Computers and Mathematics with Applications 62 (2011) 3591–3602

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Some series identities involving the generalized Apostol type and related polynomials

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ARTICLE INFO

Article history:
Received 22 May 2011
Received in revised form 2 September 2011
Accepted 6 September 2011

Keywords:
Generalized Apostol type polynomials
Generalized Apostol–Euler polynomials
Generalized Apostol–Bernoulli polynomials
Genocchi polynomials of higher order
Generalized sum of integer powers
Generalized alternating sum

ABSTRACT

A unification (and generalization) of various Apostol type polynomials was introduced and investigated recently by Luo and Srivastava [Q.-M. Luo, H.M. Srivastava, Some generalizations of the Apostol–Genocchi polynomials and the Stirling numbers of the second kind, Appl. Math. Comput. 217 (2011) 5702–5728]. In this paper, we prove several symmetry identities for these generalized Apostol type polynomials by using their generating functions. As special cases and consequences of our results, we obtain the corresponding symmetry identities for the Apostol–Euler polynomials of higher order, the Apostol–Bernoulli polynomials of higher order and the Apostol–Genocchi polynomials of higher order, and also for another family of generalized Apostol type polynomials which were investigated systematically by Ozden et al. [H. Ozden, Y. Simsek, H.M. Srivastava, A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials, Comput. Math. Appl. 60 (2010) 2779–2787]. We also derive several relations between the Apostol type polynomials, the generalized sum of integer powers and the generalized alternating sum. It is shown how each of these results would extend the corresponding known identities.

1. Introduction, definitions and motivation

The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ of (real or complex) order α, are usually defined by means of the following generating functions (see, for details, [1, pp. 532–533] and [2, p. 61 *et seq.*]; see also [3] and the references cited in each of these earlier references):

$$\left(\frac{z}{e^z-1}\right)^\alpha \cdot e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!} \quad (|z|<2\pi;\ 1^\alpha := 1), \tag{1}$$

$$\left(\frac{2}{e^z+1}\right)^\alpha \cdot e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{z^n}{n!} \quad (|z|<\pi;\ 1^\alpha := 1) \tag{2}$$

and

$$\left(\frac{2z}{e^z+1}\right)^\alpha \cdot e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{z^n}{n!} \quad (|z|<\pi;\ 1^\alpha := 1), \tag{3}$$

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2011.09.010

so that, obviously, the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are given, respectively, by

$$B_n(x) := B_n^{(1)}(x), \qquad E_n(x) := E_n^{(1)}(x) \quad \text{and} \quad G_n(x) := G_n^{(1)}(x) \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{4}$$

For the classical Bernoulli numbers B_n of order n, the classical Euler numbers E_n of order n and the classical Genocchi numbers G_n of order n, we have

$$B_n := B_n(0) = B_n^{(1)}(0), \qquad E_n := E_n(0) = E_n^{(1)}(0) \quad \text{and} \quad G_n := G_n(0) = G_n^{(1)}(0), \tag{5}$$

respectively.

Some interesting analogues of the classical Bernoulli polynomials and numbers were first investigated by Apostol (see [4, p. 165, Eq. (3.1)]) and (more recently) by Srivastava (see [5, pp. 83–84]). We begin by recalling here Apostol's definitions as follows.

Definition 1 (*Apostol [4]; See also [5]*)**.** The Apostol–Bernoulli polynomials $\mathcal{B}_n(x; \lambda)(\lambda \in \mathbb{C})$ are defined by means of the following generating function:

$$\frac{ze^{xz}}{\lambda e^z - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda)\frac{z^n}{n!} \quad (|z| < 2\pi \text{ when } \lambda = 1;\ |z| < |\log \lambda| \text{ when } \lambda \neq 1) \tag{6}$$

with, of course,

$$B_n(x) = \mathcal{B}_n(x; 1) \quad \text{and} \quad \mathcal{B}_n(\lambda) := \mathcal{B}_n(0; \lambda), \tag{7}$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol–Bernoulli numbers.

Subsequently, Luo and Srivastava [6] further extended the Apostol–Bernoulli polynomials as the Apostol–Bernoulli polynomials of order α defined below.

Definition 2 (*Luo and Srivastava [6]*)**.** The Apostol–Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x; \lambda)$ $(\lambda \in \mathbb{C})$ of (real or complex) order α are defined by means of the following generating function:

$$\left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x; \lambda)\frac{z^n}{n!} \quad (|z| < 2\pi \text{ when } \lambda = 1;\ |z| < |\log \lambda| \text{ when } \lambda \neq 1;\ 1^{\alpha} := 1) \tag{8}$$

with, of course,

$$B_n^{(\alpha)}(x) = \mathcal{B}_n^{(\alpha)}(x; 1) \quad \text{and} \quad \mathcal{B}_n^{(\alpha)}(\lambda) := \mathcal{B}_n^{(\alpha)}(0; \lambda), \tag{9}$$

where $\mathcal{B}_n^{(\alpha)}(\lambda)$ denotes the Apostol–Bernoulli numbers of order α.

On the other hand, Luo [7] gave an analogous extension of the generalized Euler polynomials as the Apostol–Euler polynomials of order α defined as follows.

Definition 3 (*Luo [7]*)**.** The Apostol–Euler polynomials $\mathcal{E}_n^{(\alpha)}(x; \lambda)$ $(\lambda \in \mathbb{C})$ of (real or complex) order α are defined by means of the following generating function:

$$\left(\frac{2}{\lambda e^z + 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x; \lambda)\frac{z^n}{n!} \quad \left(|z| < |\log(-\lambda)|;\ 1^{\alpha} := 1\right) \tag{10}$$

with, of course,

$$E_n^{(\alpha)}(x) = \mathcal{E}_n^{(\alpha)}(x; 1) \quad \text{and} \quad \mathcal{E}_n^{(\alpha)}(\lambda) := \mathcal{E}_n^{(\alpha)}(0; \lambda), \tag{11}$$

where $\mathcal{E}_n^{(\alpha)}(\lambda)$ denotes the Apostol–Euler numbers of order α.

On the subject of the Genocchi polynomials $G_n(x)$ and their various extensions, a remarkably large number of investigations have appeared in the literature (see, for example, [8–18]). Moreover, Luo (see [15–17]) introduced and investigated the Apostol–Genocchi polynomials of (real or complex) order α, which are defined as follows.

Definition 4. The Apostol–Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x; \lambda)$ $(\lambda \in \mathbb{C})$ of (real or complex) order α are defined by means of the following generating function:

$$\left(\frac{2z}{\lambda e^z + 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x; \lambda)\frac{z^n}{n!} \quad \left(|z| < |\log(-\lambda)|;\ 1^{\alpha} := 1\right) \tag{12}$$

with, of course,

$$G_n^{(\alpha)}(x) = \mathcal{G}_n^{(\alpha)}(x; 1) \quad \text{and} \quad \mathcal{G}_n^{(\alpha)}(\lambda) := \mathcal{G}_n^{(\alpha)}(0; \lambda) \tag{13}$$

and

$$\mathcal{G}_n(x;\lambda) := \mathcal{G}_n^{(1)}(x;\lambda) \quad \text{and} \quad \mathcal{G}_n(\lambda) := \mathcal{G}_n^{(1)}(\lambda), \tag{14}$$

where $\mathcal{G}_n(\lambda)$, $\mathcal{G}_n^{(\alpha)}(\lambda)$ and $\mathcal{G}_n(x;\lambda)$ denote the so-called Apostol–Genocchi numbers, the Apostol–Genocchi numbers of order α and the Apostol–Genocchi polynomials, respectively.

Ozden et al. [19] investigated the following unification (and generalization) of the generating functions of the three families of Apostol type polynomials (see also [20, p. 5726, Eq. (170)]):

$$\frac{2^{1-\kappa} z^\kappa e^{xz}}{\beta^b e^z - a^b} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}(x;\kappa,a,b)\frac{z^n}{n!}$$

$$\left(|z| < 2\pi \text{ when } \beta = a;\ |z| < \left|b\log\left(\frac{\beta}{a}\right)\right| \text{ when } \beta \neq a;\ 1^\alpha := 1;\ \kappa,\beta \in \mathbb{C};\ a,b \in \mathbb{C}\setminus\{0\}\right). \tag{15}$$

Subsequently, Luo and Srivastava [20] introduced the following unification (and generalization) of the above-mentioned three families of the generalized Apostol type polynomials.

Definition 5 (*Luo and Srivastava [20, p. 5726, Definition 6]*)**.** The generalized Apostol type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)$ of (real or complex) order α are defined by means of the following generating function:

$$\left(\frac{2^u z^v}{\lambda e^z + 1}\right)^\alpha \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)\frac{z^n}{n!}$$

$$\left(|z| < |\log(-\lambda)|;\ \alpha,\lambda,u,v \in \mathbb{C};\ 1^\alpha := 1\right), \tag{16}$$

so that, by comparing Definition 5 with Definitions 2 to 4, we have

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = (-1)^\alpha \mathcal{F}_n^{(\alpha)}(x;-\lambda;0,1), \tag{17}$$

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1,0) \tag{18}$$

and

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1,1), \tag{19}$$

respectively.

Thus, if we compare the generating functions (15) and (16), we readily find that

$$\mathcal{Y}_{n,\beta}(x;\kappa,a,b) = -\frac{1}{a^b}\mathcal{F}_n^{(1)}\left(x; -\left(\frac{\beta}{a}\right)^b; 1-\kappa; \kappa\right). \tag{20}$$

For each $k \in \mathbb{N}_0$, $S_k(n)$ defined by

$$S_k(n) = \sum_{j=0}^{n-1} j^k \quad (k \in \mathbb{N}_0;\ n \in \mathbb{N})$$

is called sum of integer powers (or, simply, the power sum). The exponential generating function for $S_k(n)$ is given by [21]

$$\sum_{k=0}^{\infty} S_k(n)\frac{t^k}{k!} = 1 + e^t + e^{2t} + \cdots + e^{(n-1)t} = \frac{e^{nt}-1}{e^t-1}. \tag{21}$$

We now define the generalized sum of integer powers as follows.

Definition 6. For any arbitrary real or complex parameter λ, the generalized sum of integer powers $\mathcal{S}_k(n;\lambda)$ is defined by the following generating function:

$$\sum_{k=0}^{\infty} \mathcal{S}_k(n;\lambda)\frac{t^k}{k!} = \frac{\lambda e^{nt}-1}{\lambda e^t - 1}. \tag{22}$$

By comparing the generating function (22) of Definition 6 with the generating function in (21), it is easily seen that

$$\mathcal{S}_k(n;1)=S_k(n)\quad(k\in\mathbb{N}_0;\ n\in\mathbb{N}).$$

For $k\in\mathbb{N}_0$ and $n\in\mathbb{N}$, $T_k(n)$ defined by

$$T_k(n)=\sum_{k=0}^{n-1}(-1)^k n^k\quad(k\in\mathbb{N}_0;\ n\in\mathbb{N})$$

is called the *alternating sum*. The exponential generating function for $T_k(n)$ is given by

$$\sum_{k=0}^{\infty}T_k(n)\frac{t^k}{k!}=\frac{1-(-1)^n e^{nt}}{1+e^t}. \tag{23}$$

A generalized alternating sum of order α is now defined as follows.

Definition 7. For any arbitrary real or complex parameter λ, the generalized alternating sum of order $\alpha\mathcal{T}_k^{(\alpha)}(n;\lambda)$ is defined by the following generating function:

$$\sum_{k=0}^{\infty}\mathcal{T}_k^{(\alpha)}(n;\lambda)\frac{t^k}{k!}=\left(\frac{1-\lambda(-1)^n e^{nt}}{1+\lambda e^t}\right)^{\alpha}. \tag{24}$$

It is easily observed from the generating functions (23) and (24) that

$$\mathcal{T}_k^{(1)}(n;1)=T_k(n)\quad(k\in\mathbb{N}_0;\ n\in\mathbb{N}).$$

In recent years, several authors obtained many interesting results involving various other relatives of the Bernoulli polynomials and Euler polynomials [22–26,19,5,27,18,28–30]. In this paper, we obtain a number of series identities involving the generalized Apostol type and related polynomials. In Section 2, we prove several symmetry identities for the generalized Apostol type polynomials by using the method of generating functions. In Section 3, we obtain relations between the Apostol type polynomials, the generalized sum of integer powers and the generalized alternating sum. The various results presented in this paper would extend the corresponding known identities (see, for example, [24,28–30]).

2. Symmetry identities for the generalized Apostol type polynomials

In this section, we prove several symmetry identities for the generalized Apostol type polynomials by using the method of generating functions. As special cases, we obtain the corresponding symmetry identities for the generalized Apostol–Euler polynomials, the generalized Apostol–Bernoulli polynomials, Apostol–Genocchi polynomials of higher order and another family of generalized Apostol type polynomials defined by the generating function (15). These results provide extensions of some known identities (see, for details, [24,28–30]).

Theorem 1. *For $m\in\mathbb{N}_0$, $a,b\in\mathbb{N}$, $\lambda\in\mathbb{C}$ and $\alpha\geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m}\binom{m}{k}a^k b^{v\alpha+m-k}\mathcal{F}_k^{(\alpha)}(bx;\lambda;u;v)\mathcal{T}_{m-k}^{(\alpha)}(a;\lambda)=\sum_{k=0}^{m}\binom{m}{k}b^k a^{v\alpha+m-k}\mathcal{F}_k^{(\alpha)}(ax;\lambda;u;v)\mathcal{T}_{m-k}^{(\alpha)}(b;\lambda)$$
$$\left(m\in\mathbb{N}_0;\ a,b\in\mathbb{N};\ \lambda\in\mathbb{C};\ \alpha\geqq 1\right). \tag{25}$$

Proof. Let the function $f(t)$ be given by

$$f(t)=\frac{t^{v\alpha}\left[1-\lambda(-1)^a e^{abt}\right]^{\alpha}e^{abxt}}{(\lambda e^{at}+1)^{\alpha}(\lambda e^{bt}+1)^{\alpha}}. \tag{26}$$

We first use (16) and (24) to expand the function $f(t)$ as follows:

$$\begin{aligned}f(t)&=\frac{1}{(2^u a^v)^{\alpha}}\cdot\left(\frac{2^u(at)^v}{\lambda e^{at}+1}\right)^{\alpha}\cdot e^{abxt}\cdot\left(\frac{1-\lambda(-1)^a e^{abt}}{\lambda e^{bt}+1}\right)^{\alpha}\\&=\frac{1}{(2^u a^v)^{\alpha}}\left(\sum_{m=0}^{\infty}\mathcal{F}_m^{(\alpha)}(bx;\lambda;u;v)\frac{(at)^m}{m!}\right)\left(\sum_{m=0}^{\infty}\mathcal{T}_m^{(\alpha)}(a;\lambda)\frac{(bt)^m}{m!}\right)\\&=\frac{1}{(2^u a^v)^{\alpha}}\sum_{m=0}^{\infty}\left[\sum_{k=0}^{m}\binom{m}{k}a^k b^{m-k}\mathcal{F}_k^{(\alpha)}(bx;\lambda;u;v)\mathcal{T}_{m-k}^{(\alpha)}(a;\lambda)\right]\frac{t^m}{m!}.\end{aligned} \tag{27}$$

We now observe that, if a and b have the same parity, then the function $f(t)$ given by (26) is symmetric in a and b. Therefore, we may also expand $f(t)$ as follows:

$$f(t) = \frac{1}{(2^u b^v)^\alpha} \sum_{m=0}^{\infty} \left[\sum_{k=0}^{m} \binom{m}{k} b^k a^{m-k} \mathcal{F}_k^{(\alpha)}(ax; \lambda; u; v) \mathcal{T}_{m-k}^{(\alpha)}(b; \lambda) \right] \frac{t^m}{m!}. \tag{28}$$

Equating the coefficients of $\frac{t^m}{m!}$ on the right-hand sides of the last two Eqs. (27) and (28), we get the identity (25) asserted by Theorem 1. □

Upon setting

$$\lambda \mapsto -\lambda, \qquad u = 0 \quad \text{and} \quad v = 1$$

in Theorem 1, if we multiply both sides of the resulting equation by $(-1)^\alpha$, we obtain the following corollary.

Corollary 1. *For $m \in \mathbb{N}_0$, $a, b \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and $\alpha \geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} a^k b^{\alpha+m-k} \mathcal{B}_k^{(\alpha)}(bx; \lambda) \mathcal{T}_{m-k}^{(\alpha)}(a; -\lambda) = \sum_{k=0}^{m} \binom{m}{k} b^k a^{\alpha+m-k} \mathcal{B}_k^{(\alpha)}(ax; \lambda) \mathcal{T}_{m-k}^{(\alpha)}(b; -\lambda)$$
$$\big(m \in \mathbb{N}_0;\ a, b \in \mathbb{N};\ \lambda \in \mathbb{C};\ \alpha \geqq 1\big). \tag{29}$$

In its special case when $u - 1 = v = 0$, Theorem 1 yields the following result.

Corollary 2. *For $m \in \mathbb{N}_0$, $a, b \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and $\alpha \geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k} \mathcal{E}_k^{(\alpha)}(bx; \lambda) \mathcal{T}_{m-k}^{(\alpha)}(a; \lambda) = \sum_{k=0}^{m} \binom{m}{k} b^k a^{m-k} \mathcal{E}_k^{(\alpha)}(ax; \lambda) \mathcal{T}_{m-k}^{(\alpha)}(b; \lambda)$$
$$\big(m \in \mathbb{N}_0;\ a, b \in \mathbb{N}; \lambda \in \mathbb{C}; \alpha \geqq 1\big). \tag{30}$$

For $\alpha = \lambda = 1$ in (30), we get the following known result.

Corollary 3 (*See [29, Theorem 2.2, Eq. (18)]*)**.** *For $m \in \mathbb{N}_0$ and $a, b \in \mathbb{N}$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k} E_k(bx) T_{m-k}(a) = \sum_{k=0}^{m} \binom{m}{k} b^k a^{m-k} E_k(ax) T_{m-k}(b) \quad \big(m \in \mathbb{N}_0;\ a, b \in \mathbb{N}\big). \tag{31}$$

By setting $u = v = 1$ in Theorem 1, we are led to the following identity.

Corollary 4. *For $m \in \mathbb{N}_0$, $a, b \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and $\alpha \geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} a^k b^{\alpha+m-k} \mathcal{G}_k^{(\alpha)}(bx; \lambda) \mathcal{T}_{m-k}^{(\alpha)}(a; \lambda) = \sum_{k=0}^{m} \binom{m}{k} b^k a^{\alpha+m-k} \mathcal{G}_k^{(\alpha)}(ax; \lambda) \mathcal{T}_{m-k}^{(\alpha)}(b; \lambda)$$
$$\big(m \in \mathbb{N}_0;\ a, b \in \mathbb{N};\ \lambda \in \mathbb{C};\ \alpha \geqq 1\big). \tag{32}$$

A special case of the identity (32) when $x = 0$ yields the following result.

Corollary 5. *For $m \in \mathbb{N}_0$, $a, b \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and $\alpha \geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} a^k b^{\alpha+m-k} \mathcal{G}_k^{(\alpha)}(\lambda) \mathcal{T}_{m-k}^{(\alpha)}(a; \lambda) = \sum_{k=0}^{m} \binom{m}{k} b^k a^{\alpha+m-k} \mathcal{G}_k^{(\alpha)}(\lambda) \mathcal{T}_{m-k}^{(\alpha)}(b; \lambda)$$
$$\big(m \in \mathbb{N}_0;\ a, b \in \mathbb{N};\ \lambda \in \mathbb{C};\ \alpha \geqq 1\big). \tag{33}$$

By setting $\alpha = \lambda = 1$ in (33), we obtain the following known identity.

Corollary 6 (*See [29, Theorem 3.1,Eq. (22)]*). *For $m \in \mathbb{N}_0$ and $a, b \in \mathbb{N}$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k+1} G_k T_{m-k}(a) = \sum_{k=0}^{m} \binom{m}{k} b^k a^{m-k+1} G_k T_{m-k}(b) \quad \bigl(m \in \mathbb{N}_0;\ a, b \in \mathbb{N}\bigr). \tag{34}$$

Finally, in Theorem 1, if we first set

$$\alpha = 1, \qquad \lambda = -\left(\frac{\beta}{c}\right)^d, \qquad u = 1 - \kappa \quad \text{and} \quad v = \kappa$$

and then multiply both sides of the resulting equation by $-\frac{1}{c^d}$, we are led to the following corollary.

Corollary 7. *For $m \in \mathbb{N}_0$, $a, b \in \mathbb{N}$, $c, d \in \mathbb{C} \setminus \{0\}$ and $\kappa, \beta \in \mathbb{C}$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} a^{k-m} \mathcal{Y}_{k,\beta}(bx; \kappa, c, d) \mathcal{T}_{m-k}\left(a; -\left(\frac{\beta}{c}\right)^d\right) = \sum_{k=0}^{m} \binom{m}{k} b^{k-m} \mathcal{Y}_{k,\beta}(ax; \kappa, c, d) \mathcal{T}_{m-k}\left(b; -\left(\frac{\beta}{c}\right)^d\right)$$
$$\bigl(m \in \mathbb{N}_0;\ a, b \in \mathbb{N};\ c, d \in \mathbb{C} \setminus \{0\};\ \kappa, \beta \in \mathbb{C}\bigr). \tag{35}$$

Our next main result is asserted by Theorem 2.

Theorem 2. *For $m \in \mathbb{N}_0$, $a, b \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and $\alpha \geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} \sum_{\ell=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{\ell+j} a^k b^{m-k} \mathcal{F}_k^{(\alpha)}\left(bx + \frac{b}{a}\ell + j; \lambda; u; v\right) \mathcal{F}_{m-k}^{(\alpha)}(ay; \lambda; u; v)$$
$$= \sum_{k=0}^{m} \binom{m}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{\ell+j} b^k a^{m-k} \mathcal{F}_k^{(\alpha)}\left(ax + \frac{a}{b}\ell + j; \lambda; u; v\right) \mathcal{F}_{m-k}^{(\alpha)}(by; \lambda; u; v)$$
$$\bigl(m \in \mathbb{N}_0;\ a, b \in \mathbb{N};\ \lambda \in \mathbb{C};\ \alpha \geqq 1\bigr). \tag{36}$$

Proof. Let the function $g(t)$ be given by

$$g(t) = \frac{t^{2v\alpha} e^{abxt} e^{abyt} \left[1 - (-\lambda e^{bt})^a\right] \left[1 - (-\lambda e^{at})^b\right]}{(\lambda e^{at} + 1)^{\alpha+1} (\lambda e^{bt} + 1)^{\alpha+1}}, \tag{37}$$

which, in view of (16), can be expanded as follows:

$$\begin{aligned}
g(t) &= \frac{1}{(2^{2u} a^v b^v)^\alpha} \left(\frac{2^u (at)^v}{\lambda e^{at} + 1}\right)^\alpha e^{abxt} \left(\frac{1 - (-\lambda e^{bt})^a}{\lambda e^{bt} + 1}\right) \left(\frac{1 - (-\lambda e^{at})^b}{\lambda e^{at} + 1}\right) \left(\frac{2^u (bt)^v}{\lambda e^{bt} + 1}\right)^\alpha e^{abyt} \\
&= \frac{1}{(2^{2u} a^v b^v)^\alpha} \left(\frac{2^u (at)^v}{\lambda e^{at} + 1}\right)^\alpha e^{abxt} \left(\sum_{\ell=0}^{a-1} (-\lambda)^\ell e^{bt\ell}\right) \left(\sum_{j=0}^{b-1} (-\lambda)^j e^{atj}\right) \left(\frac{2^u (bt)^v}{\lambda e^{bt} + 1}\right)^\alpha e^{abyt} \\
&= \frac{1}{(2^{2u} a^v b^v)^\alpha} \left[\sum_{\ell=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{\ell+j} \left(\frac{2^u (at)^v}{\lambda e^{at} + 1}\right)^\alpha e^{\left(bx + \frac{b}{a}\ell + j\right)at}\right] \left(\frac{2^u (bt)^v}{\lambda e^{bt} + 1}\right)^\alpha e^{abyt} \\
&= \frac{1}{(2^{2u} a^v b^v)^\alpha} \left[\sum_{\ell=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{\ell+j} \sum_{m=0}^{\infty} \mathcal{F}_m^{(\alpha)}\left(bx + \frac{b}{a}\ell + j; \lambda; u; v\right) \frac{(at)^m}{m!}\right] \left(\sum_{m=0}^{\infty} \mathcal{F}_m^{(\alpha)}(ay; \lambda; u; v) \frac{(bt)^m}{m!}\right) \\
&= \frac{1}{(2^{2u} a^v b^v)^\alpha} \sum_{m=0}^{\infty} \left[\sum_{k=0}^{m} \binom{m}{k} \sum_{\ell=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{\ell+j} a^k b^{m-k} \mathcal{F}_k^{(\alpha)}\left(bx + \frac{b}{a}\ell + j; \lambda; u; v\right) \mathcal{F}_{m-k}^{(\alpha)}(ay; \lambda; u; v)\right] \frac{t^m}{m!}.
\end{aligned} \tag{38}$$

Since a and b have the same parity, the function $g(t)$ given by (37) is symmetric in a and b. Consequently, we may also expand $g(t)$ as follows:

$$g(t)=\frac{1}{\left(2^{2u}a^{v}b^{v}\right)^{\alpha}}\sum_{m=0}^{\infty}\left[\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{\ell+j}b^{k}a^{m-k}\mathcal{F}_{k}^{(\alpha)}\left(ax+\frac{a}{b}\ell+j;\lambda;u;v\right)\cdot\mathcal{F}_{m-k}^{(\alpha)}(by;\lambda;u;v)\right]\frac{t^{m}}{m!}. \tag{39}$$

Now, by equating the coefficients of $\frac{t^m}{m!}$ on the right-hand sides of these last two (38) and (39), we get the identity (36) asserted by Theorem 2. □

Just as in our derivations of the various corollaries and consequences of Theorem 1, we can deduce each of the following corollaries by appropriately specializing Theorem 2.

Corollary 8 (*See [30, Theorem 2.10, Eq. (23)]*). *For $m\in\mathbb{N}_0$, $a,b\in\mathbb{N}$, $\lambda\in\mathbb{C}$ and $\alpha\geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{a-1}\sum_{j=0}^{b-1}\lambda^{i+j}a^{k}b^{m-k}\mathcal{B}_{k}^{(\alpha)}\left(bx+\frac{b}{a}\ell+j;\lambda\right)\mathcal{B}_{m-k}^{(\alpha)}(ay;\lambda)$$
$$=\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{b-1}\sum_{j=0}^{a-1}\lambda^{\ell+j}b^{k}a^{m-k}\mathcal{B}_{k}^{(\alpha)}\left(ax+\frac{a}{b}\ell+j;\lambda\right)\mathcal{B}_{m-k}^{(\alpha)}(by;\lambda)$$
$$\left(m\in\mathbb{N}_0;\ a,b\in\mathbb{N};\ \lambda\in\mathbb{C};\ \alpha\geqq 1\right). \tag{40}$$

Corollary 9. *For $m\in\mathbb{N}_0$, $a,b\in\mathbb{N}$, and $\alpha\geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{a-1}\sum_{j=0}^{b-1}a^{k}b^{m-k}B_{k}^{(\alpha)}\left(bx+\frac{b}{a}\ell+j\right)B_{m-k}^{(\alpha)}(ay)$$
$$=\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{b-1}\sum_{j=0}^{a-1}b^{k}a^{m-k}B_{k}^{(\alpha)}\left(ax+\frac{a}{b}\ell+j\right)B_{m-k}^{(\alpha)}(by)\quad\left(m\in\mathbb{N}_0;\ a,b\in\mathbb{N};\ \alpha\geqq 1\right). \tag{41}$$

Corollary 10 (*See [28, Theorem 2, Eq. (12)]*). *For $m\in\mathbb{N}_0$, $a,b\in\mathbb{N}$, and $\alpha\geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{a-1}a^{k}b^{m-k}B_{k}^{(\alpha)}\left(bx+\frac{b}{a}\ell\right)B_{m-k}^{(\alpha)}(ay)$$
$$=\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{b-1}b^{k}a^{m-k}B_{k}^{(\alpha)}\left(ax+\frac{a}{b}\ell\right)B_{m-k}^{(\alpha)}(by)\quad\left(m\in\mathbb{N}_0;\ a,b\in\mathbb{N};\ \alpha\geqq 1\right). \tag{42}$$

Corollary 11. *For $m\in\mathbb{N}_0$, $a,b\in\mathbb{N}$, $\lambda\in\mathbb{C}$ and $\alpha\geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{\ell+j}a^{k}b^{m-k}\mathcal{E}_{k}^{(\alpha)}\left(bx+\frac{b}{a}\ell+j;\lambda\right)\mathcal{E}_{m-k}^{(\alpha)}(ay;\lambda)$$
$$=\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{\ell+j}b^{k}a^{m-k}\mathcal{E}_{k}^{(\alpha)}\left(ax+\frac{a}{b}\ell+j;\lambda\right)\mathcal{E}_{m-k}^{(\alpha)}(by;\lambda)$$
$$\left(m\in\mathbb{N}_0;\ a,b\in\mathbb{N};\ \lambda\in\mathbb{C};\ \alpha\geqq 1\right). \tag{43}$$

Corollary 12. *For $m\in\mathbb{N}_0$, $a,b\in\mathbb{N}$, and $\alpha\geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{a-1}\sum_{j=0}^{b-1}(-1)^{\ell+j}a^{k}b^{m-k}E_{k}^{(\alpha)}\left(bx+\frac{b}{a}\ell+j\right)E_{m-k}^{(\alpha)}(ay)$$
$$=\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{b-1}\sum_{j=0}^{a-1}(-1)^{\ell+j}b^{k}a^{m-k}E_{k}^{(\alpha)}\left(ax+\frac{a}{b}\ell+j\right)E_{m-k}^{(\alpha)}(by)\quad\left(m\in\mathbb{N}_0;\ a,b\in\mathbb{N};\ \alpha\geqq 1\right). \tag{44}$$

Corollary 13 (*See [29, Theorem 2.1, Eq. (17)]*)**.** *For $m \in \mathbb{N}_0$ and $a, b \in \mathbb{N}$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{\ell=0}^{a-1}(-1)^{\ell}a^kE_k\left(bx+\frac{b}{a}\ell\right)=\sum_{\ell=0}^{b-1}(-1)^{\ell}b^kE_k\left(ax+\frac{a}{b}\ell\right)\quad\left(m\in\mathbb{N}_0;\ a,b\in\mathbb{N}\right). \tag{45}$$

Corollary 14. *For $m \in \mathbb{N}_0, a, b \in \mathbb{N}, \lambda \in \mathbb{C}$ and $\alpha \geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}a^kb^{m-k}\mathcal{G}_k^{(\alpha)}\left(bx+\frac{b}{a}\ell+j;\lambda\right)\mathcal{G}_{m-k}^{(\alpha)}(ay;\lambda)\\&=\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{\ell+j}b^ka^{m-k}\mathcal{G}_k^{(\alpha)}\left(ax+\frac{a}{b}\ell+j;\lambda\right)\mathcal{G}_{m-k}^{(\alpha)}(by;\lambda)\\&\left(m\in\mathbb{N}_0;\ a,b\in\mathbb{N};\ \lambda\in\mathbb{C};\ \alpha\geqq 1\right).\end{aligned} \tag{46}$$

Corollary 15. *For $m \in \mathbb{N}_0, a, b \in \mathbb{N}, c, d \in \mathbb{C} \setminus \{0\}$ and $\kappa, \beta \in \mathbb{C}$, if a and b have the same parity, then the following identity holds true:*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{a-1}\sum_{j=0}^{b-1}\left(\frac{\beta}{c}\right)^{d(\ell+j)}a^kb^{m-k}\mathcal{Y}_k^{(\alpha)}\left(bx+\frac{b}{a}\ell+j;\kappa,c,d\right)\mathcal{Y}_{m-k}^{(\alpha)}(ay;\kappa,c,d)\\&=\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{b-1}\sum_{j=0}^{a-1}\left(\frac{\beta}{c}\right)^{d(\ell+j)}b^ka^{m-k}\mathcal{Y}_k^{(\alpha)}\left(ax+\frac{a}{b}\ell+j;\kappa,c,d\right)\mathcal{Y}_{m-k}^{(\alpha)}(by;\kappa,c,d)\\&\left(m\in\mathbb{N}_0;\ a,b\in\mathbb{N};\ c,d\in\mathbb{C}\setminus\{0\};\ \kappa,\beta\in\mathbb{C}\right).\end{aligned} \tag{47}$$

Theorem 3. *For $m \in \mathbb{N}_0, a, b \in \mathbb{N}, \lambda \in \mathbb{C}$ and $\alpha \geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{\ell+j}a^kb^{m-k}\mathcal{F}_k^{(\alpha)}\left(bx+\frac{b}{a}\ell;\lambda;u;v\right)\mathcal{F}_{m-k}^{(\alpha)}\left(ay+\frac{a}{b}j;\lambda;u;v\right)\\&=\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{\ell+j}b^ka^{m-k}\mathcal{F}_k^{(\alpha)}\left(ax+\frac{a}{b}i;\lambda;u;v\right)\mathcal{F}_{m-k}^{(\alpha)}\left(by+\frac{b}{a}j;\lambda;u;v\right)\\&\left(m\in\mathbb{N}_0;\ a,b\in\mathbb{N};\ \lambda\in\mathbb{C};\ \alpha\geqq 1\right).\end{aligned} \tag{48}$$

Proof. Our proof of Theorem 3 is much akin to that of each of Theorems 1 and 2. Here, in the proof of Theorem 3, we first make use of (16) in order to expand the function $g(t)$ defined by (37) and then apply the symmetry of $g(t)$ in a and b to produce a second expansion of $g(t)$. The details involved are fairly straightforward and we, therefore, leave them as an exercise for the interested reader. □

Each of the following corollaries and consequences of Theorem 3 would result just as those of Theorems 1 and 2.

Corollary 16 (*See [30, Theorem 2.7, Eq. (18)]*)**.** *For $m \in \mathbb{N}_0, a, b \in \mathbb{N}, \lambda \in \mathbb{C}$ and $\alpha \geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{a-1}\sum_{j=0}^{b-1}\lambda^{\ell+j}a^kb^{m-k}\mathcal{B}_k^{(\alpha)}\left(bx+\frac{b}{a}\ell;\lambda\right)\mathcal{B}_{m-k}^{(\alpha)}\left(ay+\frac{a}{b}j;\lambda\right)\\&=\sum_{k=0}^{m}\binom{m}{k}\sum_{\ell=0}^{b-1}\sum_{j=0}^{a-1}\lambda^{\ell+j}b^ka^{m-k}\mathcal{B}_k^{(\alpha)}\left(ax+\frac{a}{b}\ell;\lambda\right)\mathcal{B}_{m-k}^{(\alpha)}\left(by+\frac{b}{a}j;\lambda\right)\\&\left(m\in\mathbb{N}_0;\ a,b\in\mathbb{N};\ \lambda\in\mathbb{C};\ \alpha\geqq 1\right).\end{aligned} \tag{49}$$

Corollary 17. *For $m \in \mathbb{N}_0$, $a, b \in \mathbb{N}$ and $\alpha \geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} \sum_{\ell=0}^{a-1} \sum_{j=0}^{b-1} a^k b^{m-k} B_k^{(\alpha)} \left(bx + \frac{b}{a}\ell \right) B_{m-k}^{(\alpha)} \left(ay + \frac{a}{b} j \right)$$

$$= \sum_{k=0}^{m} \binom{m}{k} \sum_{\ell=0}^{b-1} \sum_{j=0}^{a-1} b^k a^{m-k} B_k^{(\alpha)} \left(ax + \frac{a}{b}\ell \right) B_{m-k}^{(\alpha)} \left(by + \frac{b}{a} j \right) \quad \left(m \in \mathbb{N}_0;\ a, b \in \mathbb{N};\ \alpha \geqq 1 \right). \tag{50}$$

Corollary 18. *For $m \in \mathbb{N}_0$, $a, b \in \mathbb{N}$ and $\alpha \geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} \sum_{\ell=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{\ell+j} a^k b^{m-k} \mathcal{E}_k^{(\alpha)} \left(bx + \frac{b}{a}\ell; \lambda \right) \mathcal{E}_{m-k}^{(\alpha)} \left(ay + \frac{a}{b} j; \lambda \right)$$

$$= \sum_{k=0}^{m} \binom{m}{k} \sum_{\ell=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{\ell+j} b^k a^{m-k} \mathcal{E}_k^{(\alpha)} \left(ax + \frac{a}{b}\ell; \lambda \right) \mathcal{E}_{m-k}^{(\alpha)} \left(by + \frac{b}{a} j; \lambda \right)$$

$$\left(m \in \mathbb{N}_0;\ a, b \in \mathbb{N};\ \lambda \in \mathbb{C};\ \alpha \geqq 1 \right). \tag{51}$$

Corollary 19. *For $m \in \mathbb{N}_0$, $a, b \in \mathbb{N}$ and $\alpha \geqq 1$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} \sum_{\ell=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{\ell+j} a^k b^{m-k} \mathcal{G}_k^{(\alpha)} \left(bx + \frac{b}{a}\ell; \lambda \right) \mathcal{G}_{m-k}^{(\alpha)} \left(ay + \frac{a}{b} j; \lambda \right)$$

$$= \sum_{k=0}^{m} \binom{m}{k} \sum_{\ell=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{\ell+j} b^k a^{m-k} \mathcal{G}_k^{(\alpha)} \left(ax + \frac{a}{b}\ell; \lambda \right) \mathcal{G}_{m-k}^{(\alpha)} \left(by + \frac{b}{a} j; \lambda \right)$$

$$\left(m \in \mathbb{N}_0;\ a, b \in \mathbb{N};\ \lambda \in \mathbb{C};\ \alpha \geqq 1 \right). \tag{52}$$

Corollary 20. *For $m \in \mathbb{N}_0$, $a, b \in \mathbb{N}$, $c, d \in \mathbb{C} \setminus \{0\}$ and $\kappa, \beta \in \mathbb{C}$, if a and b have the same parity, then the following identity holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} \sum_{\ell=0}^{a-1} \sum_{j=0}^{b-1} \left(\frac{\beta}{c} \right)^{d(\ell+j)} a^k b^{m-k} \mathcal{Y}_{k,\beta} \left(bx + \frac{b}{a}\ell; \kappa, c, d \right) \mathcal{Y}_{m-k,\beta} \left(ay + \frac{a}{b} j; \kappa, c, d \right)$$

$$= \sum_{k=0}^{m} \binom{m}{k} \sum_{\ell=0}^{b-1} \sum_{j=0}^{a-1} \left(\frac{\beta}{c} \right)^{d(\ell+j)} b^k a^{m-k} \mathcal{Y}_{k,\beta} \left(ax + \frac{a}{b}\ell; \kappa, c, d \right) \mathcal{Y}_{m-k,\beta} \left(by + \frac{b}{a} j; \kappa, c, d \right)$$

$$\left(m \in \mathbb{N}_0;\ a, b \in \mathbb{N};\ c, d \in \mathbb{C} \setminus \{0\};\ \kappa, \beta \in \mathbb{C} \right). \tag{53}$$

3. Miscellaneous results

In this section, we obtain several further relationships between the Apostol type polynomials, the generalized sum of integer powers and the generalized alternating sum.

Theorem 4. *For $m, n \in \mathbb{N}$, the following relationship holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} \mathcal{S}_{m-k}(n; \lambda) \mathcal{T}_k(n; \lambda) = \begin{cases} 2^m \mathcal{S}_m(n; \lambda^2) & (n \text{ odd}) \\ \displaystyle\sum_{k=0}^{m} \binom{m}{k} \mathcal{S}_{m-k}(n; \lambda) \mathcal{E}_k(\lambda) - 2^m \mathcal{S}_m(n; \lambda^2) & (n \text{ even}). \end{cases} \tag{54}$$

Proof. If $n \in \mathbb{N}$ is odd, then we write

$$\left(\sum_{m=0}^{\infty} \mathcal{S}_m(n;\lambda)\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty} \mathcal{T}_m(n;\lambda)\frac{t^m}{m!}\right) = \frac{\lambda e^{nt}-1}{\lambda e^t - 1}\cdot\frac{1-\lambda(-1)^n e^{nt}}{1+\lambda e^t} = \frac{\lambda^2 e^{2nt}-1}{\lambda^2 e^{2t}-1} = \sum_{m=0}^{\infty} 2^m \mathcal{S}_m(n;\lambda^2)\frac{t^m}{m!}. \tag{55}$$

By using the multiplication rule of formal power series on the left-hand side of in (55), and then comparing the coefficients of $\frac{t^m}{m!}$ in the two resulting equations, we get the first part of the desired result (54).

Similarly, if $n \in \mathbb{N}$ is even, by considering

$$\begin{aligned}\left(\sum_{m=0}^{\infty} \mathcal{S}_m(n;\lambda)\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty} \mathcal{T}_m(n;\lambda)\frac{t^m}{m!}\right) &= \frac{\lambda e^{nt}-1}{\lambda e^t - 1}\cdot\frac{1-\lambda e^{nt}}{1+\lambda e^t}\\ &= \frac{2}{\lambda e^t+1}\cdot\frac{\lambda e^{nt}-1}{\lambda e^t-1} - \frac{\lambda^2 e^{2nt}-1}{\lambda^2 e^{2t}-1}\\ &= \left(\sum_{m=0}^{\infty}\mathcal{E}_m(\lambda)\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}\mathcal{S}_m(n;\lambda)\frac{t^m}{m!}\right) - \left(\sum_{m=0}^{\infty}\mathcal{S}_m(n;\lambda^2)\frac{(2t)^m}{m!}\right)\\ &= \sum_{m=0}^{\infty}\left[\sum_{k=0}^{m}\binom{m}{k}\mathcal{S}_{m-k}(n;\lambda)\mathcal{E}_k(\lambda) - 2^m\mathcal{S}_m(n;\lambda^2)\right]\frac{t^m}{m!}\end{aligned} \tag{56}$$

we are led to the second part of the desired result (54). □

By taking $\lambda = 1$ in Theorem 4, we can deduce the following relationship.

Corollary 21 (*See, for the First Relationship, [29, Theorem 3.2, Eq. (23)]*)**.** *Let $m, n \in \mathbb{N}$. Then*

$$\sum_{k=0}^{m}\binom{m}{k} S_{m-k}(n)T_k(n) = \begin{cases} 2^m S_m(n) & (n \text{ odd})\\ \displaystyle\sum_{k=0}^{m}\binom{m}{k} S_{m-k}(n)E_k - 2^m S_m(n) & (n \text{ even}).\end{cases} \tag{57}$$

If we compare the second relationship in (57) with a known result [29, Theorem 3.2, Eq. (24)], we obtain a new identity asserted by the following corollary.

Corollary 22. *Let $m, n \in \mathbb{N}$ and let n be an even integer. Then*

$$\sum_{k=0}^{m}\binom{m}{k} S_{m-k}(n)E_k = 2^{m+1}S_m\left(\frac{n}{2}\right). \tag{58}$$

Finally, we prove the following result.

Theorem 5. *For $m, n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$, the following identity holds true:*

$$\sum_{k=0}^{m}\binom{m}{k}\mathcal{F}_k(n;\lambda;u;v)\mathcal{S}_{m-k}(n,\lambda) = \frac{2^{m-v+u}m!}{(m-v+1)!}\left[\lambda\mathcal{B}_{m-v+1}(n;\lambda^2) - \mathcal{B}_{m-v+1}\left(\frac{n}{2};\lambda^2\right)\right]$$
$$\big(m, n \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\big). \tag{59}$$

Proof. For the function $h(t)$ given by

$$h(t) = \frac{2^u t^v e^{nt}(\lambda e^{nt}-1)}{(\lambda e^t+1)(\lambda e^t-1)}, \tag{60}$$

we have

$$h(t) = \frac{2^u t^v e^{nt}}{\lambda e^t + 1} \cdot \frac{\lambda e^{nt} - 1}{\lambda e^t - 1}$$
$$= \left(\sum_{m=0}^{\infty} \mathcal{F}_m(n; \lambda; u; v) \frac{t^m}{m!}\right) \left(\sum_{m=0}^{\infty} \mathcal{S}_m(n; \lambda) \frac{t^m}{m!}\right)$$
$$= \sum_{m=0}^{\infty} \left[\sum_{k=0}^{m} \binom{m}{k} \mathcal{F}_k(n; \lambda; u; v) \mathcal{S}_{m-k}(n; \lambda)\right] \frac{t^m}{m!} \tag{61}$$

and

$$h(t) = \frac{2^u t^v \lambda e^{2nt} - 2^u t^v e^{nt}}{\lambda^2 e^{2t} - 1}$$
$$= 2^{u-1} t^{v-1} \lambda \left(\frac{2te^{2nt}}{\lambda^2 e^{2t} - 1}\right) - 2^{u-1} t^{v-1} \left(\frac{2te^{nt}}{\lambda^2 e^{2t} - 1}\right)$$
$$= 2^{u-1} t^{v-1} \lambda \sum_{m=0}^{\infty} \mathcal{B}_m(n; \lambda^2) \frac{(2t)^m}{m!} - 2^{u-1} t^{v-1} \sum_{m=0}^{\infty} \mathcal{B}_m\left(\frac{n}{2}; \lambda^2\right) \frac{(2t)^m}{m!}$$
$$= \lambda \sum_{m=0}^{\infty} 2^{m+u-1} \mathcal{B}_m(n; \lambda^2) \frac{t^{m+v-1}}{m!} - \sum_{m=0}^{\infty} 2^{m+u-1} \mathcal{B}_m\left(\frac{n}{2}; \lambda^2\right) \frac{t^{m+v-1}}{m!}. \tag{62}$$

Now, upon equating the coefficients of $\frac{t^m}{m!}$ in the two expansions of $h(t)$ in (61) and (62), we are easily led to the identity (59) asserted by Theorem 5. □

It is not difficult to deduce each of the following corollaries and consequences of Theorem 5.

Corollary 23. *Let $m, n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. Then*

$$\sum_{k=0}^{m} \binom{m}{k} \mathcal{B}_k(n; \lambda) \mathcal{S}_{m-k}(n, -\lambda) = 2^{m-1} \left[\mathcal{B}_m\left(\frac{n}{2}; \lambda^2\right) + \lambda \mathcal{B}_m(n; \lambda^2)\right] \quad \big(m, n \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\big). \tag{63}$$

Corollary 24. *Let $m, n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. Then*

$$(m+1) \sum_{k=0}^{m} \binom{m}{k} \mathcal{E}_k(n; \lambda) \mathcal{S}_{m-k}(n, \lambda) = 2^{m+1} \left[\lambda \mathcal{B}_{m+1}(n; \lambda^2) - \mathcal{B}_{m+1}\left(\frac{n}{2}; \lambda^2\right)\right] \quad \big(m, n \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\big). \tag{64}$$

Corollary 25. *If $m, n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$, then we have the following identity:*

$$\sum_{k=0}^{m} \binom{m}{k} \mathcal{G}_k(n; \lambda) \mathcal{S}_{m-k}(n, \lambda) = 2^{m} \left[\lambda \mathcal{B}_m(n; \lambda^2) - \mathcal{B}_m\left(\frac{n}{2}; \lambda^2\right)\right] \quad \big(m, n \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\big). \tag{65}$$

Corollary 26. *For $m, n \in \mathbb{N}_0$, the following identity holds true:*

$$\sum_{k=0}^{m} \binom{m}{k} G_k(n) S_{m-k}(n) = 2^{m} \left[B_m(n) - B_m\left(\frac{n}{2}\right)\right] \quad \big(m, n \in \mathbb{N}_0\big). \tag{66}$$

Corollary 27. *Let $m \in \mathbb{N}_0$, $c, d \in \mathbb{C} \setminus \{0\}$ and $\kappa, \beta \in \mathbb{C}$. Then*

$$\sum_{k=0}^{m} \binom{m}{k} \mathcal{Y}_{k,\beta}(n; \kappa; c; d) \mathcal{S}_{m-k}\left(n, -\left(\frac{\beta}{c}\right)^d\right)$$
$$= -\frac{2^{m-2k+1} m!}{(m-k+1)!} \left[\left(\frac{\beta}{c}\right)^d \mathcal{B}_{m-k+1}\left(n; \left(\frac{\beta}{c}\right)^{2d}\right) + \mathcal{B}_{m-k+1}\left(\frac{n}{2}; \left(\frac{\beta}{c}\right)^{2d}\right)\right]$$
$$\big(m \in \mathbb{N}_0;\ c, d \in \mathbb{C} \setminus \{0\};\ \kappa, \beta \in \mathbb{C}\big). \tag{67}$$

We conclude our present investigation by remarking that a number of *further* properties and identities involving the general families of Apostol type polynomials, which we have studied in this paper, can be derived in analogous manners.

Acknowledgements

The first-named author would like to thank Professor Qiu-Ming Luo for many helpful suggestions. This work was supported by the *Shanghai Natural Science Foundation of the People's Republic of China* under Grant 10ZR1409100.

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