance it follows from the theory of matroid quotients [19] that, as is true of any induced suborder of Π_n that includes all atoms and all elements of rank 2 that cover more than two atoms, its "geometric closure" in the sense of line-closure [14] is Π_n. It follows that any geometric lattice of rank $n-1$ into which NC_n can be embedded as an order necessarily contains a restriction that is isomorphic to Π_n. The counterparts of these statements hold in the type-B setting. We note, however, that NC_n and NC_n^B are not minimal lattices that are suborders of Π_n and Π_n^B, respectively, that have these property. Of course, NC_n and NC_n^B have a number of other important properties that may be relevant: they share their rank functions with Π_n and Π_n^B, they have the same meets as Π_n and Π_n^B, and they are self-dual. We note one more connection between NC_n and Π_n that has a counterpart in the type-B setting and much more generally. Not only is there the natural inclusion map of NC_n into Π_n in which order and meets are preserved, but there is also a natural closure map of Π_n for which the (order-theoretic) quotient is NC_n, namely $\alpha \mapsto \bar{\alpha}$ where

$$\bar{\alpha} = \wedge_{NC_n}\{\beta \mid \beta \in NC_n \text{ and } \alpha \leqslant \beta\}.$$

From this, it follows from Rota's fundamental result on the Möbius function of a quotient [22], that the sum $\sum \mu(\hat{0}, x)$ of the Möbius values $\mu(\hat{0}, x)$, computed in Π_n, over all spanning crossing partitions (that is, elements x of Π_n such that $\bar{x} = \hat{1}_{NC_n}$) is the Möbius value $\mu(\hat{0}, \hat{1})$ of NC_n. This raises the question of giving a closed formula or generating function for the number of spanning crossing partitions in Π_n. This sequence begins $1, 1, 1, 2, 6, 21, 85$. We have seen that $\det(M(L, \vee_{\Pi_n}, q))$ has interesting factorization for both $L = NC_n$ and $L = L_n$, the lattice of modular elements of Π_n; we remark that both of these lattices are quotients of Π_n. (See [15] for the lattice of all closures of an ordered set.)

In [2], Rodica proved relationships between the determinant of the matrix of chromatic joins and the determinants of matrices that arise in topology and algebra. The matrix $L_n(q)$ arose in Lickorish's work [17] on the existence of the Witten–Reshetikhin–Turaev invariants for 3-manifolds. The matrix $L_n(q)$ is indexed by noncrossing perfect matchings, that is, noncrossing partitions of $\{1, 2, \ldots, 2n\}$ in which each block has size two. (See [24, Section 4.3].) The entry in row α and column β is $q^{\langle \alpha, \beta \rangle}$; this exponent $\langle \alpha, \beta \rangle$, the Lickorish bilinear form of α and β, is the number of closed curves that are formed when the arcs joining elements matched by α are drawn above the elements $\{1, 2, \ldots, 2n\}$, listed in a line, and the corresponding arcs for β are drawn below these elements. Based on the results of [6,28], Rodica proved the following equation in [2]:

$$\det\bigl(T_n\bigl(q^2\bigr)\bigr) = q^{C_n} \det\bigl(L_n(q)\bigr). \tag{8}$$

She also showed that the two determinants are directly related to a determinant associated with a certain irreducible representation of the Hecke algebra of type A, specifically,

$$\det\bigl(L_n\bigl(q^{1/2} + q^{-1/2}\bigr)\bigr) = q^{e_n} \det\bigl(S^{(n,n)}(q)\bigr), \tag{9}$$

for some integer e_n, where $S^{(n,n)}(q)$ denotes the Gram matrix (as in [8]) for the inner product on the Specht module of the type-A Hecke algebra indexed by the partition $\lambda = (n, n)$.

Thus, Rodica's aim here was to unify various determinant results in the literature. However, the proofs in [2] are not the conceptual proofs that Rodica would have preferred. Thus, she was interested in the following problem.

Problem 7. Find proofs of the results in [2], such as Eqs. (8) and (9) above, that are more algebraic or combinatorial.

In [6], the Temperley–Lieb algebra is used to compute the determinant of $L_n(q)$. It is well known that the Temperley–Lieb algebra is a quotient of the Hecke algebra; also, Gram determinants for Hecke algebras have been computed both for type A and for type B (see [8,9]). This connection remains to be exploited to prove results such as those in [2].

One corollary of the formula for $\det(L_n(q))$ is that there is a bijection σ of noncrossing perfect matchings such that the Lickorish bilinear form $\langle \alpha, \sigma(\alpha) \rangle$ is always 1. However, explicitly finding such a bijection σ is still open. Note that for $n \geqslant 2$, any such σ is necessarily a derangement since $\langle \alpha, \alpha \rangle = n$ for every noncrossing perfect matching α.

Problem 8. Find a derangement σ of the set M of all noncrossing perfect matchings of $\{1, 2, \ldots, 2n\}$ such that $\langle\alpha, \sigma(\alpha)\rangle = 1$ for all $\alpha \in M$.

There are several possible type-B counterparts for the meander and Lickorish determinants. See, for example, [5].

Problem 9. Do Eqs. (8) and (9) have type-B counterparts? Is there a counterpart of Problem 8 in type B?

In support of a type-B counterpart of Eqs. (8), we note the following examples that Rodica worked out. Consider centrally symmetric noncrossing perfect matchings of the set

$$[\pm 2n] = \{+1, +2, \ldots, +2n, -1, -2, \ldots, -2n\},$$

placed clockwise in order around a circle. Draw the arcs for one such matching, α, inside the circle; draw the arcs for a second such matching, β, outside the circle. Let $\langle\alpha, \beta\rangle$ be the number of pairs of components $C, -C$ in the resulting diagram for which $C \neq -C$. (Below we will also want to consider the number of components in this diagram with $C = -C$; we will denote this by $g(\alpha, \beta)$.) Let such matchings index the rows and columns of a matrix $L_n^B(q)$, and let the entry in row α and column β be $q^{\langle\alpha,\beta\rangle}$. The matrices $L_n^B(q)$ are a possible type-B counterpart of the Lickorish matrices. The determinants of the matrices $L_n^B(q)$ in the cases $n = 1, 2, 3$ are, respectively,

$$q^2 - 1,$$
$$\bigl(q^2 - 1\bigr)^5\bigl(q^2 - 3\bigr),$$
$$\bigl(q^2 - 1\bigr)^{21}\bigl(q^2 - 3\bigr)^6\bigl(q^6 - 6q^4 + 9q^2 - 1\bigr).$$

This supports the natural conjecture that $\det(M(NC_n^B, \vee_{\Pi_n^B}, q^2)) = \det(L_n^B(q))$. One may usefully consider two-variable extensions of all the matrices mentioned in this paper. In particular, again let centrally symmetric noncrossing perfect matchings on $[\pm 2n]$ index the rows and columns of a matrix $L_n^B(q, z)$, and let the entry in row α and column β be $q^{\langle\alpha,\beta\rangle} z^{g(\alpha,\beta)}$. The matrix $L_n^B(q, z)$ is equivalent to the two-variable annular skein determinant considered in [3]. The determinants of $L_n^B(q, z)$ for $n = 1, 2, 3$ are

$$(q - z)(q + z),$$
$$(q - z)^4(q + z)^4\bigl(q^2 - 2 - z\bigr)\bigl(q^2 - 2 + z\bigr),$$
$$(q - z)^{15}(q + z)^{15}\bigl(q^2 - 2 - z\bigr)^6\bigl(q^2 - 2 + z\bigr)^6\bigl(q^3 - 3q - z\bigr)\bigl(q^3 - 3q + z\bigr).$$

One hope is that at least one of the type-B determinants is a specialization of one of the Gram determinants found in [9]. Presumably the correct determinant would come from a representation of the Hecke algebra of degree $\binom{2n}{n}$; thus, prime candidates would be the representations indexed by the bipartitions (n, n), $(n, 1^n)$, $(1^n, n)$, and $(1^n, 1^n)$. This connection, in type-A, was shown by Rodica in [2].

Rodica was at a very productive point in her career the decade before her passing, and she had time to pursue only a relatively small number of the many problems that arose naturally out of her work. In closing, we mention several of these problems that are somewhat more remotely related to the main themes of this paper. (See [23] for problems of a different flavor that Rodica also developed.)

The type-B matrix of chromatic joins reflects a broad interest that Rodica had in type-B objects. Parts of her program for developing type-B counterparts of type-A objects are contained in [25]. Some of the results in [25] suggest there should be simpler combinatorial proofs than Rodica had time to find. In particular, we mention that she was interested in finding a shorter, more elegant proof of [25, Proposition 1].

The lattice of noncrossing partitions, NC_n, was one of the main themes in much of Rodica's work (see [24]). The lattice NC_n is known to have the k-Sperner property. The LYM property is stronger than the Sperner property. Recall (e.g., [13]) that a ranked poset P has the LYM property if for every antichain A of P, we have

$$\sum_{a\in A} \frac{1}{W_{\mathrm{rk}(a)}} \leqslant 1,$$

where $W_{\mathrm{rk}(a)}$ is the number of elements of P that have rank $\mathrm{rk}(a)$.

Problem 10. Does NC_n have the LYM property?

It is easy to check that the answer to Problem 10 is affirmative for $n \leqslant 4$.

It is natural to ask which attractive lattice properties NC_n and Π_n share. Consider, for instance, the following problem.

Problem 11. Is NC_n universal in the same sense that the partition lattice Π_n is universal? (See [20].)

By the results in [20], an affirmative resolution to this problem is equivalent to showing that any partition lattice can be embedded in some lattice of noncrossing partitions.

Acknowledgments

The author gratefully thanks Joseph Bonin for assistance in preparing the manuscript and for discussions about the content. Thanks are also due to Mietek Dabkowski and Jozef Przytycki for sharing their results and observations.

References

[1] G.D. Birkhoff, D.C. Lewis, Chromatic polynomials, Trans. Amer. Math. Soc. 60 (1946) 355–451.
[2] A. Copeland, F. Schmidt, R. Simion, Note on two determinants with interesting factorizations, Discrete Math., in press.

[3] M. Dabkowski, J. Przytycki, Personal communications, 2002.
[4] R. Dahab, The Birkhoff–Lewis equations, PhD dissertation, University of Waterloo, 1993.
[5] T. tom Dieck, Symmetrische Brücken und Knotentheorie zu den Dynkin-Diagrammen von Typ-B, J. Reine Angew. Math. 451 (1994) 71–88.
[6] P. Di Francesco, O. Golinelli, E. Guitter, Meanders and the Temperley–Lieb algebra, Comm. Math. Phys. 186 (1997) 1–59.
[7] P. Di Francesco, Meander determinants, Comm. Math. Phys. 191 (1998) 543–583.
[8] R. Dipper, G. James, Blocks and idempotents of Hecke algebras of general linear groups, Proc. London Math. Soc. 54 (1987) 57–82.
[9] R. Dipper, G. James, E. Murphy, Gram determinants of type-B_n, J. Algebra 189 (1997) 481–505.
[10] T.A. Dowling, A class of geometric lattices based on finite groups, J. Combin. Theory Ser. B 14 (1973) 61–86.
[11] P. Flajolet, Combinatorial aspects of continued fractions, Discrete Math. 32 (1980) 125–161.
[12] C.D. Godsil, Algebraic Combinatorics, Chapman and Hall, New York, 1993.
[13] C. Greene, D.J. Kleitman, Proof techniques in the theory of finite sets, in: G.C. Rota (Ed.), Studies in Combinatorics, Math. Assoc. America, Washington, DC, 1978, pp. 22–79.
[14] M. Halsey, Line-closed combinatorial geometries, Discrete Math. 65 (1987) 245–248.
[15] M. Hawrylycz, V. Reiner, The lattice of closure relations on a poset, Algebra Universalis 30 (1993) 301–310.
[16] G. Kreweras, Sur les partitions non croisées d'un cycle, Discrete Math. 1 (1972) 333–350.
[17] W.B.R. Lickorish, Invariants for 3-manifolds from the combinatorics of the Jones polynomial, Pacific J. Math. 149 (1991) 337–347.
[18] B. Lindstrom, Determinants on semilattices, Proc. Amer. Math. Soc. 20 (1969) 207–208.
[19] J.G. Oxley, Matroid Theory, Oxford Univ. Press, Oxford, 1992.
[20] P. Pudlák, J. Tůma, Every finite lattice can be embedded in a finite partition lattice, Algebra Universalis 10 (1980) 74–95.
[21] V. Reiner, Noncrossing partitions for classical reflection groups, Discrete Math. 177 (1997) 195–222.
[22] G.-C. Rota, On the foundations of combinatorial theory. I. Theory of Möbius functions, Z. Wahrscheinlichkeits. Verw. Gebiete 2 (1964) 340–368.
[23] F. Schmidt, Integer partitions and binary trees, Adv. Appl. Math. 28 (2002) 592–601.
[24] R. Simion, Noncrossing partitions, Discrete Math. 217 (2000) 367–409.
[25] R. Simion, A type-B associahedron, Adv. Appl. Math., to appear.
[26] R. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge Univ. Press, Cambridge, 1997.
[27] W.T. Tutte, On the Birkhoff–Lewis equations, Discrete Math. 92 (1991) 417–425.
[28] W.T. Tutte, The matrix of chromatic joins, J. Combin. Theory Ser. B 57 (1993) 269–288.