A Note on Idempotent Matrices

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ABSTRACT

Let H be an $n\times n$ matrix, and let the trace, the rank, the conjugate transpose, the Moore-Penrose inverse, and a g-inverse (or an inner inverse) of H be respectively denoted by $\operatorname{tr} H$, $\rho(H)$, H^*, $H^\dagger$, and H^-. This note develops two results: (i) the class of idempotent g-inverse of an idempotent matrix, and (ii) if H is an $n\times n$ matrix and $\rho(H)=\operatorname{tr} H$, then $\operatorname{tr}(H^2H^\dagger H^*)\geqslant\rho(H)$, and the equality holds iff H is idempotent. This result is compared with the previous result of Khatri (1983), and some consequences of (i) and (ii) are given.

1. IDEMPOTENT MATRICES AND g-INVERSES

Let H be an $n\times n$ idempotent matrix. Then any g-inverse H^- of H is given by

$$H^- = H+(I-H)Z_1+Z_2(I-H)\qquad \text{for some matrices } Z_1 \text{ and } Z_2.$$

This can be rewritten as

$$H^- = H_1 + H_2,$$

$$H_1=[I+(I-H)Z_1]H[I+Z_2(I-H)] \quad\text{and}\quad H_2=(I-H)Z_3(I-H), \tag{1}$$

where Z_1, Z_2 and Z_3 are arbitrary matrices. Notice that $\rho(H_1)=\rho(H)$. Observe that $H^- = H_1+H_2$ is idempotent iff

$$H_1^2+H_1H_2+H_2H_1+H_2^2=H_1+H_2.$$

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This condition implies

$$0 = HZ_2(I-H)Z_1H = HZ_2(I-H)Z_3(I-H) = (I-H)Z_3(I-H)Z_1H,$$

$$H_2^2 = H_2. \tag{2}$$

The conditions (2) imply that $H_i^2 = H_i$ $(i = 1,2)$ and $H_1H_2 = H_2H_1 = 0$. Thus we get

THEOREM 1. *Let H be an idempotent matrix. Then H^- is idempotent iff $H^- = H_1 + H_2$, $H_1 = [I + (I-H)Z_1]H[I + Z_2(I-H)]$, $H_2 = (I-H)Z_3(I-H)$, and Z_1, Z_2, and Z_3 satisfy the conditions* (2).

Notice that H_1 is a reflexive idempotent g-inverse of H (that is, $H_1HH_1 = H_1$, $HH_1H = H$, and $H_1^2 = H_1$).

LEMMA 1. *Let H be an idempotent matrix and H^* be a g-inverse of H. Then, $H = H^*$ is a Hermitian idempotent matrix.*

Proof. This follows from H, HH^*, H^*H, and H^* being idempotent and $(H - HH^*)(H^* - HH^*) = (H - HH^*)(H - HH^*)^* = 0$. ■

NOTE 1. Lemma 1 can be rewritten in the following way: Let H be a non-Hermitian idempotent matrix. Then H^* cannot be a g-inverse of H.

LEMMA 2. *Let H be an idempotent matrix and H^-H be Hermitian idempotent. Then*

$$H^- = H_1 + H_2,$$

with $H_1 = H^*(HH^*)^-H[I + Z_2(I-H)]$ and $H_2 = (I-H)Z_3(I-H)$, where Z_2 and Z_3 are arbitrary.

Proof. Notice that from (1), we get that

$$H^-H = (I + (I-H)Z_1)H = H^*(I + Z^*(I-H)^*)$$

is Hermitian, so that $[I + (I-H)Z_1]HH^* = H^*$, or $(I + (I-H)Z_1)H =$

$H^*(HH^*)^- H$. Hence, $H^- = H_1 + H_2$ gives

$$H_1 = H^*(HH^*)^- H[I + Z_2(I - H)] \quad \text{and} \quad H_2 = (I - H)Z_3(I - H).$$

■

NOTE 2. If H and H^- are idempotent and $H^- H$ is Hermitian, then

$$HZ_2(I - H)H^* = 0,$$

$$HZ_2(I - H)Z_3(I - H) = 0,$$

$$(I - H)Z_3(I - H)H^* = 0.$$

These give $HZ_2(I - H) = HW_2R$ with $R = \{I - (I - H)H^*(H^* - HH^*)^-\}(I - H)$, and $H_2 = TW_3R$ is idempotent, where W_2 and W_3 are arbitrary and $T = (I - H)\{I - (HW_2R)^-(HW_2R)\}$.

Similarly, we can establish

LEMMA 3. *Let H be an idempotent matrix and HH^- be Hermitian. Then*

$$H^- = H_1 + H_2,$$

$$H_1 = \{I + (I - H)Z_1\}H(H^*H)^- H^* \quad \text{and} \quad H_2 = (I - H)Z_3(I - H),$$

where Z_1 and Z_3 are arbitrary.

Further, if H^- is idempotent, then

$$(I - H)Z_1H = R_1W_1H,$$

$$R_1 = (I - H)\{I - (H^* - H^*H)^-(H^* - H^*H)\},$$

and

$$H_2 = R_1W_3T, \qquad T = \{I - (R_1W_1H)(R_1W_1H)^-\}(I - H),$$

where W_1 and W_3 are arbitrary matrices such that H_2 is idempotent.

LEMMA 4. *Let H be an idempotent matrix, and let HH^- and H^-H be Hermitian idempotent. Then $H^- = H_1 + H_2$,*

$$H^\dagger = H_1 = H^*(HH^*)^- H(H^*H)^- H^* \quad \text{and} \quad H_2 = (I-H)Z_3(I-H).$$

Further, if H^- is idempotent, then H must be Hermitian and

$$H^- = H + H_2, \qquad \textit{where} \quad H_2 \textit{ is idempotent}.$$

Proof. HH^- and H^-H are Hermitian $\Rightarrow$ HH_1 and H_1H are Hermitian. By Lemmas 2 and 3, we get $H_1 = H^*(HH^*)^- H(H^*H)^- H^* = H^\dagger$.

Now, for H^- to be idempotent, we must have H_1, H_2 idempotent with $H_1H_2 = H_2H_1 = 0$. Now, $H_1^2 = H_1 \Rightarrow H_1 = H^*$. Further,

$$\begin{aligned}(H^* - H^*H)(H^* - H^*H)^* &= (H^* - H^*H)(H - H^*H)\\ &= H^*H - H^*H^2 - H^{*2}H + (H^*H)^2 = 0\end{aligned}$$

because $HH^*H = H$ and H^* are idempotent. Hence $H = H^*H = H^*$. This proves the lemma. ■

NOTE 3. Let H be a non-Hermitian idempotent matrix. Then there does not exist an idempotent g-inverse H^- of H such that H^-H and HH are both Hermitian idempotent.

NOTE 4. If $H^- = H_1 + H_2$ is defined in (1) and H is idempotent, then H^- is reflexive g-inverse of H iff $H_2 = 0$. Hence, $H^\dagger$ is idempotent iff H is Hermitian idempotent. If H is not a Hermitian matrix and H is idempotent, then $H^\dagger$ cannot be idempotent.

3. CONDITIONS FOR AN IDEMPOTENT MATRIX

Khatri [1] has established the following:

LEMMA 5. *Let $H_1, H_2, \ldots, H_k$ and $H = \sum_{i=1}^k H_i$ be $n \times n$ matrices. Now consider the conditions*

(a) $H_i^2 = H_i$ *for all* i,
(b) $H_iH_j = 0$ *for all* $i \neq j$,
(c) $H^2 = H$,
(d) $\rho(H) = \sum_{i=1}^k \rho(H_i)$, *and*
(e) *either* $\rho(H_i^2) = \rho(H_i)$ *or* $\operatorname{tr} H_i = \rho(H_i)$ *for all* i.

Then

(i) (a) *and* (b) $\Rightarrow$ *all conditions,*
(ii) (a) *and* (c) $\Rightarrow$ *all conditions,*
(iii) (b), (c) *and* (e) $\Rightarrow$ *all conditions, and*
(iv) (c) *and* (d) $\Rightarrow$ *all conditions.*

Khatri [2] considered the situation (a) and (d) for Hermitian matrices $H_1, H_2, \dots, H_k$. In this case, he established

LEMMA 6. *Let* $H_1, H_2, \dots, H_k$ *be Hermitian idempotent matrices and* $\sum_{i=1}^{k} \rho(H_i) = \rho(H)$ *with* $H = \sum_{i=1}^{k} H_i$. *Then the product of the nonzero eigenvalues of* H *is* $\Pi\lambda_{\mathrm{NE}}(H) \leqslant 1$, *and the equality holds iff* $H^2 = H$ *or* $H_i H_j = 0$ *for all* $i \neq j$.

In this note, we try to delete the condition of Hermitian matrices. For this, we establish

THEOREM 2. *Let* H *be an* $n \times n$ *matrix such that* $\rho(H) = \operatorname{tr} H$. *Then* $\operatorname{tr}(H^2 H^\dagger H^*) \geqslant \rho(H)$, *and the equality holds iff* $H^2 = H$.

Proof. Let H be an $n \times n$ matrix of rank t. Then we can write

$$H = BC \quad \text{and} \quad H^\dagger = C^*(CC^*)^{-1}(B^*B)^{-1}B^* = C^\dagger B^\dagger,$$

where C and B are $t \times n$ and $n \times t$ matrices of rank t. Let $Y = I_t - CB$. Then $\operatorname{tr} Y = t - \operatorname{tr} CB = t - \operatorname{tr} H = 0$ and so $\operatorname{tr} Y^* = 0$. Now, $HH^\dagger = B(B^*B)^{-1}B^*$ and so

$$\begin{aligned} H^2 H^\dagger H^* &= B(I - Y)(B^*B)^{-1}(I - Y)^* B^* \\ &= B(B^*B)^{-1}B^* - BY(B^*B)^{-1}B^* - B(B^*B)^{-1}Y^*B^* \\ &\quad + BY(B^*B)^{-1}(BY)^*. \end{aligned}$$

Hence, on account of $BY(B^*B)^{-1}(BY)^*$ being positive semidefinite, we get

$$\operatorname{tr}(H^2 H^\dagger H^*) = t + \operatorname{tr}\{BY(B^*B)^{-1}(BY)^*\} \geqslant t,$$

and the equality holds iff $BY = 0$ or $B = HB$ or $H^2 = H$. This proves the required result. ∎

NOTE 5. Let H be a Hermitian matrix. Then $H^2 H^{\dagger} H^* = H^2$ and if $\rho(H) = \operatorname{tr} H$, then $\operatorname{tr} H^2 \geqslant \rho(H)$, and the equality holds iff H is idempotent.

THEOREM 3.

(a) *Let H be an $n \times n$ matrix such that the nonzero eigenvalues of H are real and* $\operatorname{tr} H = \rho(H) = \rho(H^2)$. *Then* $\operatorname{tr} H^2 \geqslant \rho(H)$, *and the equality holds iff* $\lambda_{\mathrm{NE}}(H) = 1$.

(b) *Let H be an $n \times n$ matrix such that the nonzero eigenvalues of H are real and positive, and* $\operatorname{tr} H = \rho(H) = \rho(H^2)$. *Then* $\prod \lambda_{\mathrm{NE}}(H) \leqslant 1$, *and the equality holds iff* $\lambda_{\mathrm{NE}}(H) = 1$.

Proof. (a): Let $\lambda_1, \lambda_2, \ldots, \lambda_t$ be the nonzero eigenvalues of H, with $t = \rho(H)$ on account of $\rho(H) = \rho(H^2)$. Now $\rho(H) = \operatorname{tr} H$ implies

$$\bar{\lambda} = \frac{1}{t} \sum_{i=1}^{t} \lambda_i = 1 \quad \text{and} \quad \sum_{i=1}^{t} (\lambda_i - \bar{\lambda})^2 \geqslant 0 \quad \Leftrightarrow \quad \sum_{i=1}^{t} \lambda_i^2 \geqslant t,$$

and the equality holds iff $\lambda_i = 1$ for all $i = 1, 2, \ldots, t$.

(b): Now, since the λ_i's are positive, we have

$$\left(\prod \lambda_{\mathrm{NE}}(H)\right)^{1/t} = \left(\prod_{i=1}^{t} \lambda_i\right)^{1/t} \leqslant \bar{\lambda} = 1,$$

and the equality holds iff $\lambda_i = 1$ for all $i = 1, 2, \ldots, k$. This proves Theorem 3. ■

NOTE 6. Notice that the nonzero eigenvalues of H can be one, but H cannot be idempotent even when the conditions $\operatorname{tr} H = \rho(H) = \rho(H^2)$ are satisfied. For example, consider

$$H = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

Then H is nonsingular and $\lambda_{\mathrm{NE}}(H) = 1$ appears twice. Notice that H is not idempotent. In this situation, H is not semisimple. Thus, in Theorem 3, if we add the condition that H is semisimple, then we get the idempotency of H if $\lambda_{\mathrm{NE}}(H) = 1$.

NOTE 7. Let $H_1, H_2, \dots, H_k$ be idempotent matrices, and let $H = \sum_{i=1}^{k} H_i$. Then H need not be semisimple even if $\rho(H) = \sum_{i=1}^{k} \rho(H_i)$.

(a) For example, let

$$H_1 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cdot & 1 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} \cdot & 1 & 1 \\ \cdot & 1 & 1 \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

Then

$$\begin{pmatrix} \cdot & 1 & 1 \\ \cdot & 1 & 1 \\ 1 & \cdot & 1 \end{pmatrix} = H,$$

and $\rho(H^2) = 1$, while $\rho(H) = \operatorname{tr} H = 2 = \rho(H_1) + \rho(H_2)$.

(b) Let

$$H_1 = \begin{pmatrix} \cdot & 1 \\ \cdot & 1 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} \cdot & \cdot \\ -1 & 1 \end{pmatrix}.$$

Then,

$$H = \begin{pmatrix} \cdot & 1 \\ -1 & 2 \end{pmatrix}$$

is not a semisimple matrix even though

$$\rho(H) = \rho(H_1) + \rho(H_2) = \rho(H^2) = \operatorname{tr} H.$$

NOTE 8. For getting an idempotent matrix, Theorem 3 can be rewritten as

THEOREM 3′.

(a) *Let H be a semisimple matrix with real eigenvalues and $\rho(H) = \operatorname{tr} H$. Then $\operatorname{tr} H^2 \geqslant \rho(H)$, and the equality holds iff H is idempotent.*

(b) *Let H be a semisimple matrix with nonnegative eigenvalues and $\rho(H) = \operatorname{tr} H$. Then $\prod \lambda_{\mathrm{NE}}(H) \leqslant 1$, and the equality holds iff H is idempotent.*

Notice that Theorem 3′(b) generalizes Lemma 6 of Khatri [2] in connection with Lavoie's inequality.

NOTE 9. Let $A' = (A'_1, A'_2, \ldots, A'_k)$ and $B = (B_1, B_2, \ldots, B_k)$ be such that B_i is a g-inverse of A_i (or $H_i = B_i A_i$ is an idempotent matrix of rank A_i) for all $i = 1, 2, \ldots, k$. Let $BA = H$, and assume that H is a semisimple with nonnegative eigenvalues and $\rho(H) = \sum_{i=1}^k \rho(A_i)$ [or $\rho(H) = \sum_{i=1}^k \rho(H_i)$]. Then $\prod \lambda_{\mathrm{NE}}(H) \leqslant 1$, and the equality holds iff $H^2 = H$ or $A_i B_j A_j = 0$ for all $i \neq j$.

If $H = BA$ is semisimple with real eigenvalues and $\rho(H) = \sum_{i=1}^k \rho(A_i)$, then $\operatorname{tr} H^2 \geqslant \rho(H)$ and the equality holds iff $H^2 = H$ or $A_i B_j A_j = 0$ for all $i \neq j$.

If $H = BA$ and $\rho(H) = \sum_{i=1}^k \rho(A_i)$, then $\operatorname{tr}(H^2 H^\dagger H^*) \geqslant \rho(H)$ and the equality holds iff $H^2 = H$ or $A_i B_j A_j = 0$ for all $i \neq j$. Further, if Rank $B_i =$ Rank A_i for all $i = 1, 2, \ldots, k$, then $A_i B_j A_j = 0 \Rightarrow A_i B_j = 0$ for all $i \neq j$, and AB is a diagonal idempotent matrix.

NOTE 10. Let A and B be $n \times n$ square matrices such that

$$\rho\begin{pmatrix} A \\ B \end{pmatrix} = \rho(A, B) = \rho(B).$$

This condition is equivalent to $A = AB^- B = BB^- A$ for any g-inverse B^- of B. Let $H = AB^-$ and $H_0 = AB^\dagger$. Then $\rho(H) = \rho(A)$, and for any nonzero λ

$$|H - \lambda I| = |BB^\dagger AB^- - \lambda I| = |(AB^- B)B^\dagger - \lambda I| = |AB^\dagger - \lambda I| = |H_0 - \lambda I|,$$

and hence the eigenvalues of H are the same as those of H_0, so the eigenvalues of H are invariant under any choice of g-inverse B^- of B. In particular, if B is idempotent, then the eigenvalues of H are the same as those of A.

Now, we shall give some sufficient conditions on A and B so that $H = AB^-$ is semisimple with real eigenvalues.

CONDITION i. Let

$$\rho(A, B) = \rho\begin{pmatrix} A \\ B \end{pmatrix} = \rho(B)$$

and B be idempotent. Let $B = B_1 B_2$ with $B_2 B_1 = I_t$ and $t = \rho(B)$. Let

$$(B_1, B_3) = B_{(1)} \quad \text{and} \quad B_{(2)} = \begin{pmatrix} B_2 \\ B_4 \end{pmatrix}$$

be nonsingular matrices such that $B_{(2)} B_{(1)} = I_n$. Then $B_{(2)} = B_{(1)}^{-1}$ and $B_4 B_1 = 0$.

Now

$$B_{(1)}^{-1}HB_{(1)} = B_{(1)}^{-1}AB_2^*\left(B_2B_2^*\right)^{-1}B_2B^-B_{(1)}$$
$$= \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_t & C \\ 0 & 0 \end{pmatrix},$$

where $A_0 = B_2AB_2^*(B_2B_2^*)^{-1}$ and $C = B_2B^-B_3$. If A is a Hermitian matrix, then there exists a nonsingular matrix R such that $A_0 = RD_\lambda R^{-1}$, where D_λ is a diagonal matrix with real diagonal elements. Let us denote

$$R_0 = \begin{pmatrix} R & -C \\ 0 & I_{n-t} \end{pmatrix} \quad \text{and} \quad R_0^{-1} = \begin{pmatrix} R^{-1} & R^{-1}C \\ 0 & I_{n-t} \end{pmatrix}.$$

Then

$$B_{(1)}^{-1}HB_{(1)} = R_0\begin{pmatrix} D_\lambda & 0 \\ 0 & 0 \end{pmatrix}R_0^{-1},$$

and hence H is semisimpie with real eigenvalues. Notice that the eigenvalues of H are the eigenvalues of $AB_2^*(B_2B_2^*)^{-1}B_2$ $(= AB^+B = A)$.

Condition ii. Let A and B be Hermitian matrices such that $\rho(A, B) = \rho(B)$. Then $\rho(H) = \rho(AB^-) = \rho(A)$.

(a) Let A be nonnegative definite of rank r. Then $\rho(H) = r$. Let us denote $A = YY^*$ and $I - AA^\dagger = ZZ^*$, where $Y^*Z = 0$, $Z^*Z = I_{n-r}$, and Y is an $n \times r$ matrix of rank r. Let $(Y, Z) = Y_0$. Then Y_0 is nonsingular and

$$Y_0^{-1} = \begin{pmatrix} (Y^*Y)^{-1}Y^* \\ Z^* \end{pmatrix}.$$

Now

$$Y_0^{-1}HY_0 = \begin{pmatrix} Y^*B^-Y & Y^*B^-Z \\ 0 & 0 \end{pmatrix}.$$

and $r = \rho(H) = \rho(YY^*B^-) = \rho(Y^*B^-)$. Now, assume that $\rho(H^2) = \rho(H) = r$.

Then Y^*B^-Y is nonsingular and Hermitian for any B^-. Under these conditions,

$$Y_0^{-1}HY_0 = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & C \\ 0 & 0 \end{pmatrix}$$

with

$$A_0 = Y^*B^-Y \quad \text{and} \quad C = A_0^{-1}Y^*B^-Z.$$

Then, using arguments similar to those given for Condition i, we see that H is semisimple with real eigenvalues.

(b) Let B be nonnegative definite of rank t. Let $B = YY^*$ and $I - BB^\dagger = ZZ^*$, where $Y^*Z = 0$, $Z^*Z = I_{n-t}$, and Y is an $n \times t$ matrix of rank t. Let $Y_0 = (Y, Z)$. Then Y_0 is nonsingular,

$$Y_0^{-1} = \begin{pmatrix} (Y^*Y)^{-1}Y^* \\ Z^* \end{pmatrix},$$

and

$$Y_0^{-1}HY_0 = \left(Y_0^{-1}H_0Y_0\right)\left(Y_0^{-1}BB^-Y_0\right) = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_t & C \\ 0 & 0 \end{pmatrix},$$

where $A_0 = (Y^*Y)^{-1}Y^*AY(Y^*Y)^{-1}$ and $C = Y^*BZ$. Then, arguing as for Condition i, we see that H is semisimple with real eigenvalues. For this situation, one can refer to Theorem 6.2.2 and Theorem 6.4.2 (ii) of Rao and Mitra [3].

The above results can be summarized as follows:

THEOREM 5. *Let A be a Hermitian matrix and B be an idempotent matrix such that*

$$\rho(A, B) = \rho\begin{pmatrix} A \\ B \end{pmatrix} = \rho(B).$$

Then H is semisimple with real eigenvalues which are the same as those of A. If $\rho(A) = \rho(H) = \operatorname{tr} H = \operatorname{tr} A$, then $\operatorname{tr} H^2 = \operatorname{tr} A^2 \geqslant \rho(A)$, and the equality holds iff H is idempotent. Further, if A is nonnegative definite, then $\prod \lambda_{\mathrm{NE}}(H) = \prod \lambda_{\mathrm{NE}}(A) \leqslant 1$, and the equality holds iff H is idempotent.

(a) *If A is nonnegative definite and* $\rho(H^2) = \rho(H)$, *then H is semisimple with real eigenvalues. Further, if* $\rho(A) = \operatorname{tr} H$, *then* $\operatorname{tr} H^2 \geqslant \rho(H)$ *and the equality holds iff H is idempotent.*

(b) *If B is nonnegative definite, then H is semisimple with real eigenvalues. Further, if* $\rho(A) = \operatorname{tr} H$, *then* $\operatorname{tr} H^2 \geqslant \rho(H)$, *and the equality holds iff H is idempotent. Moreover, if A is nonnegative definite and* $\rho(A) = \operatorname{tr} H$, *then* $\prod \lambda_{\mathrm{NE}}(H) \leqslant 1$, *and the equality holds iff H is idempotent.*

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REFERENCES

1 C. G. Khatri, Some results for the singular normal multivariate regression models, *Sankhyā Ser. A* 30:267–280 (1968).

2 C. G. Khatri, A generalization of Lavoie's inequality concerning the sum of idempotent matrices, *Linear Algebra Appl.* 54:97–108 (1983).

3 C. R. Rao and S. K. Mitra, *Generalized Inverse of Matrices and its Applications*, Wiley, New York, 1971.

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