1. INTRODUCTION

For the study of the completely specified, deterministic finite automata, let us introduce here a finite characteristic semigroup \( I(A) \) (or \( I \)). In this paper, we consider the case that the input semigroup \( I \) of automata is a free semigroup \( \Sigma^+ \) generated by \( \Sigma \), a finite set of input symbols.

Our discussion in this paper is mainly on the transition diagram (hereafter referred to as the transition structure) of the automata. As the transition structure is independent of the null word, we consider the case \( I = \Sigma^+ \) only.

We introduce the concept of the amalgamation of weakly connected automata to characterize the transition structure of automata.

First, we show that there is an isomorphism between \( I(A) \) of automata represented by an amalgamation of weakly connected automata and that of a direct product of the weakly connected automata. From this fact automata represented by the amalgamation can be decomposed into a direct product of automata with the isomorphic characteristic semigroup.

Second, in the light of Oehmke's work [9], we present the transition structure of automata using an amalgamation and a direct sum of quotient automata of that generated by the characteristic semigroup. The result holds for those automata in which each \( I \) contains a left identity.

Finally, we classify automata according to their characteristic semigroups, and arrive at the following new results in the inclusion of the classes of automata: The class of state independent automata is properly included by that of right group type automata. The class of quasi-perfect automata is equivalent to that of weakly connected state independent automata under the condition that the characteristic semigroup contains an identity such that it can induce only the state transition to itself for all states.

The other inclusions are also discussed.
2. Preliminaries

**Definition 1.** An automaton is a triple $A = (Q, I, M)$, where $Q$ is a nonempty finite set of states, $I$ is an input semigroup, and $M$, called the state transition function of $A$, is a mapping of $Q \times I$ into $Q$, which is defined by $M(q, xy) = M(M(q, x), y)$ for all $q \in Q$ and $x, y \in I$.

In this paper $I$ is a free semigroup $\Sigma^+$ generated by $\Sigma$, a finite set of input symbols.

**Definition 2.** Let $\rho_q$ be a relation on $I$ defined as follows:

$$x \rho_y y(x, y \in I) \iff M(q, x) = M(q, y) \quad \text{for } q \in Q.$$

Let $\rho = \bigcap_{q \in Q} \rho_q$. Then the quotient semigroup $\bar{I}(A)$ (or for short, $\bar{I}$) of $I$ by $\rho$ is called the characteristic semigroup of $A$.

The fundamental automaton $\bar{A} = (Q, \bar{I}, \bar{M})$ of $A = (Q, I, M)$ is defined as

$$\bar{M}(q, [x]) = M(q, x) \quad \text{for all } q \in Q \text{ and } x \in I.$$

($\bar{A}$ is often used in this paper.)

An alternating sequence of states and input symbols (or input words) $q_0x_1q_1 \cdots x_nq_n$ in which $M(q_{i-1}, x_i) = q_i$ or $M(q_i, x_i) = q_{i-1}$ for each $q_i \in Q$ and $x_i \in \Sigma$ (or $I$), where $i \neq j$ implies $q_i \neq q_j$, is called a semipath from $q_0$ to $q_n$.

**Definition 3.** $A = (Q, I, M)$ is called weakly connected iff there exists a semipath from $q$ to $q'$ for all $q, q' \in Q$.

**Definition 4.** $A = (Q, I, M)$ is a direct sum of $A_i = (Q_i, I, M_i)$ $(i = 1, \ldots, t \geq 1)$, denoted by $A = \bigoplus_{i=1}^t A_i$, iff (1) $Q = \bigcup_{i=1}^t Q_i$, where $i \neq j$ implies $Q_i \cap Q_j = \emptyset$, and (2) $q \in Q_i$ implies $M(q, x) = M_i(q, x)$ for all $q \in Q$ and $x \in I$.

**Proposition 1.** Let $A = \bigoplus_{i=1}^m A_i$ and let $\bar{I}(A_i) = I/\rho_i$ for $i = 1, \ldots, m$. Then $\bar{I}(A) = I/\bigcap_{i=1}^m \rho_i$.

A partition $\tau$ on $Q$ is called a partition with the substitution property (referred to as an SP partition) iff $q \tau q'$ implies $M(q, x) \tau M(q', x)$ for all $x \in I$. Then $A_\tau = (Q_\tau, I, M_\tau)$ is called the quotient automaton of $A = (Q, I, M)$ by $\tau$, where $Q_\tau = \{[q], q \in Q\} \{[q], \text{ means a block of } \tau \text{ containing } q \in Q \text{ and is often abbreviated to } q\tau\}$, and $M_\tau([q], x) = [M(q, x)]$, for all $q \in Q$ and $x \in I$.

Given automata $A_1 = (Q_1, I, M_1)$ and $A_2 = (Q_2, I, M_2)$, a mapping $f$ of $Q_1$ into (onto) $Q_2$ is called a homomorphism of $A_1$ into (onto) $A_2$ iff $M_2(f(q), x) = M_2(f(q), x)$ for all $q \in Q$ and $x \in I$. Such $f$ is denoted as $f: A_1 \xrightarrow{\sim} A_2$. If $f: A_1 \xrightarrow{\text{onto}} A_2$ is bijective, then it is called an isomorphism of $A_1$ onto $A_2$, which is denoted as $f: A_1 \cong A_2$. $A_1 \xrightarrow{\sim} A_2$ and $A_1 \cong A_2$ means $A_1$ is homomorphic into (onto) $A_2$ and $A_1$ is isomorphic onto $A_2$, respectively.
DEFINITION 5. Let $S$ be an arbitrary finite semigroup. Then an automaton $\mathfrak{A}_S = (S, S, M_S)$, defined by $M_S(u, v) = u \cdot v$ ($\cdot$ is an operation of $S$ and in what follows $\cdot$ is often omitted), is called an automaton generated by $S$.

It is known that $I(\mathfrak{A}_S) \cong S$ iff $S$ is left reductive and that if $S$ has a left identity then $S$ is left reductive [2, 7].

The concept of $\mathfrak{A}_S$ can be found in [2].

Remark 1. In general, $I(\mathfrak{A}_S)$ is a homomorphic image of $S$ (in the semigroup theoretical sense).

DEFINITION 6. $A' = (Q', I, M')$ is a subautomaton of $A = (Q, I, M)$, denoted as $A' \subset A$, iff (1) $Q' \subseteq Q$, (2) $M' = M \mid (Q' \times I)$ (the restriction of $M$ to $Q' \times I$), and (3) $M'(q, x) \in Q'$ for all $q \in Q'$ and $x \in I$.

Remark 2. If $A' \subset A$ then $I(A')$ is a homomorphic image of $I(A)$.

Remark 3. There is a one-to-one correspondence between subautomata of $\mathfrak{A}_S$ and right ideals of $S$.

An equivalence relation $\pi$ on $S$ is called a right congruence iff $s \pi s' \implies s \pi t s' t$ for all $t \in S$. The definition of a quotient automaton of $\mathfrak{A}_S$ by $\pi$, denoted by $(\mathfrak{A}_S)_\pi$ or $\mathfrak{A}_\pi$, is similar to $A_\pi$.

The next theorem can be easily verified.

THEOREM 1. Let $A$ be any finite automaton. Then $A = \bigoplus_{i=1}^{t} A_i$ for some $t$, where each $A_i$ is weakly connected.

THEOREM 2. Let $A = \bigoplus_{i=1}^{t} A_i$ be any finite automaton. Then $I(A) = S$ is isomorphic to a subdirect product of $I(A_i)$, $i = 1, \ldots, t$.

The proof of Theorem 2 is obvious by Proposition 1.

Remark 4. Let $\omega$ be the universal relation on $S$ and let $\square$ be the composition of relations. Let $\sigma_i = \rho_i \rho$. If each $\sigma_i$ satisfies $(\sigma_1 \cap \cdots \cap \sigma_i) \square \sigma_{i+1} = \omega$ and $\bigwedge_{i=1}^{t} \sigma_i = \tau_S$, then $I/\rho$ is isomorphic to the direct product of $I/\rho_i$, $i = 1, \ldots, t$ [7]. If $A_i \cong A_j$ ($i, j = 1, \ldots, t$; $i \neq j$), then $\rho_i = \rho_j$, and $I/\rho \cong I/\rho_i$. And this implies that $I(A)$ is not necessarily isomorphic to the direct product of $I(A_i)$, $i = 1, \ldots, t$.

3. AMALGAMATION OF AUTOMATA

An amalgamation of automata, which is regarded as a generalization of the direct sum, is defined and using the direct sum and the amalgamation the transition structure of automata is characterized algebraically.

3.1. Amalgamation of Automata

Roughly speaking, the amalgamation is to join some weakly connected automata by identifying the isomorphic subautomata of them.
DEFINITION 7. Let $A_i = (Q_i, I, M_i)$ $(i = 1, 2)$ be weakly connected. Then $A = (Q, I, M)$ is called an automaton composed of the amalgamation of $A_1$ and $A_2$ (with respect to $A_1'$ and $A_2'$), denoted by $A = A_1(A_1') \boxplus A_2(A_2')$, iff the following conditions are satisfied.

1. For $A_i' \subseteq A_i$; $i = 1, 2$, $f_{21}: A_2' \cong A_1'$.
2. $Q = Q_1 \cup Q_2$, where $Q_2 = Q_2 - Q_2' = \{q \in Q_2 : q \notin Q_2'\}$.
3. (i) $M(q, x) = M_1(q, x)$ for all $q \in Q_1$ and $x \in I$, (ii) $M(q, x) = M_2(q, x)$ if $M_2(q, x) \in Q_2'$ for all $q \in Q_2$ and $x \in I$, (iii) $M(q, x) = f_{21}(M_2(q, x))$ if $M_2(q, x) \in Q_2'$ for all $q \in Q_2$ and $x \in I$.

From this it is clear that $M$ is well defined.

It should be indicated that the operation of amalgamation depends on the choice of the isomorphism $f_{21}: A_2' \rightarrow A_1'$.

The amalgamation of $A_1, \ldots, A_k (k \geq 3)$ is defined recursively:

$$\bigoplus_{i=1}^{k-1} A_i(A_{1i}', A_{2i}', i = 1, k - 1) \bigoplus A_k(A_{2k}', i = 1, k - 1)$$

where $A_{1j}' \subseteq \bigoplus_{i=1}^{j-1} A_i(A_{1i}', A_{2i}', i = 1, j - 1)$, $A_{2j}' \subseteq A_j$ for $j = 2, \ldots, k - 1$ and $\bigoplus_{i=1}^{j-2} A_i(A_{1i}', A_{2i}', i = 1, j - 1) = A_1(A_{11}) \boxplus A_2(A_{21})$.

In what follows the amalgamation $A_1(A_1') \boxplus A_2(A_2')$ and $\bigoplus_{i=1}^{k-1} A_i(A_{1i}', A_{2i}', i = 1, k - 1)$ will often be abbreviated as $A_1 \boxplus A_2$ and $\bigoplus_{i=1}^{k-1} A_i$, respectively.

PROPOSITION 2. Let $A = \bigoplus_{i=1}^{k} A_i$ and let $I(A_i) = I(p_i)$ for $i = 1, \ldots, k$. Then $I(A) = I(\bigcap_{i=1}^{k} p_i)$.

The proof of this statement, since it is clear from Definition 7.

Remark 5. $I(\bigoplus_{i=1}^{k} A_i) \cong I(\bigoplus_{i=1}^{k} A_i)$ by Propositions 1 and 2.

Let $V : S \cup T$, where $S$ is a semigroup, $T$ any set. Let $\varphi$ be an arbitrary mapping from $V$ onto $S$ such that $\varphi(s) = s$ for $s \in S$ and $\varphi(t) = s_t$ for $t \in T$ and $s_t \in S$. Define an operation $\circ$ on $V$ as follows:

$$x \circ y = \varphi(x) \cdot \varphi(y) \quad \text{for all } x, y \in V.$$ 

It is easily shown that $V$ is a semigroup with respect to $\circ$. $V$ (with $\circ$) is called an inflation of $S$ (with respect to $T$ and $\varphi$) and is denoted by $V = \text{Inf}(S, T, \varphi)$, $V = \text{Inf}(S, T)$, or $V = \text{Inf}(S)$. $V = \text{Inf}(S)$ is a special case of an ideal extension of $S$ [7].

Since the proofs of Propositions 3 and 4 require routine computation only, they are omitted.

PROPOSITION 3. If $V = \text{Inf}(S, T, \varphi)$ then $I(\mathcal{M}_V) \cong I(\mathcal{M}_S)$. 

Let $B$ and $P_S$ be the set of all binary relations on $S$ and that of right congruences on $S$, respectively. We define a mapping $L$ of $B$ into $B$ as

$$L(\pi) = \{(x, y) \mid sx \pi sy \text{ for all } s \in S\},$$
where $\pi \in B$ and $x, y \in S$.

Remark 6. If $S$ has a left identity, then $L$ is a closure operator such that $L(\pi) \subseteq \pi$. If $\pi \in P_S$ then $L(\pi)$ is the unique maximal congruence contained in $\pi$ and $L(\pi_1 \cap \pi_2) = L(\pi_1) \cap L(\pi_2)$ for $\pi_1$ and $\pi_2$ in $P_S$ [2].

Remark 7. Let $\tau \in P_S$ and $\mathcal{U}_\tau = (S_\tau, S, M_\tau)$. Then $I(\mathcal{U}_\tau) = S/L(\tau)$.

The definition of $L$ and the result of Remark 7 can be found in [2]. Let us define a relation $\theta$ on $V = \text{Inf}(S, T, \varphi)$ by $\tau \in P_S$ as

(i) $s \theta s'$ iff $s \tau s'$ for all $s, s' \in S$,
(ii) $t \theta t'$ iff $t = t'$ for all $t, t' \in T$,
(iii) $\theta$ is a null relation in other cases.

We denote $\theta = \text{ex}(\tau, T)$ or $\theta = \text{ex}(\tau)$. Clearly $\theta$ is a right congruence on $V$.

**Proposition 4.** Let $\mathcal{U}_\theta = (\mathcal{U}_{1 \theta} \circ) = (V_\theta, S, M_\theta), \text{ where } V = \text{Inf}(S, T, \varphi), \theta = \text{ex}(\tau, T), S = I(\mathcal{U}_\theta)$, and $\tau \in P_S$. Then $I(\mathcal{U}_\theta) \cong S/L(\tau)$.

The following propositions are verified from Propositions 3 and 5, and from Propositions 2 and 4 and Remark 6, respectively.

**Proposition 5.** $I(\mathcal{U}_\theta) = I(\mathcal{U}_{1 \theta})$ iff $L(\tau) = \rho_S$, where $S/\rho_S = I(\mathcal{U}_\theta)$.

**Proposition 6.** Let $\mathcal{U} = \bigoplus_{i=1,k} \mathcal{U}_{\theta_i}$. Then $I(\mathcal{U}) = S/L(\bigcap_{i=1,k} \tau_i)$.

### 3.2. Direct Product and Amalgamation of Automata

A direct product of automata is defined in this section, and it is shown that the characteristic semigroup of a direct product of automata is isomorphic to that of the amalgamation. The direct product decomposition of automata is discussed.

**Definition 8.** $A = (Q, I, M)$ is a direct product of $A_i = (Q_i, I, M_i)$ ($i = 1, \ldots, k \geq 2$), denoted by $A = \bigotimes_{i=1,k} A_i$, iff $A$ satisfies

1. $Q = Q_1 \times \cdots \times Q_k$,
2. $M(q, x) = (M_1(q_1, x), \ldots, M_k(q_k, x))$ for all $q = (q_1, \ldots, q_k) \in Q$ ($q_i \in Q_i$, $i = 1, \ldots, k$) and $x \in I$.

**Definition 9.** $A^D = (Q^D, I^D, M^D)$ is called a direct product decomposition of $A = (Q, I, M)$ iff $A^D$ satisfies

1. $A^D = \bigotimes_{i=1,k} A_i$ ($k \geq 2$), $|Q_i| \leq |Q|$ for $i = 1, \ldots, k$,
2. $I(A^D) \cong I(A)$. 


In the following $S$ is assumed to have a left identity.

**Proposition 7.** Let $\mathcal{D} = \otimes_{i=1}^{k} \mathcal{D}_i$, where \( \mathcal{D}_i = (S_i, \cdot, M_i, r_i) \), $i = 1, \ldots, k$ ($\geq 2$). Then \( \bar{I}(\mathcal{D}) \cong S/L(\cap_{i=1}^{k} \mathcal{T}_i) \).

**Proof.** Let \( \bar{I}(\mathcal{D}) = S/\rho \). Then $apb$ for $a, b \in S$ implies $M^D((s_1 \mathcal{T}_1, \ldots, s_k \mathcal{T}_k), a) = M^D(s_1 \mathcal{T}_1, \ldots, s_k \mathcal{T}_k)$ for all $(s_1 \mathcal{T}_1, \ldots, s_k \mathcal{T}_k) \in S_1 \times \cdots \times S_k$, where $s_1, \ldots, s_k \in S$ imply $s_1 \mathcal{T}_1, \ldots, s_k \mathcal{T}_k$ for all $(s_1 \mathcal{T}_1, \ldots, s_k \mathcal{T}_k) \in S_1 \times \cdots \times S_k$, and $s_k \mathcal{T}_k = s_k \mathcal{T}_k$ for all $s_1 \mathcal{T}_1 \in S_1$, $\ldots$, $s_k \mathcal{T}_k \in S_k$. This implies \( \bar{I}(\mathcal{D}) \cong S/L(\cap_{i=1}^{k} \mathcal{T}_i) \). Q.E.D.

The next proposition follows from Remark 5 and Propositions 4, 6, and 7.

**Proposition 8.** \( \bar{I}(\otimes_{i=1}^{k} \mathcal{D}_i) \cong \bar{I}(\otimes_{i=1}^{k} \mathcal{D}_i) \cong I(\otimes_{i=1}^{k} \mathcal{D}_i) \cong S/L(\cap_{i=1}^{k} \mathcal{T}_i) \), where $\mathcal{D}_i = (\mathcal{D}_i, \cdot, \theta_i, \mathcal{V}_i)$, $\mathcal{V}_i = \text{Inf}(S_i, \mathcal{T}_i)$, $\theta_i = \text{ex}(\mathcal{T}_i)$.

The next theorem follows from Proposition 8.

**Theorem 3.** If $A = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} A_{ij}$, where $m + \sum_{i=1}^{m} k_i \geq 3$, then $A$ has a direct product decomposition $A^D = \otimes_{i=1}^{m} \otimes_{j=1}^{k_i} A_{ij}$. And furthermore if $A \cong \mathfrak{A} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} \mathcal{D}_{ij}$, then $L(\bigcap_{i=1}^{m} \bigcap_{j=1}^{k_i} \mathcal{T}_{ij}) = \mathcal{T}$, where $\bar{I}(A) = S$, $\mathcal{D}_{ij} = (\mathcal{D}_{ij}, \cdot, \theta_{ij}, \mathcal{V}_{ij})$, $V_{ij} = \text{Inf}(S_i, \mathcal{T}_i)$, $\theta_{ij} = \text{ex}(\mathcal{T}_i)$, and $\theta_{ij} = \text{ex}(\mathcal{T}_i)$ for $i = 1, \ldots, m$, $r$, and conversely.

The next proposition is easily established. This describes the direct product decomposition of $\mathcal{D}_{ij}$ in Theorem 3. In the following $\mathcal{D}_{ij}$ is denoted as $\mathfrak{A}$ for simplicity.

**Proposition 9.** If $\mathfrak{A}$ has a direct product decomposition of the form $\mathcal{D}^D = \otimes_{i=1}^{r} \mathcal{D}_i$, then $\mathcal{D}_i$, $i = 1, \ldots, r$, can be chosen such that $L(\bigcap_{i=1}^{r} \mathcal{T}_i) = \mathcal{T}$, where $| V_{\theta} | \geq | V_{\theta_i} |$, $V = \text{Inf}(S, T)$, $V_i = \text{Inf}(S_i, T_i)$, $\theta = \text{ex}(\mathcal{T})$, and $\theta_i = \text{ex}(\mathcal{T}_i)$ for $i = 1, \ldots, r$, and conversely.

**Remark 8.** It is well known that $\bar{I}(\bigotimes_{i=1}^{k} A_i)$ is isomorphic to the subdirect product of $\bar{I}(A_i)$, $i = 1, \ldots, k$ ($\geq 2$).

4. **Representation of Automata by Amalgamation and Direct Sum**

In this section we show that an automaton whose characteristic semigroup has a left identity is represented by the amalgamation and the direct sum of weakly connected automata.

Our main purpose in this section is to prove the following theorem.

**Theorem 4.** Let $A = (Q, I, M)$ be weakly connected in which $\bar{I}(A)$ has a left identity. Then $A = \bigoplus_{i=1}^{k} A_i$ and $\mathcal{D} \cong \bigoplus_{i=1}^{k} \mathcal{D}_i$ for some $k$ ($\geq 1$) determined by $A$, where
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\[ A_i = (Q_i, I, M_i), V_i = \text{Inf}(S_i, T_i, \nu_i), \theta_i = \text{ex}(\tau_i, T_i), \tau_i \in P_S, |Q_i| \geq 1, |S_i| \geq 1 \text{ for } i = 1, \ldots, k, \text{ and } L(\bigcap_{i=1}^{k} \tau_i) = \iota_S. \]

This theorem is proved by the following lemmas.

Let \( S = I(A) = \{s_1, \ldots, s_n\} \), and let \( e_i \) be a left identity of \( S \). Choose an element \( q_1 \in Q \) such that \( M(q_1, e_i) = q_1 \). There exists such a state since \( e_i e_i = e_i \).

Let \( \tau_i \) be a relation on \( S \) such that \( s_i \tau_s \tau_j \iff M(q_1, s_i) = M(q_1, s_j) \). Let \( S/\tau_i = \{S_1^{(i)}, \ldots, S_1^{(p_i)}\} = S_{1_i} \). Let \( \bar{A}_i = (Q_i, I, M_i) \) be the subautomaton of \( A \) generated by the state \( q_1 \), where \( Q_1 = \{q_{1i}, \ldots, q_{1p_i}\} \) \( (q_{1i} \in Q_1) \) and let \( q_{1v} = M(q_{1i}, s) \) for \( s \in S_1^{(i)} \), \( \gamma = 1, \ldots, p_i \). Then we have Lemma 1.

**LEMMA 1** [9]. \( A_i = (Q_i, I, M_i) \approx \mathfrak{U}_{r_i} = (S_{1_i}, S, M_{ri}). \)

Let \( S_e = \{e_1, \ldots, e_s\} \subseteq S \) be the set of all left identities of \( S \), and let \( D_i = \{q \in Q | M(q, e_i) = q\} \) and \( Q_e = \bigcup_{i=1}^{s} D_i \), respectively, where \( i \neq j \) does not always imply \( D_i \cap D_j = \emptyset \). Select \( q_1 \in Q_e \) and let \( \bar{A}_i = (Q_i, I, M_i) \) be the subautomaton of \( A \) generated by the state \( q_1 \), where \( Q_{1i} = \{q_{1i}, \ldots, q_{1p_i}\} \) and \( q_{1v} = M(q_{1i}, s) \) for \( s \in S_1^{(i)} \), \( \gamma = 1, \ldots, p_i \).

Construct \( Q_i \) for each \( q_{1i} \in Q_i \) and delete any \( Q_j \) such that \( Q_j \subseteq Q_i \) for some \( Q_j \). Let \( Q_1, \ldots, Q_k \) \( (k \geq 1) \) be the remainder of the deletion. If we put \( Q_e = \bigcup_{i=1}^{k} Q_i \) and \( T = Q - Q_e \), respectively, then \( M(t, s_j) \in Q_i \) for all \( t \in T \) and \( s_j \in S \). More precisely, if \( M(t, e_i) = q_1 \in L \subseteq Q \) \( (1 \leq L \leq k) \) (such \( Q_L \) may exist more than once) for \( e_i \in S_e \) such that \( M(q_{1i}, e_i) = q_{1i} \), then \( M(t, s_j) = M(t, e_i s_j) = M(M(t, e_i), s_j) = M(q_{1i}, s) = M(M(q_{1i}, s), s_j) = M(q_{1i}, s s_j) = q_{1i} \) for \( s \in S_1^{(i)} \) and \( s' = s s_j \in S_1^{(i)} \). Thus we have the next lemma.

**LEMMA 2.** Let \( S_e \) and \( T \) be defined above. Then for each \( t \in T \) there exist \( e_i \in S_e \) and an index \( L \) such that \( M(t, e_i) = q_1 \in Q \) \( (1 \leq L \leq k) \) where \( M(q_{1i}, e_i) = q_{1i} \), and \( M(t, s_j) = q_{1i} \in Q \) \( s_j \in S \) where \( s' = s s_j \in S_1^{(i)} \) and \( s \in S_1^{(i)} \).

Let us define \( T_i = \{t \in T | M(t, e_i) \in Q_i \} \) for some \( e_i \in S_e \) for \( i = 1, \ldots, k \). Then we can put \( T = Q - Q_e = \bigcup_{i=1}^{k} T_i \), where \( i \neq j \) does not always imply \( T_i \cap T_j = \emptyset \).

By the above consideration we have the following lemma.

**LEMMA 3.** \( A_{T_i} = (Q_{T_i}, I, M_{T_i}) \approx \mathfrak{U}_{q_i} = (V_{q_i}, S, M_{q_i}), \) where \( Q_{T_i} = Q \cup T_i, M_{T_i} = M(T_i \times I), V_i = \text{Inf}(S, T_i, \nu_i), \theta_i = \text{ex}(\tau_i, T_i), \) and \( \tau_i \in P_S, \tau_i \) and \( \nu_i \) are defined as \( s_i \tau_i s_k \iff M(q_{1i}, s_i) = M(q_{1i}, s_k) \), and \( \nu_i(t) = s \iff M(t, e') = M(q_{1i}, s) = q_{1i} \in Q_i \)

for \( s \in S_1^{(i)} \), \( t \in T \), and a fixed \( e' \in S_e \) such that \( M(q_{1i}, e') = q_{1i} \).

**Proof.** Define a mapping \( g_i: Q_{T_i} \rightarrow V_{q_i} \) as

\[
g_i(q_{1i}) = S_1^{(i)} \text{ for all } q_{1i} \in Q_i \text{ and } g_i(t) = [t]_{q_i} \text{ for all } t \in T_i.
\]

(g_i | Q_i): \( \bar{A}_i = (Q_i, I, M_i) \approx \mathfrak{U}_{q_i} = (S_{1_i}, S, M_{1_i}) \) by Lemma 1 and \( g_i | T_i \) is bijective by the definition of \( g_i \), so that \( g_i \) is bijective. Let \( M(t, e') = q_{1i} = M(q_{1i}, s) \) for \( t \in T_i \).
and \(s \in S\) given in the definition of \(q_i\). Then \(\overline{M}_{\mathcal{T}}(t, s_k) = q_{4L} = \overline{M}_{\mathcal{T}}(q_i, s')\) for all \(s_k \in S\), where \(s' = s_{sl} \in S_{i}^{(2)}\). Consequently, \(g_i(\overline{M}_{\mathcal{T}}(t, s_k)) = g_i(q_{4L}) = S_{i}^{(2)}\), while \(M_{\mathcal{S}}(g_i(t), s_k) = M_{\mathcal{S}}([t \circ s_k]_{q_i} = [s_{sl}]_{q_i} = [s']_{q_i} = S_{i}^{(2)}\).

This implies \(g_i: \overline{\mathcal{A}}_{\mathcal{T}} \cong \mathcal{A}_{\mathcal{T}}\).

Remark 9. \(A_{\mathcal{T}} \subset \overline{\mathcal{A}}\). And if \(\overline{M}(q_i, s) = \overline{M}(q_i, s')\), then we can define \(g(t) = s'\) since \(s \sim s'\) implies \([s_{sl}]_{q_i} = [s's_{sl}]_{q_i}\) for all \(s_{sl} \in S\).

Remark 10. Let \(C = \{Q_1, \ldots, Q_k\}\) be defined above. Since \(A\) is weakly connected, there exists \(Q_i \in C\) such that \(Q_i \cap Q_j \neq \emptyset\) for each \(Q_i \in C\). Furthermore, select an arbitrary element \(Q_i \in C\) and set \(Q_1 = Q_i\). And set the element of \(i\)th selection as \(Q_i\) for \(i = 1, \ldots, k\). Set \(Q^{(i)} = Q_1\) and \(Q^{(i+1)} = Q^{(i)} \cup Q_{i+1}\) for \(i = 1, \ldots, k - 1\). Then we can select \(Q_{i+1}\) such that \(Q^{(i)} \cap Q_{i+1} \neq \emptyset\) for \(i = 1, \ldots, k - 1\), if \(k \geq 2\).

Let \(Q_{i+1}^{(12)} = Q_{i+1}^{(12)} \cap Q_{i+1}^{(12)}\), where \(Q_{i+1}^{(12)} = Q_i \cup T_1\), \(Q_i \in C\) \((i = 1, 2)\) and \(Q_1 \cap Q_2 \neq \emptyset\). Then \(\overline{A}_{12} = (Q_{i+1}^{(12)}, I, \overline{M}_{i+1}) \subset \overline{\mathcal{A}}\), where \(\overline{M}_{i+1} = \overline{M} | (Q_{i+1}^{(12)} \times I)\). \(Q_{i+1}^{(12)}\) corresponds to \(Q_i\) of Definition 7.

The next lemma is easily established.

**Lemma 4.** Let \(\overline{A}_{12}\) be defined above. Then \(\overline{A}_{12} = (Q_{i+1}^{(12)}, I, \overline{M}_{i+1}) \cong \mathcal{A}_{\mathcal{T}}(\mathcal{U}_i) \cap \mathcal{A}_{\mathcal{T}}(\mathcal{U}_2),\) where \(Q_{i+1}^{(12)} = Q_i \cup T_1\), \(Q_i \in C\) \((i = 1, 2)\), \(\mathcal{U}_i \subset \mathcal{U}_2\) such that \(\mathcal{U}_i \cong \mathcal{A}_{\mathcal{T}}(\mathcal{U}_1)\) and \(\mathcal{U}_i\) is defined in Lemma 3 for \(i = 1, 2\). And \(\overline{M}_{i+1} = \overline{M} | (Q_{i+1}^{(12)} \times I)\).

We propose the following algorithm to establish Theorem 4.

**Algorithm 0.** Let \(A = (Q, I, M)\) be weakly connected and \(I(A) = S = \{s_1, \ldots, s_n\}\) \((n \geq 1)\).

**Step 1.** Let \(S_s = \{e_1, \ldots, e_a\} \subseteq S\) \((1 \leq a \leq n)\) be the set of all left identities of \(S\), and define \(D_i = \{q \in Q | q = M(q, e_i)\}\) for each \(e_i \in S_s\) and let \(Q_e = \bigcup_{i=1}^a D_i\).

**Step 2.** For each \(q_i \in Q_e\), construct \(Q_i\) and set \(C^* = \{Q_i | i = 1, \ldots, |Q_e|\}\).

**Step 3.** Delete \(Q_j\) from \(C^*\) such that \(Q_j \subseteq Q_i\) for some \(Q_i \in C^*\). Let \(Q_1, \ldots, Q_k\) \((k \geq 1)\) be the remainder and put \(C = \{Q_1, \ldots, Q_k\}, Q_s = \bigcup_{i=1}^k Q_i\) and \(T = Q - Q_s\), respectively.

**Step 4.** Set \(C^{(0)} = C, Q^{(0)} = \emptyset, C_0 = \emptyset\).

**Step 5.** Set \(r = 1\).

**Step 6.** If \(r = 1\) go to Step 7. Otherwise go to Step 8.

**Step 7.** Select any \(Q_i \in C^{(0)}\). Go to Step 9.

**Step 8.** Select \(Q_i \in C^{(r-1)}\) such that \(Q^{(r-1)} \cap Q_i \neq \emptyset\).

**Step 9.** Set \(\tilde{q}_i = q_i\) and \(q_i = g_i\), where \(q_i\) is the state by which the subautomaton \(\overline{A}\) of \(\overline{A}\) is generated. Define a relation \(\tau_r\) on \(S\) as

\[s \tau_r s_k \Leftrightarrow \overline{M}(\tilde{q}_r, s) = \overline{M}(\tilde{q}_r, s_k)\]
Step 10. Set $Q^{(r)} = Q^{(r-1)} \cup \tilde{Q}_r$ and $C_r = C_{r-1} \cup \{\tilde{Q}_r\}$.

Step 11. Define $T_r = \{ t \in T \mid \tilde{M}(t, e_s) \in \tilde{Q}_r \text{ for some } e_s \in S_e \}$.

Step 12. Construct $\mathcal{A}_{\theta_r} = (V_{\theta_r}, S, M_{\theta_r})$ for $V_{\theta_r} = \text{Inf}(S, T_r, \varphi_r)$ and $\theta_r = \text{ex}(\tau_r, T_r)$, where $\varphi_r$ is defined for each $t \in T_r$ as

$$\varphi_r(t) = s \iff \tilde{M}(t, e_s) = \tilde{M}(q_r, s) \in \tilde{Q}_r$$

such that $\tilde{M}(q_r, e_s) = \tilde{q}_r$, where $s \in S$. Set $\mathcal{A}_r = \mathcal{A}_{\theta_r}$.

Step 13. Set $Q_{T_r} = \tilde{Q}_r \cup T_r$.

Step 14. Set $C^{(r)} = C - C_r$.

Step 15. Set $\tilde{A}_r = (Q_{T_r}, I, \tilde{M}_{T_r})$, where $\tilde{M}_{T_r} = \tilde{M} | (Q_{T_r} \times I)$.

Step 16. If $r = 1$ go to Step 17. Otherwise go to 20.

Step 17. Set $\mathcal{A}^{(r)} = \mathcal{A}_r$.

Step 18. Set $\tilde{A}^{(r)} = \tilde{A}_r$.

Step 19. Set $Q_T(r) = Q_{T_r}$. Go to Step 27.

Step 20. Set $Q_T(r) = Q_T(r - 1) \cup Q_{T_r}$.

Step 21. Set $\tilde{A}^{(r)} = (Q_T(r), I, \tilde{M}_T(r))$, where $\tilde{M}_T(r) = \tilde{M} | (Q_T(r) \times I)$.

Step 22. Set $Q_T(r - 1)' = Q_T(r - 1) \cap Q_{T_r}$.

Step 23. Set $\tilde{A}^{(r-1)'} = (Q_T(r - 1)' , I, \tilde{M}_T(r - 1)' )$, where $\tilde{M}_T(r - 1)' = \tilde{M} | (Q_T(r - 1)' \times I)$.

Step 24. Set $\tilde{A}^{(r-1)'}(\tilde{A}^{(r-1)}) \boxplus \tilde{A}_r(\tilde{A}^{(r-1)})' = \tilde{A}^{(r)}$.

Step 25. Construct $\mathcal{A}^{(r-1)}(\mathcal{A}^{(r-1)'}) \boxplus \mathcal{A}_r(\mathcal{A}_r')$, where $\mathcal{A}^{(r-1)'} \subseteq \mathcal{A}^{(r-1)}$ and $\mathcal{A}_r' \subseteq \mathcal{A}_r$, such that $\mathcal{A}^{(r-1)'} \cong \tilde{A}^{(r-1)'} \cong \mathcal{A}_r'$.

Step 26. Set $\mathcal{A}^{(r)} = \mathcal{A}^{(r-1)} \boxplus \mathcal{A}_r$.

Step 27. If $r < k$ go to Step 28. Otherwise go to Step 29.

Step 28. Set $r = r + 1$ and return to Step 6.

Step 29. The algorithm terminates.

The next lemma follows from Lemmas 2, 3, and 4.

Lemma 5. If all the elements in $C$ are selected then the algorithm terminates. At that time we have $A = \bigoplus_{i=1,k} A_i$ and $A \cong \bigoplus_{i=1,k} \mathcal{A}_i$, where each $A_i = (Q_i, I, M_i) \cong \mathcal{A}_i = \mathcal{A}_{\theta_i} = (V_{\theta_i}, S, M_{\theta_i})$.

Proof of Theorem 4. The theorem follows from Lemma 5, the algorithm above, and Propositions 2 and 4. Q.E.D.
The next theorem follows from Theorems 1 and 4 and Propositions 1 and 4.

**Theorem 5.** Let \( A = (Q, I, M) \) be any finite automaton such that \( I(A) = S \) has a left identity. Then \( A = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_t} (A_{ij} \cong \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_t} (V_{\theta_{ij}}, S, M_{\theta_{ij}})) \) for some \( m \) and \( k_t \) \((i = 1, \ldots, m)\), where \( A_{ij} \cong \mathfrak{U}_{ij} = \mathfrak{U}_{\theta_{ij}} = (V_{\theta_{ij}}, S, M_{\theta_{ij}}) \) and

\[
L\left( \bigcap_{i=1}^{m} \left( \bigcap_{j=1}^{k_t} \tau_{ij} \right) \right) = i_s.
\]

5. **Classification of Automata**

Finite automata whose characteristic semigroups have left identities are classified according to their characteristic semigroups using the results of the last section.

Let \( S \) be a semigroup. As stated in Remark 3, there is a one-to-one correspondence between subautomata of \( \mathfrak{U}_S \) and right ideals of \( S \). \( S \) is called right simple if \( S \) has no right ideals except \( S \) itself. And \( S \) is right simple iff \( aS = S \) for any \( a \in S \) [7], so that the next proposition can be obtained.

**Proposition 10.** Let \( S \) be a semigroup. Then \( \mathfrak{U}_S \) is strongly connected iff \( S \) is right simple.

\( A = (Q, I, M) \) is strongly connected iff \( M(q, x) = q' \) for all \( q, q' \in Q \) and some \( x \in I \).

The next proposition follows from Definition 7 and Proposition 10.

**Proposition 11.** Let \( S \) be a right simple semigroup and let \( V_i = \text{Inf}(S, T_i, q_i), \quad \theta_i = \text{ex}(\tau_i, T_i) \) for \( i = 1, \ldots, k \) \((\geq 2)\), where \( \tau_i \in P_S \). Then there exists no nontrivial amalgamation of \( \mathfrak{U}_{\theta_i}, i = 1, \ldots, k \).

A semigroup \( S \) is called a right group iff there exists only one \( c \in S \) such that \( ac = b \) for all \( a, b \in S \) [7]. There are many equivalent definitions of a right group [7]; for example,

**RG-1.** \( S \) is a right group iff \( S \) is isomorphic to the direct product of a group and a right zero semigroup,

**RG-2.** \( S \) is a right group iff \( S \) is a right simple semigroup with a left identity.

**Definition 10.** \( A = (Q, I, M) \) is called left identity type \((A^L)\), identity type \((A^I)\), right simple type \((A^{RS})\), right group type \((A^{RG})\), group type \((A^G)\) iff \( I(A) \) is a semigroup with a left identity, a semigroup with an identity, right simple, a right group and a group, respectively.

The class of each type automata is denoted by \( K \) such as \( K(A^L) \) for \( A^L \).

**Remark 11.** For the explanation of the inclusions, we shall use the class of the right simple type automata, \( K(A^{RS}) \).

The following discussion is on \( K(A^G) \).
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DEFINITION 11 [11]. \( A = (Q, I, M) \) is called a permutation automaton (\( A^p \)) iff \( q \neq q' \) implies \( \bar{M}(q, [x]) \neq \bar{M}(q', [x]) \) for all \( q, q' \in Q \) and \( [x] \in I \).

DEFINITION 12 [6, 10]. \( A = (Q, I, M) \) is called quasi-perfect (\( A^{qp} \)) iff \( A \) is strongly connected, state independent (Definition 13), and group type.

It has been proved in [1, 6] that \( A^{qp} \cong \mathfrak{U}_G \) for some group \( G \). And it has already been shown in [8] that if \( A^p \) is strongly connected then \( A^p \cong \mathfrak{U}_\tau = (\mathfrak{U}_G)_\tau \), where \( G \cong I(A^p) \) and \( \tau \) is a right congruence on \( G \). The more general result on \( A^p \) is described in [11].

Remark 12. It is well known that there is a one-to-one correspondence between right congruences (congruences) on a group \( G \) and subgroups (normal subgroups) of \( G \). And \( I(A_H) \cong G/N \) for \( A_H = (G_H, G, M_H) \), where \( G_H \) is the set of all right cosets of \( G \) modulo \( H \) and \( N = \bigcap_{a \in G} a^{-1}Ha \) [2]. \( M_H \) is defined by \( M_H(Ha, b) = Hab \) for all \( Ha \in G_H \) and \( b \in G \).

We have already established the following lemma in [11].

**Lemma 6** [11]. \( K(A^G) \supseteq K(A^p) \supseteq K(A^{qp}) \).

\( A^{qp} \) is called perfect (\( A^{pA} \)) and a counter (\( A^c \)) iff \( G \) is abelian and cyclic, respectively, where \( A^{qp} \cong \mathfrak{U}_G \). It is well known that \( K(A^{qp}) \supseteq K(A^{pA}) \supseteq K(A^c) \).

The next lemma follows from the definition of a right group, RG-2.

**Lemma 7.** \( K(A^L) \supseteq K(A^{RG}) \supseteq K(A^{RS}) \) and \( K(A^{RG}) = K(A^L) \cap K(A^{RS}) \).

Remark 13. A right group is a group if it has an identity [7].

Next we discuss the state independency of automata introduced by Trauth [10].

**Definition 13.** \( A = (Q, I, M) \) is called state independent, denoted by \( A^{si} \), iff \( \rho_q = \rho_{q'} \) for all \( q, q' \in Q \).

**Remark 14.** \( I(A^{si}) = I_{\rho_q} \) for any \( q \in Q \).

The next theorem has already been obtained in [6].

**Theorem 6** [6]. \( A = (Q, I, M) \) is strongly connected and state independent iff \( \bar{A} \cong \mathfrak{U}_S \), where \( S \) is a right group.

The following lemma has already been verified in [5].

**Lemma 8** [5]. Let \( S \) be a right group, and let \( \tau \) be a right congruence on \( S \). Then \( \mathfrak{U}_\tau \), is state independent iff \( \tau \) is a congruence on \( S \).

We will show \( K(A^{RG}) \supseteq K(A^{si}) \) in the following.

**Lemma 9.** Let \( A = (Q, I, M) \) be state independent. Then \( I(A) \) is a right group.

**Proof.** By the definition, if \( \bar{M}(q, s_i) = \bar{M}(q, s_k) \) for some \( q \in Q \) then \( \bar{M}(q', s_i) = \).
$\bar{M}(q', s_k)$ for all $q' \in Q$, where $s_i$, $s_j$, and $s_k\in I/\rho_q = I(A)$. Let $q \in Q$ be arbitrarily fixed, and let $I(A) = I/\rho_q = \{s_1, \ldots, s_n\} = S(n \geq 1)$. Let $Q_q = \{q_i \mid q_i = \bar{M}(q, s_i), i = 1, \ldots, n\}$. Then $|Q_q| = n$ since $i \neq j$ implies $\bar{M}(q, s_i) \neq \bar{M}(q, s_j)$. There are two cases for each $q \in Q$.

1. $q$ has a self-loop such that $\bar{M}(q, s_i) = q$, and $s_i \in S$ is uniquely determined by $q$ since $A$ is state independent.

2. $q$ has no self-loop.

Case 1. (In this case $q \in Q_q$.) Clearly, $s_i$ is a left identity of $S$ since $\bar{M}(q, s_i) = M(q, s_i)$ for all $s_i \in S$ (in Fig. 1, $s_1$ is such an element). Let $\bar{A}_q = (Q_q, I, \bar{M}_q)$, where $\bar{M}_q = \bar{M} | (Q_q \times I)$. Then $\bar{A}_q \subseteq \bar{A}$, and there exists a right congruence $\tau$ on $S$ such that $\bar{A}_q \cong \mathcal{A}_\tau = (S_\tau, S, M_\tau)$ as in Lemma 1. $\tau = \tau_S$ since $|Q_q| = n$, so that $\bar{A}_q \cong \mathcal{A}_S$. Let $A$ be state independent, $|Q_q| = n$ and $\bar{A}_q \subseteq \bar{A}$. Let $q_k = \bar{M}(q, s_k) \in Q_q$. Then there exists $s_{j(k)} \in S$ such that $\bar{M}(q_k, s_{j(k)}) = q_k$, and $\bar{M}(q_k, s_\alpha) \neq \bar{M}(q_k, s_\beta)$ if $\alpha \neq \beta$ for $s_\alpha, s_\beta \in (S \setminus \{s_{j(k)}\})$. Thus $\mathcal{A}_S$ is strongly connected, which implies $S$ is right simple by Proposition 10. Since $S$ has a left identity, $S$ is a right group by RG-2.

![Fig. 1. Case 1 in the proof of Lemma 11.](image1)

Case 2. (In this case $q \notin Q_q$.) Refer to Fig. 2.) Let $q_i \in Q_q$. Then $\bar{M}(q_i, s_i) = \bar{M}(q, s_i) = \bar{M}(q, s_{j(k)}) = q_k \in Q_q$ for all $s_j \in S$, where $s_k = s_{j(k)}$. Thus $\bar{A}_q = (Q_q, I, \bar{M}_q) \subseteq \bar{A}$, where $\bar{M}_q = \bar{M} | (Q_q \times I)$. This implies that there exist no $q_i \in Q_q$ and no $s_i \in S$ such that $\bar{M}(q_i, s_i) = q$. Since $A$ is state independent and $|S| = |Q_q| = n$, there exists $s_{L(i)} \in S$ such that $\bar{M}(q_i, s_{L(i)}) = q_i$ for each $q_i \in Q_q$. If we consider $q_i$ as $q$ in Case 1, a verification similar to that for Case 1 will complete the proof. Q.E.D.

![Fig. 2. Case 2 in the proof of Lemma 11.](image2)
Lemma 9 shows $K(A_{RG}) \supseteq K(A_{SI})$. Now let us prove the proper inclusion by the following counterexample.

Let $D_4$ be a dihedral group of order 8. The defining relation of $D_4$ by the generator $x$ and $y$ is $x^4 = y^2 = (xy)^2 = e$, where $e$ is an identity of $D_4$ [14]. Let $H = \{e, y\}$. $H$ is a subgroup of $D_4$ such that $N = \bigcap_{a \in D_4} a^{-1}Ha = \{e\}$. This implies $I(A_H) \cong I(D_4) \cong D_4$ by Remark 12. Since $H$ is not normal in $D_4$, $A_H$ is not state independent by Lemma 8 and Remark 12.

Thus we have,

**Lemma 10.** $K(A_{RG}) \supseteq K(A_{SI})$.

**Lemma 11.** Let $V = \text{Inf}(S, T, \varphi)$, where $S$ is a right group. Then $\mathcal{U}_V$ is state independent.

**Proof.** Let $a$ and $b$ be arbitrary elements in $V$. For all $x, y \in V$, $x \rho_a y \Rightarrow M_V(a, x) = M_V(a, y) \Rightarrow a \circ x = a \circ y \Rightarrow \varphi(a) \varphi(x) = \varphi(a) \varphi(y) \Rightarrow M_S(\varphi(a), \varphi(x)) = M_S(\varphi(a), \varphi(y)) \Rightarrow M_V(\varphi(a), \varphi(x)) = M_V(\varphi(a), \varphi(y)) \Rightarrow \varphi(x) \rho_{\varphi(a)} \varphi(y)$. Similarly $x \rho_b y \Rightarrow \varphi(x) \rho_{\varphi(b)} \varphi(y)$. $\varphi(V) = \{\varphi(a) | a \in V\} = S$ by the definition of $\varphi$. Since $\mathcal{U}_S$ is state independent $\rho_{\varphi(a)} = \rho_{\varphi(b)}$, so that $\rho_a = \rho_b$.

By Proposition 11, Theorem 6, and Lemmas 8, 9, and 11, we obtain the next theorem.

**Theorem 7.** Let $A$ be weakly connected. Then $A$ is state independent iff $\bar{A} \cong \mathcal{U}_V$ for $V = \text{Inf}(S, T, \varphi)$, where $S$ is a right group.

It is obvious that if $A = \bigoplus_{i=1}^m A_i$ is state independent then so are $A_i$, $i = 1, \ldots, m$ ($\geq 2$). And it is easily established that $\mathcal{U} = \bigoplus_{i=1}^m \mathcal{U}_{V_i}$ is state independent for $V_i = \text{Inf}(S, T_i, \varphi_i)$, $i = 1, \ldots, m$, if $S$ is a right group.

The next theorem follows from Theorems 5 and 7 and the fact stated above.

**Theorem 8.** Let $\bar{I}(A) = S$. Then $A = \bigoplus_{i=1}^m A_i$ is state independent iff $\bar{A} \cong \bigoplus_{i=1}^m \mathcal{U}_{V_i}$ for $V_i = \text{Inf}(S, T_i, \varphi_i)$, $i = 1, \ldots, m$, where $S$ is a right group and $\bar{A}_i \cong \mathcal{U}_{V_i}$.

The next lemma is also obvious by Remark 13.

**Lemma 12.** $K(A_{RG}) \supseteq K(A^G)$.

**Remark 15.** The transition structure of $A_{RG}$ is given by Proposition 11 as the special case of Theorems 4 and 5.

Now we obtain the next theorem by these results.

**Theorem 9.** $K(A^1) \supseteq K(A^1), K(A^1) \cap K(A_{RG}) = K(A^G), K(A^1) \supseteq K(A_{RG}) \supseteq K(A^G) \supseteq K(A^P) \supseteq K(A^{PA}) \supseteq K(A^C), K(A_{RG}) = K(A^1) \cap K(A^{RS}) \supseteq K(A^{SI}), K(A^{SI}) = K(A^P) \not\subseteq \$ and $K(A^{SI}) \supseteq K(A^{AP})$, where $A^8$ means $A^G$ or $A^P$.

The inclusion in Theorem 9 is shown in Fig. 3.
In the rest of this section we discuss the representation and the classification of automata under the condition that $I = \Sigma^+$ of $A = (Q, I, M)$ has an element $x_\epsilon$ such that $M(q, x_\epsilon) = q$ for all $q \in Q$.

Then $I(A)$ is a monoid with identity $[x_\epsilon]_\rho \equiv e$.

Under this condition $T = Q - Q_{\epsilon} = \varnothing$ in Algorithm 0, so that we have the following corollary by Theorem 5.

**Corollary 1.** Let $A = (Q, I, M)$ be any finite automaton in which $I$ has an element $x_\epsilon$ such that $M(q, x_\epsilon) = q$ for all $q \in Q$. Then $A = \bigoplus_{i=1,m}(\bigoplus_{j=1,k_i} A_{ij}) \cong \mathcal{U} = \bigoplus_{i=1,m}(\bigoplus_{j=1,k_i} \mathcal{U}_{ij})$ for some $m$ and $k_i (i = 1,\ldots, m)$, where $\mathcal{U}_{ij} \equiv \mathcal{U}_{ij} = (S_{ij}, S, M_{ij}) \cong A_{ij}$ and $L(\bigcap_{i=1,m}(\bigcap_{j=1,k_i} \tau_{ij})) = \iota_S$.

On the basis of Corollary 1, the classification of automata is also obtained as a corollary of Theorem 9. $A$ in the case of $I = \Sigma^+$ is denoted as $A_{\lambda}$ for the distinction in this case.

The following lemma is obvious by Corollary 1.

**Lemma 13.** $K(A_{\lambda}) = K(A_{\lambda}^I)$, where $A_{\lambda}$ denotes a finite automaton in this case.

As stated in Remark 13, a right group is a group if it has an identity, so that the next lemma follows.

**Lemma 14.** $K(A_{\lambda}^R) = K(A_{\lambda}^G)$.

Furthermore we obtain the next lemma by Theorem 7 and Lemma 14.

**Lemma 15.** In the class of weakly connected automata, $K(A_{\lambda}^S) = K(A_{\lambda}^R)$.

**Remark 16.** It is easy to see that in the class of weakly connected automata a state independent automaton $A = (Q, I, M)$ is quasi-perfect iff $I$ has an element $x_\epsilon$ such that $M(q, x_\epsilon) = q$ for all $q \in Q$. 

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**Fig. 3.** The inclusion in Theorem 9.
We have already established the following lemma in [11].


The following lemma is obtained by Theorem 8, Lemma 15, and the result [11] that \( A^P \) is a direct sum of some strongly connected \( A^P \)'s.

**Lemma 17.** \( K(A^P) \supseteq K(A^{SI}) \supseteq K(A^{QP}). \)

**Remark 17.** \( A^{SI} = \oplus_{i=1}^{m} A_i \), where \( A_i \cong \mathcal{H}_G \) for \( i = 1, \ldots, m \), and \( G \) is a group. By these results we obtain the following corollary.

**Corollary 2.** \( K(A^F) = K(A^I) \supseteq K(A^{RG}) = K(A^G) = K(A^P) = K(A^P) \supset K(A^{SI}) \supset K(A^{RP}) \supset K(A^{PA}) \supset K(A^G). \)

The inclusion in Corollary 2 is shown in Fig. 4.

![Diagram](image)

**Fig. 4.** The inclusion in Corollary 2. \( K(A^{PA}) \) and \( K(A^P) \) are omitted here.

6. **Conclusion**

We have developed here the algebraic structure theory of automata by introducing the concept of an amalgamation of automata and have established the relationship between the direct product decomposability and the amalgamation of automata.

We have shown that the transition structure of an automaton whose characteristic semigroup has a left identity can be represented by the amalgamation and the direct sum.

In addition, we have considered the classification of automata according to the characteristic semigroups, and have presented the complete hierarchy of automata for the first time.

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