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Edge-disjoint spanners in Cartesian products of graphs

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Abstract

A spanning subgraph $S=(V,E')$ of a connected graph $G=(V,E)$ is an $(x+c)$-spanner if for any pair of vertices u and v, $d_S(u,v)\leqslant d_G(u,v)+c$ where d_G and d_S are the usual distance functions in G and S, respectively. The parameter c is called the delay of the spanner. We study edge-disjoint spanners in graphs, focusing on graphs formed as Cartesian products. Our approach is to construct sets of edge-disjoint spanners in a product based on sets of edge-disjoint spanners and colorings of the component graphs. We present several results on general products and then narrow our focus to hypercubes.

Keywords: Subgraph; Distance; Spanner; Product; Hypercube

1. Introduction

A spanner of a graph is a spanning subgraph in which the distance between any pair of vertices approximates the distance in the original graph. Although spanners were introduced by Peleg and Ullman [20] for simulation of synchronous distributed systems, they are an

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interesting graph theoretical structure with application to many problems in interconnection networks [4,5,18,19]. The use of spanners as a network topology (as a substitute for an expensive original topology) was suggested by Richards and Liestman [21] and further studied in a series of papers by Liestman and Shermer [15,13,16,14,17] and Heydemann et al. [8]. Algorithms for constructing spanners have also been studied [3,6,9,10].

One problem encountered in parallel computing is to share the resources among several users concurrently. One way to approach this problem is to multitask on the computers but to dedicate each link to an individual user. In graph-theoretic terms, this corresponds to partitioning the edges into a set of edge-disjoint spanners. Laforest et al. [11] studied edge-disjoint spanners in complete graphs and in complete digraphs. Laforest et al. [12] studied edge-disjoint spanners in complete bipartite graphs. In this paper, we continue this line of study, investigating edge-disjoint spanners in Cartesian products of graphs (and specifically in hypercubes). The remainder of this paper is organized as follows: in Section 2, along with other definitions and notation, we define our problem. In Section 3, we investigate edge-disjoint spanners in general Cartesian products. In Section 4, we restrict our attention to hypercubes.

2. Definitions

A network is represented by a connected simple graph $G=(V(G),E(G))$. We use $d_G(u,v)$ to denote the distance from vertex u to vertex v in graph G. A spanner S of a connected simple graph G is an $f(x)$-*spanner* if for any pair of vertices u and v, $d_S(u,v)\leqslant f(d_G(u,v))$. We call $d_S(u,v)-d_G(u,v)$ the *delay between vertices u and v in S*. For an $f(x)$-spanner S, we refer to $f(x)-x$ as the *delay* of the spanner. Note that $f(x)-x$ is an upper bound (but not necessarily a tight bound) on the maximum delay in S between any pair of vertices at distance x in G.

We use $H\times G$ to denote the Cartesian product of base graphs H and G. The vertex set $V(H\times G)$ is $V(H)\times V(G)=\{[u,v]:u\in V(H)$ and $v\in V(G)\}$. The edge set $E(H\times G)$ contains all pairs $([u,v],[u',v'])$ such that either (1) $u=u'$ and $(v,v')\in E(G)$, or (2) $v=v'$ and $(u,u')\in E(H)$. The definition easily extends to the product of n base graphs $G_1\times G_2\times\cdots\times G_n$ which will be denoted by $\prod_{i=1}^{n}G_i$. The following generalization of the Cartesian product is useful in constructing edge-disjoint spanners. Given a coloring of vertices of H, the *color-i product* of graphs H and G, written $H\overset{i}{\times}G$, is the graph with vertex set $V(H)\times V(G)$ and all edges $([u,v],[u',v'])$ such that (1) $u=u'$, the color of u in H is i, and $(v,v')\in E(G)$, or (2) $v=v'$ and $(u,u')\in E(H)$. Note that if all vertices of H are colored i, then the color-i product is simply the Cartesian product. For technical reasons while performing the operation of the color-i product we extend the coloring of H to $H\overset{i}{\times}G$ by assigning the color of $u\in V(H)$ to every vertex $[u,v]\in V(H\overset{i}{\times}G)$.

The (*closed*) *neighborhood* of a vertex v in graph G, denoted $N_G[v]$, is $\{x\in V : d_G(v,x)\leqslant 1\}$. More generally, the *d-neighborhood*, $N_G^d[v]$ of v in G is $\{x\in V : d_G(v,x)\leqslant d\}$.

A *d-dominating set* of vertices in graph G is a set $S\subseteq V$ such that every vertex in V is in the d-neighborhood of some element of S. A *d-domatic coloring* of G is a vertex coloring of G such that each color class constitutes a d-dominating set of G. A d-domatic coloring need not be a proper vertex coloring; we allow adjacent vertices to be assigned the same color.

The maximum number of colors in any d-domatic coloring of a fixed graph G is called the *d-domatic number* of G. The 1-domatic number of a graph G is the well-known domatic number of G and will be denoted by $dom(G)$.

Let G be a graph and let $S_1, S_2, \ldots, S_k$ be edge-disjoint subgraphs of G. A vertex coloring of G is called an *all-factor d-domatic coloring of G with respect to* $S_1, S_2, \ldots, S_k$ if the vertices of each color constitute a d-dominating set in each S_j for $1 \leqslant j \leqslant k$. In contrast, a vertex coloring of G with k colors is called a *matched-factor d-domatic coloring of G with respect to* $S_1, S_2, \ldots, S_k$ if the vertices of each color i constitute a d-dominating set of the subgraph S_i. These colorings were studied by Alon et al. [1] and we will use the results of that paper below.

Our goal is to investigate small delay spanners of Cartesian products. We are particularly interested in those spanners with constant delay, i.e. $(x + c)$-spanners for constant c. More precisely, given a constant c, we are interested in the maximum number of edge-disjoint $(x + c)$-spanners that can be found in G. We let $\mathrm{EDS}(G, c)$ denote this number.

3. General Cartesian products

In this section, we present several results on the number of edge-disjoint spanners that can be found in graphs that are the Cartesian product of other graphs. Typically, these results are lower bounds on the number of spanners in $H \times G$, based on the number of spanners of H and some properties of H or its spanners. We start with a preliminary lemma concerning the delay of a spanner constructed as the Cartesian product of spanners.

Lemma 1. *Let $G_1, G_2, \ldots, G_\alpha$ be graphs and let S_i be a delay c_i spanner of G_i for $i = 1, 2, \ldots, \alpha$. Then $S = \prod_{i=1}^{\alpha} S_i$ is a delay c spanner of $G = \prod_{i=1}^{\alpha} G_i$, where $c = \sum_{i=1}^{\alpha} c_i$.*

Proof. As each S_i is a spanner of G_i, it follows that S is a spanner of G. Let $u = [u_1, u_2, \ldots, u_\alpha]$ and $v = [v_1, v_2, \ldots, v_\alpha]$ be two vertices of G such that $u_i, v_i \in V(G_i)$ for each i. Then, $d_S(u, v) = \sum_{i=1}^{\alpha} d_{S_i}(u_i, v_i) \leqslant \sum_{i=1}^{\alpha} (d_{G_i}(u_i, v_i) + c_i) = d_G(u, v) + \sum_{i=1}^{\alpha} c_i$. Thus, S is a delay c spanner of G as claimed. □

The constructions in the remainder of this section follow a central scheme which is illustrated in Fig. 1. In particular, we construct spanners of $H \times G$ by taking a color product of a spanner of H with G. That is, each spanner of $H \times G$ will include the same edges in each copy of H, some entire copies of G, and no other edges. The edges included in each copy of H are the edges of some spanner of H. The copies of G included in a particular spanner correspond to a color class in a coloring of H. In a particular spanner S of $H \times G$, those vertices of H corresponding to a copy of G that is included in S are called *hubs*. We can bound delays of such spanners in $H \times G$ by delays along paths in H that include a hub. In general, the number of spanners of $H \times G$ that we obtain depends on the number of color classes in the coloring of H and the number of spanners of H in the set of edge-disjoint spanners of H. The delay of the spanners of $H \times G$ depends on the coloring of H and the properties of the spanners in the set of edge-disjoint spanners of H.

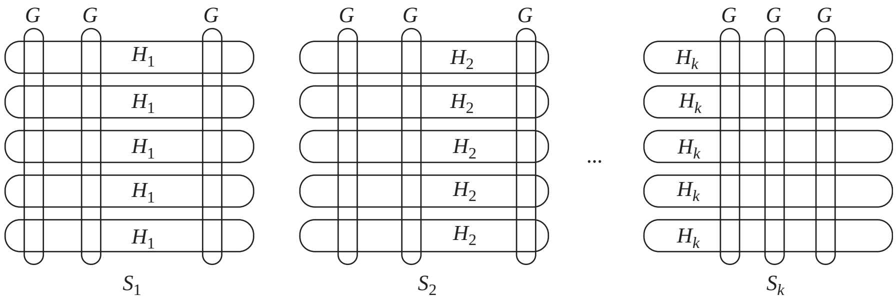

Fig. 1. A central scheme for the construction of spanners of $H \times G$. Three spanners are shown. Each spanner contains some copies of G indicated by the vertical ovals and a spanner in each copy of H indicated by the horizontal ovals. The spanners H_i of H are chosen from a set of edge-disjoint spanners of H.

In the following theorem, we use the connectivity of the spanners of H to bound the delay of the spanners of $H \times G$.

Theorem 1. *Let H be a graph on n vertices with k edge-disjoint delay c spanners, each of which is ρ-connected. Then for any connected graph G,*

$$\mathrm{EDS}\left(H \times G, c + 2\left\lfloor \frac{n - \lfloor n/k \rfloor - 1}{\rho} \right\rfloor + 2\right) \geqslant k.$$

Proof. Let $H_1, H_2, \ldots, H_k$ be ρ-connected edge-disjoint delay c spanners of H. Color the vertices of H with colors $1, 2, \ldots, k$, in such a way that every color class has cardinality at least $\lfloor n/k \rfloor$ (where we allow adjacent vertices to receive the same color). For $i = 1, 2, \ldots, k$, let $S_i = H_i \overset{i}{\times} G$. These k graphs are edge-disjoint spanners of $H \times G$.

Consider the spanner S_i for some i. Let $u = [h_1, g_1]$ and $v = [h_2, g_2]$ be two vertices of S_i, where $h_1, h_2 \in V(H)$ and $g_1, g_2 \in V(G)$. Let h_1' be a vertex of color i that is closest to h_1 in H_i. The vertex h_1' is a hub of S_i. There is a path P in S_i from $u = [h_1, g_1]$ to $[h_1', g_1]$ to $[h_1', g_2]$ to $[h_1, g_2]$ to $[h_2, g_2] = v$ with length $d_{H_i}(h_1, h_1') + d_G(g_1, g_2) + d_{H_i}(h_1', h_1) + d_{H_i}(h_1, h_2) = 2d_{H_i}(h_1, h_1') + d_G(g_1, g_2) + d_{H_i}(h_1, h_2)$. Since H_i has delay c, $d_{H_i}(h_1, h_2) \leqslant d_H(h_1, h_2) + c$, and the delay of P is at most $2d_{H_i}(h_1, h_1') + c$.

We now bound $d_{H_i}(h_1, h_1')$. If $h_1 = h_1'$, then $d_{H_i}(h_1, h_1') = 0$, and the lemma follows. Therefore we may assume that $h_1 \neq h_1'$. Let h^* be any vertex of color i in H_i. By our assumption, $h_1 \neq h^*$. Since H_i is ρ-connected, there are ρ vertex-disjoint paths from h_1 to h^* in H_i. As there are at most $n - 1 - \lfloor n/k \rfloor$ vertices of H_i different from h_1 and of color other than i, one of these vertex-disjoint paths contains at most $\lfloor (n - 1 - \lfloor n/k \rfloor)/\rho \rfloor$ such vertices. Since h^* has color i, on this path, there must be a vertex of color i at distance at most $\lfloor (n - 1 - \lfloor n/k \rfloor)/\rho \rfloor + 1$ from h_1. As a consequence,

$$d_{H_i}(h_1, h_1') \leqslant \left\lfloor \frac{n - 1 - \lfloor n/k \rfloor}{\rho} \right\rfloor + 1,$$

and the delay of P is at most $2\lfloor (n - 1 - \lfloor n/k \rfloor)/\rho \rfloor + 2 + c$, giving the result. $\square$

By using the same construction but considering the diameters of the spanners of H rather than their connectivities, one obtains the following result.

Theorem 2. *Let H be a graph with k edge-disjoint spanners each of diameter at most d. Then for any connected graph G,* $\mathrm{EDS}(H \times G, 2d) \geqslant k$.

In the previous theorems, we have placed relatively few conditions on the coloring used in our central scheme. In what follows, we make use of more sophisticated colorings to obtain better bounds.

The following theorem uses a natural proper coloring of H and will be a useful starting point for our investigation of hypercube spanners in Section 4.

Theorem 3. *Let H be any bipartite graph and let* $c \geqslant 2$ *be an integer. If* $\mathrm{EDS}(H, c) \geqslant 2$, *then for any connected graph G,* $\mathrm{EDS}(H \times G, c) \geqslant 2$.

Proof. We can properly color vertices of H with colors 1 and 2. Let H_1 and H_2 be edge-disjoint delay c spanners of H. For $i = 1, 2$, let $S_i = H_i \overset{i}{\times} G$. S_1 and S_2 are two edge-disjoint spanners of $H \times G$.

In the following, we bound the delay of S_1, the case of S_2 is similar. Let $u = [h_1, g_1]$ and $v = [h_2, g_2]$ be two vertices of S_1. If $g_1 = g_2$, then both u and v are in some copy of H_1 in $H \times G$ and their delay is at most c. Otherwise $g_1 \neq g_2$.

First consider the case when $h_1 = h_2$. Let h_1' be a neighbor of h_1 in H_1. Either h_1 or h_1' is of color 1. If h_1' has color 1, then there is a path P from $u = [h_1, g_1]$ to $[h_1', g_1]$ to $[h_1', g_2]$ to $[h_1, g_2] = v$ with length $1 + d_G(g_1, g_2) + 1 \leqslant 2 + d_G(g_1, g_2)$. If h_1 is the vertex of color 1, there is a path from u to v with length $d_G(g_1, g_2)$. In either case, the delay is at most $2 \leqslant c$.

Now consider the case when $h_1 \neq h_2$. If h_1 has color 1, then there is a path in S_1 from $u = [h_1, g_1]$ to $[h_1, g_2]$ to $[h_2, g_2] = v$ with length $d_G(g_1, g_2) + d_{H_1}(h_1, h_2) \leqslant d_{H \times G}(u, v) + c$. Otherwise, h_1 has color 2. Let h_1' be a neighbor of h_1 on a shortest path from h_1 to h_2 in H_1. There is a path in S_1 from $u = [h_1, g_1]$ to $[h_1', g_1]$ to $[h_1', g_2]$ to $[h_2, g_2] = v$ with length $1 + d_G(g_1, g_2) + (d_{H_1}(h_1, h_2) - 1) \leqslant d_G(g_1, g_2) + d_H(h_1, h_2) + c = d_{H \times G}(u, v) + c$. Thus, in either case, the delay is at most c. $\square$

Next, we use a matched factor domatic coloring for the coloring of H in the central scheme. This type of coloring was devised specifically for use in this construction.

Theorem 4. *Let H be a graph with k edge-disjoint delay c spanners* $H_1, H_2, \ldots, H_k$. *If H has a matched factor l-domatic coloring with respect to* $H_1, H_2, \ldots, H_k$, *then for any connected graph G,* $\mathrm{EDS}(H \times G, 2l + c) \geqslant k$.

Proof. Consider a matched factor l-domatic coloring of H with respect to $H_1, H_2, \ldots, H_k$ with colors $1, 2, \ldots, k$. For $i = 1, 2, \ldots, k$, let $S_i = H_i \overset{i}{\times} G$. The graphs S_i are edge-disjoint spanners of $H \times G$.

Consider S_i for some i. Let $u = [h_1, g_1]$ and $v = [h_2, g_2]$ be two vertices of S_i. Let h_1' be a vertex of color i that is closest to h_1 in H_i. The vertex h_1' is a hub of S_i. There is a path in S_i from $u = [h_1, g_1]$ to $[h_1', g_1]$ to $[h_1', g_2]$ to $[h_1, g_2]$ to $[h_2, g_2] = v$, with length $d_{H_i}(h_1, h_1') +$

$d_G(g_1, g_2) + d_{H_i}(h'_1, h_1) + d_{H_i}(h_1, h_2) = 2d_{H_i}(h_1, h'_1) + d_G(g_1, g_2) + d_{H_i}(h_1, h_2)$. Since we started with a matched factor l-domatic coloring, $d_{H_i}(h'_1, h_1) \leqslant l$, and the length of this path is at most $2l + d_G(g_1, g_2) + d_H(h_1, h_2) + c = d_{H\times G}(u, v) + 2l + c$. □

With Alon [1], we established that every graph with k edge-disjoint spanners has a matched factor $\lceil (3k-1)/2 \rceil$-domatic coloring. Combining this result with Theorem 4, we obtain:

Corollary 1. *Let H be a graph such that* $\mathrm{EDS}(H, c) \geqslant k$, *and let G be any graph. Then* $\mathrm{EDS}(H \times G, 2\lceil (3k-1)/2 \rceil + c) \geqslant k$.

In the previous constructions, we built a set of good spanners in $H \times G$ from a set of spanners in H, *all* of which have low delay. By using an all-factor domatic coloring, we may build a set of good spanners for $H \times G$ from a set of spanners in H, *one* of which has low delay. To do this, we slightly modify our central scheme, placing one copy of the low delay spanner of H in each spanner of $H \times G$.

Theorem 5. *Let H be a graph with k edge-disjoint spanners* $H_1, H_2, \ldots, H_k$ *such that* H_1 *is a delay c spanner and let H have an all-factor* r_1*-domatic coloring with k colors with respect to* $H_1, H_2, \ldots, H_k$. *Let G be a graph with an* r_2*-domatic coloring with k colors. Then,* $\mathrm{EDS}(H \times G, 4r_1 + 2r_2 + c) \geqslant k$.

Proof. We divide the edges of $H \times G$ into k spanners $S_1, S_2, \ldots, S_k$ as follows: each copy of H in $H \times G$ corresponds to a vertex of G. If this vertex has color i in the domatic coloring of G, we place the edges of $H_1, H_2, \ldots, H_k$ into spanners $S_1, S_2, \ldots, S_k$, respectively, except for H_1 and H_i. We place the edges of H_1 in S_i and the edges of H_i in S_1. For each copy of G in $H \times G$, there is a corresponding vertex of H. If this vertex has color i in the all-factor coloring of H, then we place all edges of this copy of G in S_i.

Let $u = [h_1, g_1]$ and $v = [h_2, g_2]$ be two vertices of S_i. Let h'_1 be a vertex of color i that is closest to h_1 in H. Similarly, let h'_2 be a vertex of color i that is closest to h_2 in H. Let g'_2 be a vertex of color i (in G) that is closest to g_2 in G. There is a path in S_i from $u = [h_1, g_1]$ to $[h'_1, g_1]$ to $[h'_1, g_2]$ to $[h'_1, g'_2]$ to $[h'_2, g'_2]$ to $[h'_2, g_2]$ to $[h_2, g_2] = v$; let P be a shortest such path. The length of the subpaths of P from u to $[h'_1, g_1]$ and from $[h'_2, g_2]$ to v are each at most r_1 by our all-factor r_1-domatic coloring of H. The length of the subpath from $[h'_1, g_1]$ to $[h'_1, g_2]$ is $d_G(g_1, g_2)$. The length of the subpaths from $[h'_1, g_2]$ to $[h'_1, g'_2]$ and from $[h'_2, g'_2]$to $[h'_2, g_2]$ are each at most r_2 by our r_2-domatic coloring of G. The length of the subpath from $[h'_1, g'_2]$ to $[h'_2, g'_2]$ is $d_{H_i}(h'_1, h'_2) \leqslant d_H(h'_1, h'_2) + c \leqslant d_H(h'_1, h_1) + d_H(h_1, h_2) + d_H(h_2, h'_2) + c \leqslant d_H(h_1, h_2) + 2r_1 + c$. Thus, the total distance from u to v along P is $d_G(g_1, g_2) + d_H(h_1, h_2) + 4r_1 + 2r_2 + c = d_{H\times G}(u, v) + 4r_1 + 2r_2 + c$. □

With Alon, we have shown that any graph with k edge-disjoint spanners has an all-factor $(12k \log k)$-domatic coloring with k colors. (This comes from an exact, rather than asymptotic, analysis of the proof of Theorem 2 in [1].) Combining this with the previous theorem, we obtain:

Corollary 2. *Let H be a graph with k edge-disjoint spanners such that H_1 is a delay c spanner. Let G be a graph with an r-domatic coloring with k colors. Then* $\mathrm{EDS}(H \times G, 2r + 48k \log k + c) \geqslant k$.

In the preceding results, we have constructed a set of spanners in $H \times G$ using spanners of H and complete copies of G. The number of spanners of $H \times G$ that can be obtained in this manner is limited to the number of spanners of H. To obtain more spanners of $H \times G$, we can use spanners of G and spanners of H in each spanner of $H \times G$. As our constructions easily generalize to the product of an arbitrary number of base graphs $H_1, H_2, \ldots, H_\alpha$, we state them for the general case.

Theorem 6. *Let $H_1, H_2, \ldots, H_\alpha$ be graphs. Let* $\mathrm{EDS}(H_i, c_i) \geqslant k_i$ *for $i = 1, 2, \ldots, \alpha$. If for $i = 1, \ldots, \alpha$, the domatic number* $\mathrm{dom}(H_i) \geqslant k_i - \alpha + 1 > 0$, *then*

$$\mathrm{EDS}\left(\prod_{i=1}^{\alpha} H_i, \sum_{i=1}^{\alpha} c_i + 2 + \max_{i=1,\ldots,\alpha} c_i\right) \geqslant \sum_{i=1}^{\alpha} k_i - \alpha^2 + \alpha.$$

Proof. Let $G = \prod_{i=1}^{\alpha} H_i$. For $i = 1, 2, \ldots, \alpha$, let $H_{i,1}, H_{i,2}, \ldots, H_{i,k_i}$ be a set of edge-disjoint spanners of H_i each of delay c_i. We will construct $\sum_{i=1}^{\alpha} k_i - \alpha^2 + \alpha$ spanners of G. These spanners are divided into α classes, one class for each H_i. For each class i, we will construct the spanners of G using a spanner R_i of $\prod_{j \neq i} H_j$. Class i contains $m_i = k_i - \alpha + 1$ spanners $S_{i,1}, S_{i,2}, \ldots, S_{i,m_i}$. Spanner $S_{i,j}$ consists of all copies of $H_{i,j}$ connected by some copies of some R_i. The remaining spanners $H_{i,m_i+1}, H_{i,m_i+2}, \ldots, H_{i,k_i}$ of H_i are used in the construction of the differing $R_{i'}$, for $i' \neq i$. See Fig. 2 for an example.

In particular, for any class i, let

$$R_i = \left(\prod_{1 \leqslant j \leqslant i-1} H_{j,m_j+i-1}\right) \times \left(\prod_{i+1 \leqslant j \leqslant \alpha} H_{j,m_j+i}\right).$$

In each $S_{i,j}$ some copies of R_i will be used to connect the copies of $H_{i,j}$. To this end, we use a domatic coloring of H_i with colors $1, 2, \ldots, k_i - \alpha + 1$, and let

$$S_{i,j} = H_{i,j} \overset{i}{\times} R_i.$$

As $H_{i,j}$ spans H_i and R_i spans $\prod_{j \neq i} H_j$, the graph $S_{i,j}$ is a spanner of G.

We now show that all spanners $S_{i,j}$ are edge-disjoint. Consider a pair of spanners $S_{i,j}$ and $S_{i',j'}$. If $i = i'$, then $j \neq j'$. As $H_{i,j}$ and $H_{i,j'}$ are edge-disjoint, and no copy of R_i is in both $S_{i,j}$ and $S_{i,j'}$, (by the color product construction), $S_{i,j}$ and $S_{i,j'}$ are edge-disjoint. Otherwise, $i \neq i'$. In this case, $H_{i,j}$ and $H_{i',j'}$ must be edge-disjoint because H_i and $H_{i'}$ are different spanners. Furthermore, by construction R_i and $R_{i'}$ are also edge-disjoint. Thus, $S_{i,j} \subseteq H_{i,j} \times R_i$ and $S_{i',j'} \subseteq H_{i',j'} \times R_{i'}$ are edge-disjoint.

We now establish the delay of spanner $S_{i,j}$. Let $u = [u_1, u_2, \ldots, u_\alpha]$ and $v = [v_1, v_2, \ldots, v_\alpha]$. Let $u' = [u_1, u_2, \ldots, u_{i-1}, x, u_{i+1}, u_{i+2}, \ldots, u_\alpha]$, where x is a vertex of color j that is closest to u_i in $H_{i,j}$. Let $v' = [v_1, v_2, \ldots, v_{i-1}, x, v_{i+1}, v_{i+2}, \ldots, v_\alpha]$. Let P be

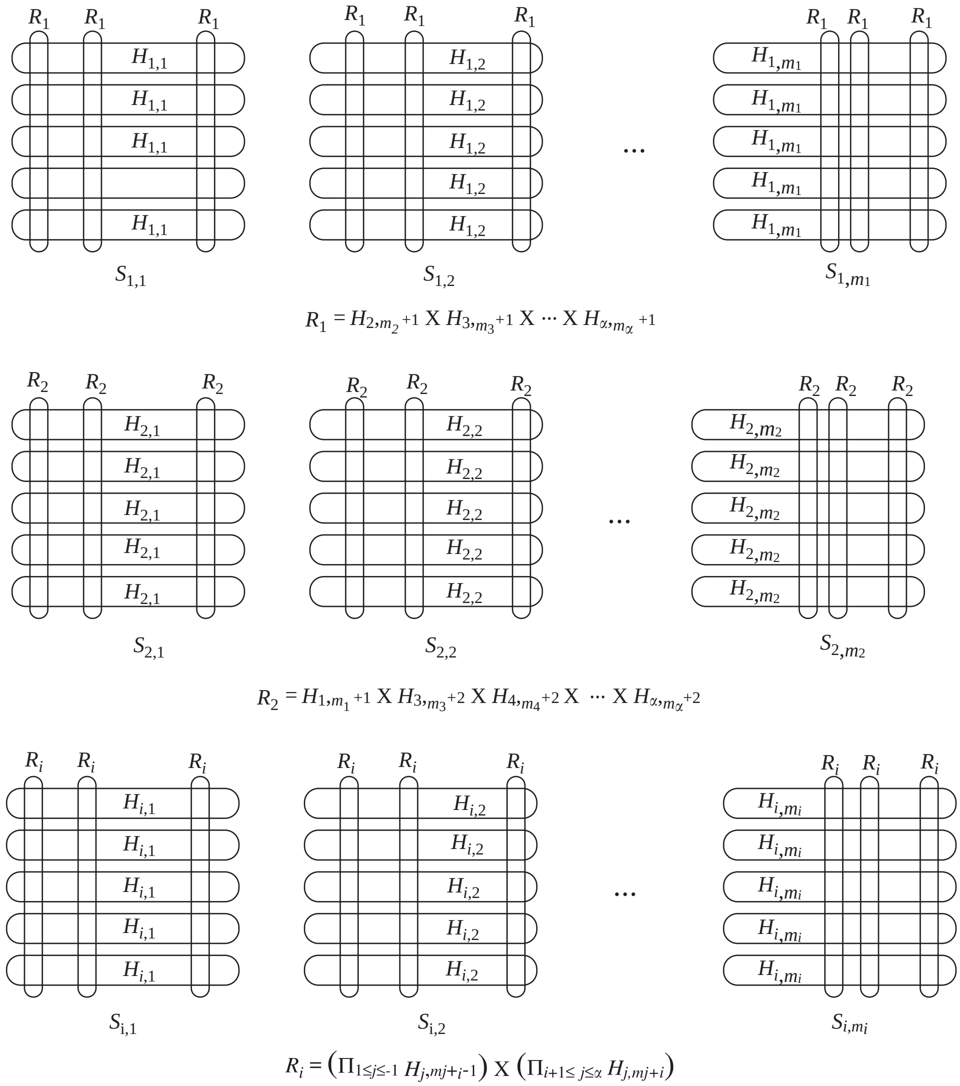

Fig. 2. Construction of spanners of $\prod H_i$. The spanners of class 1 are shown at the top, the spanners of class 2 in the middle, and those of class i at the bottom. In each spanner of class j, the horizontal ovals denote copies of H_j and vertical ovals denote copies of $\prod_{k\neq j} H_k$. Each copy of H_j is labelled with which edges of H_j it includes. Each copy of $\prod_{k\neq j} H_k$ is labelled above indicating which edges it includes.

a shortest path in $S_{i,j}$ from u to u' to v' to v. The length of the subpath of P from u to u' is at most $c_i + 1$ because u_i and x are at distance 1 in H_i and $H_{i,j}$ is a spanner of H_i with delay c_i. The length of the subpath of P from u' to v' is at most $\sum_{j\neq i}(d_{H_j}(u_j, v_j) + c_j)$ by Lemma 1. The length of the subpath of P from v' to v is at most $d_{H_i}(u_i, v_i) + 1 + c_i$

because x and v_i are at distance at most $d_{H_i}(u_i, v_i) + 1$ in H_i. Thus, the length of P is at most

$$(c_i + 1) + \sum_{j \neq i} (d_{H_j}(u_j, v_j) + c_j) + d_{H_i}(u_i, v_i) + 1 + c_i$$
$$= \sum_{j=1}^{\alpha} d_{H_j}(u_j, v_j) + \sum_{j=1}^{\alpha} c_j + c_i + 2 = d_G(u, v) + \sum_{j=1}^{\alpha} c_j + c_i + 2.$$

Therefore, every $S_{i,j}$ has delay at most $\sum_{j=1}^{\alpha} c_j + \max_{j=1,2,\ldots,\alpha} c_j + 2$. □

It is known that every graph G has domatic number approximately $\delta(G)/\ln \Delta(G)$, where $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of G, respectively [7]. Thus, the bound on $\mathrm{dom}(H_i)$ in the Theorem will hold when $\delta(H_i)$ is sufficiently large.

The previous theorem can be easily generalized, by allowing some base graphs to have unrestricted domatic number, provided that the other base graphs have suitably high domatic number. The parameter m is used to denote the number of base graphs with unrestricted domatic number; the case $m = 0$ corresponds to the previous theorem.

Theorem 7. *Let $H_1, H_2, \ldots, H_\alpha$ be graphs. Let $0 \leqslant m < \alpha$. Let $\mathrm{EDS}(H_i, c_i) \geqslant k_i$ for $1 \leqslant i \leqslant \alpha$. If for $1 \leqslant i \leqslant \alpha - m$, $\mathrm{dom}(H_i) \geqslant k_i - \alpha + m + 1 > 0$, and for $\alpha - m + 1 \leqslant i \leqslant \alpha$, the value of $k_i \geqslant \alpha - m - 1$, then*

$$\mathrm{EDS}\left(\prod_{i=1}^{\alpha} H_i, \sum_{i=1}^{\alpha} c_i + 2 + \max_{i=1,\ldots,\alpha-1} c_i\right) \geqslant \sum_{i=1}^{\alpha-m} k_i - \alpha^2 + (2m+1)\alpha - m^2 - m.$$

Proof. The proof is similar to the proof of Theorem 6. The difference is that we do not construct classes $\alpha - m + 1, \alpha - m + 2, \ldots, \alpha$, and there are no $R_{\alpha-m+1}, R_{\alpha-m+2}, \ldots, R_\alpha$. Thus, for $i \leqslant \alpha - m$, we need only use $\alpha - m - 1$ spanners $H_{i,j}$ to connect spanners of classes other than i (spanner used in $R_1, \ldots, R_{\alpha-m}$). This allows us to have $k_i - \alpha + m + 1$ spanners in each class i, giving $\sum_{i=1}^{\alpha-m} (k_i - \alpha + m + 1) = \sum_{i=1}^{\alpha-m} k_i - \alpha^2 + (2m+1)\alpha - m^2 - m$ spanners total. □

Note that Theorem 7 gives more spanners than Theorem 6 if $\sum_{i=\alpha-m+1}^{\alpha} k_i < (2m+1)\alpha - m^2 - m$.

4. Hypercubes

Let Q_d denote the d-dimensional hypercube. Note that $Q_d = Q_{d-i} \times Q_i$ for any $1 \leqslant i < d$. In this section, we prove bounds on the number of edge-disjoint spanners that can be found in hypercubes. We begin with two lemmas that show how to construct a set of spanners containing one good spanner. These lemmas will then be combined with Corollary 2 to produce the main results of this section. We use the following results from [22] and [2], respectively.

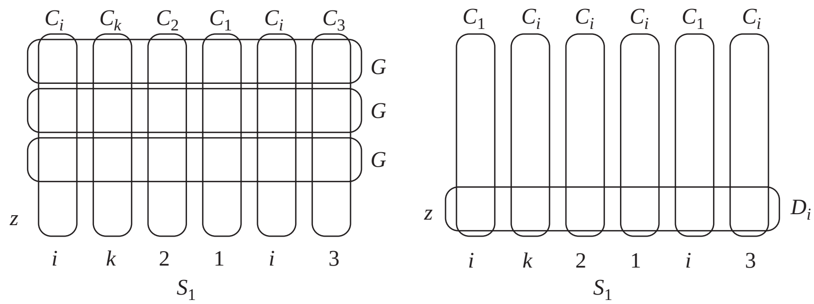

Fig. 3. Construction of spanners of $H \times G$. In each part, copies of G are drawn as horizontal ovals, and copies of H are drawn as vertical ovals. To the left of a copy of G, it is noted whether that copy corresponds to the distinguished vertex z. To the right, a label indicates which edges of G are included. In a similar way, a label below a copy of H indicates the color of the corresponding vertex of G, and a label above indicates which edges of H are included.

Lemma 2 (*Zelinka [22]*). *If m is a power of* 2, *then* $\mathrm{dom}(Q_m) = m$.

This lemma implies a slightly weaker result when m is not a power of 2.

Corollary 3. *For any integers $k \geqslant 3$ and $m \geqslant 2^{\lceil \log k \rceil}$, Q_m has a 1-domatic coloring with k colors*.

To prove this, domatically color a $2^{\lceil \log k \rceil}$-dimensional subhypercube with k colors and extend this coloring to Q_m by repeating it in each copy of the subhypercube.

Lemma 3 (*Alspach et al. [2]*). *For even integer m, Q_m can be decomposed into $m/2$ Hamilton cycles*.

Lemma 4. *For any integers $k \geqslant 2$ and $d \geqslant 4k - 2$, there exists a set of k edge-disjoint spanners $S_1, S_2, \ldots, S_k$ of Q_d such that S_1 has delay at most $4k - 2$.*

Proof. We express Q_d as the product of two graphs $H = Q_{2k}$ and $G = Q_{d-2k}$. For our construction, we want a decomposition of H into a set of k Hamilton cycles $C_1, C_2, \ldots, C_k$, a distinguished vertex z of H, a 1-domatic coloring of G with k colors, and a decomposition of G into a set of $k-1$ edge-disjoint spanners $D_2, D_3, \ldots, D_k$.

The decomposition of H is possible by Lemma 3. The distinguished vertex z is chosen arbitrarily. For $k = 2$, constructing a 1-domatic coloring of G is trivial. For $k \geqslant 3$, this coloring can be constructed by the previous corollary, since $d - 2k \geqslant 2k - 2 \geqslant 2^{\lceil \log k \rceil} \geqslant k$. The decomposition of G can be done by obtaining a set of $k-1$ edge-disjoint Hamilton cycles and dispensing the remaining edges arbitrarily.

Now, we describe the construction of spanners $S_1, S_2, \ldots, S_k$ (see Fig. 3). Each copy of H in $H \times G$ corresponds to a vertex of a particular color i in the domatic coloring of G. To construct spanner S_i when $i > 1$, we include the edges of the cycle C_i in the copy of H corresponding to each vertex of G that is not colored i. In the remaining copies of H (those corresponding to a vertex of G colored i), we include the edges of C_1. To complete S_i, we include the edges of D_i in the copy of G corresponding to the distinguished vertex z. The spanner S_1 will contain all of the remaining edges of $H \times G$ not included in $S_2, S_3, \dots, S_k$. In particular, in each copy of H corresponding to a vertex of G colored i, the spanner S_1 will contain the edges of C_i, and all edges of each copy of G, except the copy of G corresponding to the vertex z. In this copy, S_1 contains no edges.

Consider a spanner S_i, $2 \leqslant i \leqslant k$, and two arbitrary vertices u and v. There is a path from u to v in S_i that starts at u, proceeds within a copy of H to a copy of z, then proceeds within a copy of G to another copy of z, and then proceeds within a copy of H to v. Thus, S_i is connected and, therefore, a spanner.

Now consider spanner S_1. We first show that the delay between an arbitrary pair of vertices u and v is at most $4k + 4$. If u is a copy of z, let u' be a vertex adjacent to u in S_1 and otherwise let $u' = u$. Similarly, if v is a copy of z, let v' be a vertex adjacent to v in S_1, and otherwise, let $v' = v$. Let $u' = [h_1, g_1]$ and $v' = [h_2, g_2]$. Let $w = [h_1, g_2]$, that is, w is a copy of u' in the copy of H containing v'. We call the copy H'. We will construct a path from u to v that commences at u, and passes through u', w, and v' in order and then arrives at v. The subpaths from u to u' and from v' to v each contain at most one edge. The subpath form u' to w can follow any shortest path between these two vertices in the copy of G containing them. The subpath from w to v' requires further elucidation. In H', either w is v', w is adjacent to v', or there are two vertex disjoint shortest paths from w to v'. Since w and v' are not copies of z, there is a shortest path P from w to v' in H' that does not contain the copy of z. Some of the edges of P may not be in S_1. Let $e = (x, y)$ be such an edge. In the cycle decomposition of H', e belongs to some cycle C_j.In the domatic coloring of G, there is a vertex of color j adjacent to the vertex corresponding to H'. Let x' and y' be the vertices corresponding to x and y, respectively, in the copy of H corresponding to this vertex of color j. By construction, (x', y') is in S_1 and since neither x nor y is the copy of z, both (x, x') and (y, y') are edges in S_1. We use the path (x, x', y', y) to replace the edge (x, y) in the path P. Performing this replacement for each such missing edge, we obtain a path P' (in S_1) from w to v' of length at most $3d_H(w, v')$.

We have constructed a path from u to u' to w to v' to v of length at most $1 + d_G(u', w) + 3d_H(w, v') + 1$. The distance between u and v in $H \times G$ is $d_G(u, v) + d_H(u, v) \geqslant d_G(u', w) + (d_H(w, v') - 2)$, giving delay at most $2d_H(w, v') + 4$. Since $H = Q_{2k}$, $d_H(w, v') \leqslant 2k$ and we obtain a simple bound on the delay in S_1 at most $4k + 4$.

We can improve this delay to $4k - 2$ by a careful consideration of cases.

If neither u nor v is a copy of z, then the distance between them in $H \times G$ is $d_G(u, v) + d_H(u, v)$. We consider two cases. If $d_H(w, v) = 2k$, we choose P to start with an edge in S_1. This means we can construct P' of length at most $3d_H(w, v) - 1) + 1$, and we have a path from u to v in S_1 of length at most $d_G(u, w) + 3d_H(w, v) - 2$. Thus, the delay is at most $2d_H(w, v) - 2 = 4k - 2$. If $d_H(w, v) \leqslant 2k - 1$, then the length of P' is at most $3d_H(w, v) \leqslant d_H(w, v) + 2(2k - 1)$ and the delay is at most $4k - 2$.

If exactly one of u and v is a copy of z, without loss of generality v, the distance between them in $H \times G$ is $d_G(u, w) + d_H(w, v)$. If $d_H(w, v) \geqslant 2k - 1$, then we choose v' such that the edge (v, v') is in S_1 and $d_H(w, v') = d_H(w, v) - 1$. This gives a path P' of length at most $d_H(w, v') + 2(2k - 1)$. Thus, the distance from u to v in S_1 is at most $d_G(u, w) + d_H(w, v') + 2(2k - 1) + 1 = d_G(u, w) + d_H(w, v) + 2(2k - 1)$, giving delay at most $4k-2$. If $d_H(w,v) \leqslant 2k-2$, then from the general construction, we obtain a delay of at most $4k-2$ rather than the simple bound above.

If both *u* and *v* are both copies of *z*, then the delay is at most $6 \leqslant 4k-2$. □

For larger *d*, we can use a similar idea to reduce the delay on S_1 even further.

Lemma 5. *For any* $k \geqslant 2$, $m \geqslant 2$, *and* $d \geqslant 2\binom{m+k-1}{m} + 2(m+k-2)$, *there exists a set of k edge-disjoint spanners* $S_1, S_2, \ldots, S_k$ *of* Q_d *such that* S_1 *has delay at most* $\max\{6, 2\lceil (2k-1)/m \rceil + 1\}$.

Proof. We proceed as in Lemma 4 expressing Q_d as the product of $H = Q_{2(m+k-1)}$ and $G = Q_{d-2(m+k-1)}$. We decompose *H* into a set of $m+k-1$ Hamilton cycles and arbitrarily choose a vertex *z*. We also construct a 1-domatic coloring of *G* with $\binom{m+k-1}{m}$ colors. Since $d-2(m+k-1) \geqslant 2\binom{m+k-1}{m} - 2$, this is possible. As before, we decompose *G* into $k-1$ edge-disjoint spanners. We associate each of the $\binom{m+k-1}{m}$ colors with a unique choice of *m* of the cycles in the decomposition of *H*.

In the copy of *H* corresponding to a vertex of a particular color of *G*, we place the edges of the cycles associated with that color into S_1. The remaining $k-1$ cycles in this copy of *H* are each placed into one of the $k-1$ spanners $S_2, S_3, \ldots, S_k$. The edges of the copies of *G* are placed into the spanners as in the proof of Lemma 4.

The analysis of the delay of S_1 is quite similar to the proof of Lemma 4 and we only point out the major differences. To construct a path from *w* to v' (two vertices in the same copy H' of *H*), we begin by taking as many edges of S_1 as possible in the direction of v' and not leading to z'. When no such further step is possible, we are at a vertex w' such that $d_{H'}(w', v') \leqslant 2k-1$. Consider a shortest path *P* from w' to v' in H' that does not contain a copy *z*. We divide *P* into $\lceil (|V(P)|-1)/m \rceil$ subpaths of length at most *m*. Each such subpath has edges from at most *m* of the cycles in the decomposition of *H*. Thus, by the domatic coloring, there is a copy H'' of *H* adjacent to H' in which all of the edges of this subpath are in spanner S_1. We replace this subpath with an edge to H'', the corresponding subpath in H'', and an edge back to H', encountering 2 units of delay. Since $|V(P)| \leqslant 2k$, there are at most $\lceil (2k-1)/m \rceil$ subpaths giving delay at most $2\lceil (2k-1)/m \rceil$. We get delay at most $2\lceil (2k-1)/m \rceil + 1$ when one of *u* and *v* is a copy of *z*, and delay 6 when both are. □

For a fixed number of spanners *k*, increasing the parameter *m* in the previous lemma leads to lower delay for spanner S_1, but a higher lower bound on the dimension *d*. This may be continued until $m = k$ at which point the delay is 6 and cannot be further decreased.

As promised, we now combine the previous lemmas with Corollary 2 to give the main results of this section. We use $G = Q_{2k-2}$ (which has a 1-domatic coloring with *k* colors) in Corollary 2, and $H = Q_{d-2k+2}$ from Lemma 4 or 5 to obtain Theorems 8 and 9, respectively.

Theorem 8. *For* $k \geqslant 2$ *and* $d \geqslant 6k-4$, $\mathrm{EDS}(Q_d, 48k \log k + 4k) \geqslant k$.

Theorem 9. *For* $k \geqslant 2$, $m \geqslant 2$, *and* $d \geqslant \binom{m+k-1}{m} + 2m + 4k - 4$,

$$\mathrm{EDS}\left(Q_d, 48k \log k + 2 + \max\left\{6, 2\left\lceil \frac{2k-1}{m} \right\rceil + 1\right\}\right) \geqslant k.$$

These theorems show that one can find k edge-disjoint spanners with delay $O(k \log k)$ in Q_d for sufficiently large d. In particular, beyond a certain dimension, the delay depends only on the number of spanners and not the size of the cube.

Lemma 6. $\mathrm{EDS}(Q_4, 4) = 2$.

Proof. Let us consider the following decomposition of Q_4 into two Hamilton cycles, see Fig. 4. One of the Hamilton cycles is depicted in bold edges and the another in dotted edges. It is only a time-consuming exercise to check that both these Hamilton cycles are spanners of delay 4 in Q_4. Since every spanner of Q_4 must have at least 15 edges, and Q_4 has only 32 edges, $\mathrm{EDS}(Q_4, 4) \leqslant 2$. □

Theorem 10. *For* $d \geqslant 6$, $\mathrm{EDS}(Q_d, 18) \geqslant 3$.

Proof. For brevity, we here outline the general method of constructing three spanners of delay 18 in Q_6. The full details of the construction are given in the appendix. After the sketch of the proof for Q_6, we describe how to extend the construction to higher dimensions.

We view Q_6 as four copies H_1, H_2, H_3, H_4 of Q_4 interconnected by four sets of edges. We decompose each H_i into a Hamilton cycle and two matchings. Spanners S_1 and S_2 contain Hamilton cycles in H_1 and H_2, respectively, and each contains three matchings, one each in the remaining H_i's. Spanner S_3 includes the remaining two Hamilton cycles and two matchings.

Each of the four sets of interconnecting edges is divided in half. To do this, we 2-color each Q_4 in the same manner. An interconnecting edge is placed in one subset if its ends are colored 1 and placed in the other subset otherwise. The two subsets of each set of interconnecting edges will be assigned to two different spanners. In particular, we give the

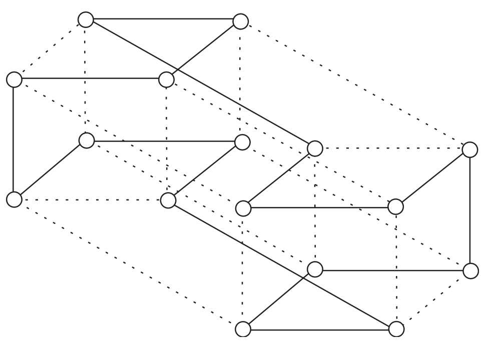

Fig. 4. Two Hamilton cycles in Q_4.

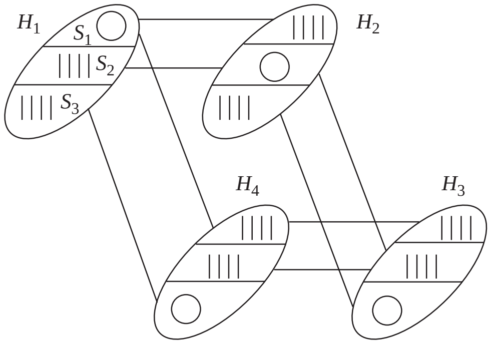

Fig. 5. Construction of three spanners of Q_6. The four ovals represent the four copies of Q_4. Each oval is divided into three parts. The top part represents edges of the spanner S_1, the middle part edges of S_2 and the bottom part edges of S_3. If a spanner S_i includes edges of the Hamilton cycle in the copy H_j, the corresponding part is marked with a circle. If a spanner S_i includes edges of a matching in the copy H_j, the corresponding part is marked with four vertical lines.

edges between H_1 and H_2 to S_1 and S_2, the edges between H_2 and H_3 to S_2 and S_3, the edges between H_3 and H_4 to S_1 and S_2, and the edges between H_4 and H_1 to S_1 and S_3, see Fig. 5.

At this point, S_1 and S_2 are connected and each contains a single cycle. The subgraph S_3, however, consists of two components, each with one cycle. To ensure that all three subgraphs are spanners, we may exchange one or more of the interconnecting edges of S_1 between H_3 and H_4 with an equal number of edges of S_3 in the Hamilton cycle of H_4. At this point, each S_i is connected and contains one cycle. In particular, each S_i contains the Hamilton cycle in H_i.

If we ignore the delays introduced by the edges exchanged between S_1 and S_3, we can easily obtain a rough estimate of the delay between any two vertices u and v in S_1 or S_2. There is a path from u to v in S_i $(i = 1, 2)$ that consists of three sections: one from u to the cycle in H_i, one around the cycle, and one from the cycle to v. To get from u to the cycle takes at most four edges, going around the cycle takes at most half the cycle length (eight edges), and to get from the cycle to v takes at most four more edges. This is a total distance of at most 16, a delay at most 14 between u and v. The actual analysis of S_1 and S_3 must take into account the exchanged edges. This analysis is tedious and contains no insight and is thus omitted here. By careful choice of the decomposition of each H_i, which matchings to assign to each spanner, which subsets of interconnecting edges to assign to each spanner, and which edges to exchange, we may obtain a set of spanners with maximum delay 18. This construction has been verified by computer and the details of the construction are given in the appendix.

To extend this construction to higher dimensions, we start with the three spanners S_1, S_2, and S_3 in Q_6 as described above. Again, we view Q_6 as four copies of Q_4, and we color the vertices of H_1 with color 1, H_2 with color 2, and $H_3 \cup H_4$ with color 3. To construct three spanners S_1', S_2', and S_3' of Q_d for $d > 6$, we group the lower 6 dimensions and the

Table 1a
Upper bounds on the delay of k spanners in Q_d for small k and d

d/k	1	2	3	4	5	6	7	8	9	10	11	12
4	0	4										
5	0	4										
6	0	4	18									
7	0	4	18									
8	0	4	18	126								
9	0	4	18	138								
10	0	4	18	138	510							
11	0	4	18	138	524							
12	0	4	18	56	524	2046						
13	0	4	18	56	524	2064						
14	0	4	18	56	272	2064	8190					
15	0	4	18	56	272	2064	8210					
16	0	4	18	56	272	380	8210	32766				
17	0	4	18	56	272	398	8210	32790				
18	0	4	18	56	132	398	1148	32790	131070			
19	0	4	18	56	132	398	1160	32790	131096			
20	0	4	18	56	132	310	1160	1532	131096	524286		
21	0	4	18	56	132	310	1160	1556	131096	524316		
22	0	4	18	56	132	310	672	1556	4604	524316	2097150	
23	0	4	18	56	132	310	672	1556	4618	524316	2097182	
24	0	4	18	56	132	170	672	888	4618	6140	2097182	8388606
25	0	4	18	56	132	170	672	900	4618	6170	2097182	8388642

Table 1b
Results used to get the corresponding entries in Table 1a

d/k	1	2	3	4	5	6	7	8	9	10	11	12
4	Trivial	Lemma 6										
5	Trivial	Theorem 3										
6	Trivial	Theorem 3	Theorem 10									
7	Trivial	Theorem 3	Theorem 10									
8	Trivial	Theorem 3	Theorem 10	Lemma 7								
9	Trivial	Theorem 3	Theorem 10	Corollary 1 4Q8								
10	Trivial	Theorem 3	Theorem 10	Corollary 1 4Q8	Lemma 7							
11	Trivial	Theorem 3	Theorem 10	Corollary 1	Corollary 1							
	Trivial	Theorem 3	Theorem 10	4Q8	5Q10							
12	Trivial	Theorem 3	Theorem 10	Theorem 6 3Q6, 3Q6	Corollary 1 5Q10	Lemma 7						
13	Trivial	Theorem 3	Theorem 10	Theorem 6 3Q6, 3Q7	Corollary 1 5Q10	Corollary 1 6Q12						
14	Trivial	Theorem 3	Theorem 10	Theorem 6 3Q6, 3Q8	Theorem 6 3Q6, 4Q8	Corollary 1 6Q12	Lemma 7					
15	Trivial	Theorem 3	Theorem 10	Theorem 6 3Q6, 3Q9	Theorem 6 3Q7, 4Q8	Corollary 1 6Q12	Corollary 1 7Q14					

16	Trivial	Theorem 3	Theorem 10	Theorem 6	Theorem 6	Theorem 6	Corollary 1	Lemma 7				
				3Q6, 3Q10	3Q8, 4Q8	4Q8, 4Q8	7Q14					
17	Trivial	Theorem 3	Theorem 10	Theorem 6	Theorem 6	Corollary 1	Corollary 1	Corollary 1				
				3Q6, 3Q11	4Q8, 3Q9	6Q16	7Q14	8Q16				
18	Trivial	Theorem 3	Theorem 10	Theorem 6	Theorem 6	Corollary 1	Theorem 6	Corollary 1	Lemma 7			
				3Q6, 3Q12	3Q6, 4Q12	6Q16	4Q8, 5Q10	8Q16				
19	Trivial	Theorem 3	Theorem 10	Theoresm 6	Theorem 6	Corollary 1	Theorem 6	Corollary 1	Corollary 1			
				3Q6, 3Q13	3Q6, 4Q13	6Q16	4Q9, 5Q10	8Q16	9Q18			
20	Trivial	Theorem 3	Theorem 10	Theorem 6	Theorem 6	Theorem 6	Theorem 6	Theorem 6	Corollary 1	Lemma 7		
				3Q6, 3Q14	3Q6, 4Q14, 56	4Q8, 4Q12	4Q10, 5Q10	5Q10, 5Q10	9Q18			
21	Trivial	Theorem 3	Theorem 10	Theorem 6	Theorem 6	Theorem 6	Theorem 6	Corollary 1	Corollary 1	Corollary 1		
				3Q6, 3Q15	3Q6, 4Q15	4Q8, 4Q13	5Q10, 4Q11	8Q20	9Q18	10Q20		
22	Trivial	Theorem 3	Theorem 10	Theorem 6	Theorem 6	Theorem 6	Theorem 6	Corollary 1	Corollary 1	Corollary 1	Lemma 7	
				3Q6, 3Q16	3Q6, 4Q16	4Q8, 4Q14	4Q8, 5Q14	8Q20	9Q18	10Q20		
23	Trivial	Theorem 3	Theorem 10	Theorem 6	Theorem 6	Theorem 6	Theorem 6	Corollary 1	Theorem 6	Corollary 1	Corollary 1	
				3Q6, 3Q17	3Q6, 4Q17	4Q8, 4Q15	4Q8, 5Q15	8Q20	5Q11, 6Q12	10Q20	11Q22	
24	Trivial	Theorem 3	Theorem 10	Theorem 6	Theorem 6	Theorem 6	Theorem 6	Theorem 6	Theorem 6	Theorem 6	Corollary 1	Lemma 7
				3Q6, 3Q18	3Q6, 4Q18	4Q12, 4Q12	4Q8, 5Q16	4Q8, 6Q16	5Q12, 6Q12	5Q12, 6Q12	11Q22	
25	Trivial	Theorem 3	Theorem 10	Theorem 6	Theorem 6	Theorem 6	Theorem 6	Theorem 6	Theorem 6	Corollary 1	Corollary 1	Corollary 1
				3Q6, 3Q19	3Q6, 4Q19	4Q12, 4Q13	4Q8, 5Q17	4Q9, 6Q16	6Q12, 5Q13	10Q24	11Q22	12Q24

upper $d-6$ dimensions and view Q_d as $Q_6 \times Q_{d-6}$. We then let each S_i' be the color-i product $S_i \underset{i}{\times} Q_{d-6}$. We now establish that each of the spanners S_1', S_2', and S_3' has delay at most 18. Consider two vertices u and v in S_i'. If u and v do not differ in any of the upper $d-6$ dimensions, then the delay between them is at most 18 by the construction above.

If u and v differ in the upper dimensions, consider a shortest path P from u to v' where v' has the same lower coordinates as v and upper coordinates as u. If P includes a vertex w of color i, then we may construct a path from u to v by following P from u to w, following edges in the upper dimensions as necessary, and then following the remainder of P projected to the copy of Q_6 containing v. Since there is no delay encountered in travelling the upper dimensions, this path has delay at most 18.

If P does not include a vertex of color i, then let w be the closest vertex of color i to u. Observe that w is also the closest vertex of color i to v'. Based on the sketch above, w is within distance 6 of both u and v'. For the exact construction presented in the appendix, these distances are at most 5. The path from u to w followed by the necessary upper dimension edges to a vertex w' and then to v has delay at most 12. □

Lemma 7. *For* $d \geqslant 2$, $\mathrm{EDS}(Q_d, 2^{d-1}-2) = \lfloor d/2 \rfloor$.

Proof. This follows from the fact that $\lfloor d/2 \rfloor$ Hamilton cycles can be found in Q_d, and the fact that the delay of any spanner of any bipartite graph must be even. □

We conclude this section with Table 1a which shows a lower bound on the delay for a set of k spanners in Q_d. These bounds were obtained by the application of various results from this paper. For each value entered in Table 1a, the corresponding entry in Table 1b indicates how the value was obtained. The bottom line indicates particular set of edge-disjoint spanners used. For example, the string aQb, cQd indicates a spanners of Q_b and c spanners of Q_d.

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Appendix

In this appendix, we include a description of a set of three edge-disjoint spanners of Q_6. These spanners have delay 14, 14, and 18. We view the set of spanners as a 3-coloring of edges and describe the construction in Table 2 by listing the label of every vertex along with the colors of its incident edges in order of increasing dimension i. For example, vertex 000000 has edges in dimension 4 and 5 in spanner S_1, in dimensions 1 and 3 in spanner S_2, and in dimension 2 and 6 in spanner S_3. The delay of each of these spanners has been verified by computer.

Table 2
Representation of three spanners in Q_6

000000	2	3	2	1	1	3	010000	3	2	1	1	1	2	100000	3	3	2	1	2	3	110000	1	3	2	2	2	2
000001	2	2	3	1	3	1	010001	3	1	2	1	3	1	100001	3	2	3	1	3	1	110001	1	2	3	2	3	1
000010	3	3	2	1	3	1	010010	3	2	1	1	3	1	100010	3	3	2	1	3	1	110010	1	3	2	2	3	1
000011	3	2	1	3	1	2	010011	3	1	1	2	1	2	100011	3	2	1	3	2	2	110011	1	2	2	3	2	2
000100	3	3	2	1	3	1	010100	3	2	1	1	3	1	100100	3	3	2	1	3	1	110100	1	3	2	2	3	1
000101	3	2	3	1	1	2	010101	3	1	2	1	1	2	100101	3	2	3	1	2	2	110101	1	2	3	2	2	2
000110	2	3	2	1	1	3	010110	3	2	1	1	1	2	100110	3	3	2	1	2	3	110110	1	3	2	2	2	2
000111	2	2	1	3	3	1	010111	3	1	1	2	3	1	100111	3	2	1	3	3	1	110111	1	2	2	3	3	1
001000	3	2	3	1	3	1	011000	3	1	2	1	3	1	101000	3	2	3	1	3	1	111000	1	2	3	2	3	1
001001	3	2	3	1	1	2	011001	3	1	2	1	1	2	101001	3	2	3	1	2	2	111001	1	2	3	2	2	2
001010	2	2	3	1	1	3	011010	3	1	2	1	1	2	101010	3	2	3	1	2	3	111010	1	2	3	2	2	2
001011	2	2	1	3	3	1	011011	3	1	1	2	3	1	101011	3	2	1	3	3	1	111011	1	2	2	3	3	1
001100	2	2	3	1	1	3	011100	1	3	2	1	1	2	101100	2	3	3	1	2	3	111100	2	1	3	2	2	2
001101	2	3	3	1	3	1	011101	1	3	2	1	3	1	101101	2	3	3	1	3	1	111101	2	1	3	2	3	1
001110	2	2	3	1	3	1	011110	1	3	2	1	3	1	101110	2	3	3	1	3	1	111110	2	1	3	2	3	1
001111	2	3	1	3	1	2	011111	1	3	1	2	1	2	101111	2	3	1	3	2	2	111111	2	1	2	3	2	2

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