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On a system of Klein–Gordon type equations with acoustic boundary conditions

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Abstract

We prove the existence, uniqueness and uniform stabilization of global solutions for a generalized system of Klein–Gordon type equations with acoustic boundary conditions on a portion of the boundary and the Dirichlet boundary condition on the rest.

1. Introduction

This paper is concerned with the existence, uniqueness and asymptotic behavior of solutions to the $k \times k$ system of Klein–Gordon type equations

$$\begin{cases} u_1'' - \Delta u_1 + \alpha_1 u_1 + a_{12} u_1 u_2^2 + a_{13} u_1 u_3^2 + \cdots + a_{1k} u_1 u_k^2 = f_1, \\ u_2'' - \Delta u_2 + \alpha_2 u_2 + a_{21} u_2 u_1^2 + a_{23} u_2 u_3^2 + \cdots + a_{2k} u_2 u_k^2 = f_2, \\ \quad \vdots \\ u_k'' - \Delta u_k + \alpha_k u_k + a_{k1} u_k u_1^2 + a_{k2} u_k u_2^2 + \cdots + a_{k(k-1)} u_k u_{k-1}^2 = f_k \end{cases} \tag{1.1}$$

in $\Omega \times (0, \infty)$ with the boundary conditions

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$$u_i = 0, \qquad \text{on } \Gamma_0 \times (0, \infty),\ 1 \leqslant i \leqslant k, \tag{1.2}$$

$$u_i' + p_i z_i'' + l_i z_i' + r_i z_i = 0, \qquad \text{on } \Gamma_1 \times (0, \infty),\ 1 \leqslant i \leqslant k, \tag{1.3}$$

$$\frac{\partial u_i}{\partial \nu} = z_i', \qquad \text{on } \Gamma_1 \times (0, \infty),\ 1 \leqslant i \leqslant k, \tag{1.4}$$

and the initial conditions

$$u_i(x, 0) = \phi_i(x), \quad u_i'(x, 0) = \psi_i(x), \quad x \in \Omega,\ 1 \leqslant i \leqslant k, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, open and connected set with smooth boundary Γ; Γ_0 and Γ_1 are subsets of Γ with positive measures such that $\Gamma = \Gamma_0 \cup \Gamma_1$; α_i $(1 \leqslant i \leqslant k)$ and a_{ij} $(1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant k)$ are nonnegative constants such that $a_{ij} = a_{ji}$ and $a_{ii} = 0$; $f_i : \Omega \times (0, \infty) \to \mathbb{R}$; $p_i : \Gamma_1 \to \mathbb{R}$; $l_i : \Gamma_1 \to \mathbb{R}$; $r_i : \Gamma_1 \to \mathbb{R}$; $\phi_i : \Omega \to \mathbb{R}$; $\psi_i : \Omega \to \mathbb{R}$ $(1 \leqslant i \leqslant k)$ are given functions; $\nu(x)$ is the outward unit normal vector on Γ and $' = \partial/\partial t$.

The $k \times k$ system (1.1) deals with the general fourth order potential energy for scalar fields, for instance, the O(N)-symmetric vector model and the SU(N)-symmetric Hermitian matrix model; see [13] and [16].

The boundary conditions (1.3) and (1.4) on the portion Γ_1 of the boundary Γ, called acoustic boundary conditions, were introduced by Beale and Rosencrans [4], see also [2,3]. They studied spectral properties for the linear scalar hyperbolic equation and proved that if $p_i > 0$ $(i = 1)$, then there is no uniform rate of decay for solutions of the initial value problem even if $l_i > 0$ $(i = 1)$ everywhere on Γ. Similar boundary conditions for a system of two one-dimensional quasilinear hyperbolic equations of first order considered Alber and Cooper in [1] and proved that the presence of the second derivative z'' in acoustic boundary conditions makes a solution to blow up.

Hyperbolic problems with nonlinear feedback on a part of the boundary were studied by Lasiecka [8], Komornik and Zuazua [7], Zuazua [17]. They proved that a dissipative boundary feedback ensures a uniform energy decay.

When $k = 2$, $\alpha_1 = \alpha^2$, $\alpha_2 = \gamma^2$ and $a_{12} = a_{21} = \theta^2$, the system (1.1) reduces to the following 2×2 system:

$$\begin{cases} u_1'' - \Delta u_1 + \alpha^2 u_1 + \theta^2 u_1 u_2^2 = f_1, \\ u_2'' - \Delta u_2 + \gamma^2 u_2 + \theta^2 u_1^2 u_2 = f_2, \end{cases}$$

proposed by Segal [12], as a model to describe the interaction of classical electromagnectic fields u_1, u_2 with masses α, γ, respectively, and the interaction constant θ.

The mixed problem for (1.1), when $k = 2$, $\alpha_1 = \alpha_2 = 0$, and $a_{12} = a_{21} = 1$ with Dirichlet boundary condition on Γ, was studied by Medeiros and Menzala [11], where the authors proved the existence and uniqueness of global weak solutions provided that $n \leqslant 3$. Nonlinear wave equations with acoustic boundary conditions were studied in [5,6].

Here is an outline of our paper. First we prove the existence of global weak solutions to (1.1)–(1.5) with no restrictions on the dimension n. Next, assuming $n \leqslant 3$, we prove the existence and uniqueness of global strong solutions to (1.1)–(1.5). In the last section we prove the exponential decay of the energy when $f_i = p_i = 0$, $1 \leqslant i \leqslant k$. In this case the acoustic boundary conditions provide an effect of a dissipative feedback on the boundary similar to those studied in [7,8,17].

2. Notation and auxiliary results

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and connected set with a smooth boundary Γ. Suppose $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are measurable subsets of Γ with positive measures. T is a positive real number, $Q = \Omega \times (0,T)$, $\Sigma_0 = \Gamma_0 \times (0,T)$ and $\Sigma_1 = \Gamma_1 \times (0,T)$. For the standard functional spaces $L^p(\Omega)$, $H^m(\Omega)$, $H_0^m(\Omega)$ and $L^p(0,T;X)$ we employ the usual notations as in [9,10]. In order to write (1.1)–(1.5) in a compact form, we use vector notations as in [15]. The inner product and norm in $\boldsymbol{L}^2(\boldsymbol{\Omega})$ are denoted by

$$(\boldsymbol{U},\boldsymbol{V}) = \sum_{i=1}^{k} \int_{\Omega} u_i(x) v_i(x)\,dx \quad \text{and} \quad |\boldsymbol{U}| = \left(\sum_{i=1}^{k} \int_{\Omega} \big(u_i(x)\big)^2\,dx \right)^{1/2}.$$

Similarly, for $\boldsymbol{L}^2(\boldsymbol{\Gamma})$ we write

$$(\boldsymbol{Z},\boldsymbol{S})_\Gamma = \sum_{i=1}^{k} \int_{\Gamma} z_i(x) s_i(x)\,d\Gamma \quad \text{and} \quad |\boldsymbol{Z}|_\Gamma = \left(\sum_{i=1}^{k} \int_{\Gamma} \big(z_i(x)\big)^2\,d\Gamma \right)^{1/2}.$$

We denote the Hilbert space $H(\Delta,\Omega) = \{u \in H^1(\Omega);\ \Delta u \in L^2(\Omega)\}$ provided with the norm $\|u\|_{\Delta,\Omega} = (\|u\|^2_{H^1(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)})^{1/2}$, where $H^1(\Omega)$ is the usual real Sobolev space of first order. The maps $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ and $\gamma_1 : H(\Delta,\Omega) \to H^{-1/2}(\Gamma)$ are the trace map of order zero and the Neumann trace map on $H(\Delta,\Omega)$, respectively. Therefore, $\boldsymbol{\gamma}_0 : \boldsymbol{H}^1(\boldsymbol{\Omega}) \to \boldsymbol{H}^{1/2}(\boldsymbol{\Gamma})$ and $\boldsymbol{\gamma}_1 : \boldsymbol{H}(\boldsymbol{\Delta},\boldsymbol{\Omega}) \to \boldsymbol{H}^{-1/2}(\boldsymbol{\Gamma})$, where

$$\boldsymbol{\gamma}_0(\boldsymbol{U}) = \big(\gamma_0(u_1),\dots,\gamma_0(u_k)\big) \quad \text{and} \quad \boldsymbol{\gamma}_1(\boldsymbol{U}) = \big(\gamma_1(u_1),\dots,\gamma_1(u_k)\big).$$

We consider $\boldsymbol{\mathcal{H}} = \{\boldsymbol{U} = (u_1,\dots,u_k) \in \boldsymbol{H}^1(\boldsymbol{\Omega})$ such that $\boldsymbol{\gamma}_0(\boldsymbol{U}) = \boldsymbol{0}$ a.e. on $\Gamma_0\}$. Then $\boldsymbol{\mathcal{H}}$ is a closed subspace of $\boldsymbol{H}^1(\boldsymbol{\Omega})$, the Poincaré inequality holds on $\boldsymbol{\mathcal{H}}$, and the inner product and norm on $\boldsymbol{\mathcal{H}}$ are denoted by

$$((\boldsymbol{U},\boldsymbol{V})) = \sum_{i=1}^{k} \sum_{j=1}^{n} \int_{\Omega} \frac{\partial u_i}{\partial x_j}(x) \frac{\partial v_i}{\partial x_j}(x)\,dx,$$

$$\|\boldsymbol{U}\| = \left(\sum_{i=1}^{k} \sum_{j=1}^{n} \int_{\Omega} \left(\frac{\partial u_i}{\partial x_j}(x) \right)^2 dx \right)^{1/2}.$$

Now we write

$$\boldsymbol{U} = (u_1,\dots,u_k) : Q \to \mathbb{R}^k, \qquad \boldsymbol{Z} = (z_1,\dots,z_k) : \Sigma_1 \to \mathbb{R}^k,$$
$$\boldsymbol{\Phi} = (\phi_1,\dots,\phi_k) : \Omega \to \mathbb{R}^k, \qquad \boldsymbol{\Psi} = (\psi_1,\dots,\psi_k) : \Omega \to \mathbb{R}^k,$$
$$\boldsymbol{F} = (f_1,\dots,f_k) : Q \to \mathbb{R}^k,$$
$$\boldsymbol{P} = \begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_k \end{pmatrix}, \qquad \boldsymbol{L} = \begin{pmatrix} l_1 & 0 & \dots & 0 \\ 0 & l_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & l_k \end{pmatrix},$$

$$\boldsymbol{R}=\begin{pmatrix} r_1 & 0 & \dots & 0\\ 0 & r_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & r_k\end{pmatrix},$$

$$\boldsymbol{G}=(g_1,\dots,g_k):\mathbb{R}^k\to\mathbb{R}^k,\quad \text{where } g_i(Y)=\alpha_i y_i+\sum_{j=1}^{k} a_{ij}y_iy_j^2,$$

$$\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{\nu}}=\left(\frac{\partial u_1}{\partial \nu},\dots,\frac{\partial u_k}{\partial \nu}\right)\quad \text{and}\quad \boldsymbol{\Delta U}=(\Delta u_1,\dots,\Delta u_k).$$

Whence the problem (1.1)–(1.5) can be rewritten as follows:

$$\boldsymbol{U}''-\boldsymbol{\Delta U}+\boldsymbol{G}(\boldsymbol{U})=\boldsymbol{F}\quad \text{in } Q, \tag{2.1}$$

$$\boldsymbol{U}=\boldsymbol{0}\quad \text{on } \Sigma_0, \tag{2.2}$$

$$\boldsymbol{U}'+\boldsymbol{P}\boldsymbol{Z}''+\boldsymbol{L}\boldsymbol{Z}'+\boldsymbol{R}\boldsymbol{Z}=\boldsymbol{0}\quad \text{on } \Sigma_1, \tag{2.3}$$

$$\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{\nu}}=\boldsymbol{Z}'\quad \text{on } \Sigma_1, \tag{2.4}$$

$$\boldsymbol{U}(x,0)=\boldsymbol{\Psi}(x),\quad \boldsymbol{U}'(x,0)=\boldsymbol{\Psi}(x),\quad x\in\Omega. \tag{2.5}$$

Definition 2.1. A pair of function $(\boldsymbol{U}(x,t),\boldsymbol{Z}(x,t))$, where $\boldsymbol{U}=(u_1,\dots,u_k):Q\to\mathbb{R}^k$ and $\boldsymbol{Z}=(z_1,\dots,z_k):\Sigma_1\to\mathbb{R}^k$, is a weak solution to (2.1)–(2.5) if $(\boldsymbol{U},\boldsymbol{Z})$ satisfies

$$\boldsymbol{U}\in L^\infty(0,T;\boldsymbol{\mathcal{H}}),\qquad \boldsymbol{U}'\in L^\infty\big(0,T;\boldsymbol{L}^2(\boldsymbol{\Omega})\big),$$
$$\boldsymbol{Z},\boldsymbol{Z}'\in L^\infty\big(0,T;\boldsymbol{L}^2(\boldsymbol{\Gamma}_1)\big), \tag{2.6}$$

$$\frac{d}{dt}\big(\boldsymbol{U}'(t),\boldsymbol{V}\big)+\big(\big(\boldsymbol{U}(t),\boldsymbol{V}\big)\big)-\big(\boldsymbol{Z}(t),\boldsymbol{\gamma}_0(\boldsymbol{V})\big)_{\Gamma_1}+\big(\boldsymbol{G}\big(\boldsymbol{U}(t)\big),\boldsymbol{V}\big)=\big(\boldsymbol{F}(t),\boldsymbol{V}\big)$$
$$\text{for all } \boldsymbol{V}\in\big(\boldsymbol{\mathcal{H}}\cap\boldsymbol{L}^\infty(\Omega)\big),\ \text{in the sense of } \mathcal{D}'(0,T), \tag{2.7}$$

$$\frac{d}{dt}\big(\boldsymbol{\gamma}_0\big(\boldsymbol{U}(t)\big)+\boldsymbol{P}\boldsymbol{Z}'(t),\boldsymbol{E}\big)_{\Gamma_1}+\big(\boldsymbol{L}\boldsymbol{Z}'(t)+\boldsymbol{R}\boldsymbol{Z}(t),\boldsymbol{E}\big)_{\Gamma_1}=0$$
$$\text{for all } \boldsymbol{E}\in\boldsymbol{L}^2(\boldsymbol{\Gamma}_1),\ \text{in the sense of } \mathcal{D}'(0,T), \tag{2.8}$$

$$\boldsymbol{U}(0)=\boldsymbol{\Phi},\qquad \boldsymbol{U}'(0)=\boldsymbol{\Psi}. \tag{2.9}$$

Remark 2.1. We observe that $T>0$ is any fixed number. Thus we are dealing with global solutions.

The following lemma due to Strauss (see [14]) plays an important role in the proof of the existence of global weak solutions; see Theorem 3.1 in the next section.

Lemma 2.1. *Let S be an open set of $\mathbb{R}^n$ with finite measure and let $(\boldsymbol{U}_m)_{m\in\mathbb{N}}$ be a sequence of measurable functions from S into $\mathbb{R}^k$. Assume that $g:\mathbb{R}^k\to\mathbb{R}$ and $h:\mathbb{R}^k\to\mathbb{R}$ satisfy the three conditions*:

(i) *If* $|h(\boldsymbol{U}_m(z))| \leqslant C$ *for all* $z \in S$ *and* $m \in \mathbb{N}$, *then there exists* $M > 0$ *such that* $|g(\boldsymbol{U}_m(z))| < M$ *for all* $z \in S$ *and* $m \in \mathbb{N}$.

(ii) $g(\boldsymbol{U}_m) : S \to \mathbb{R}$ *and* $h(\boldsymbol{U}_m) : S \to \mathbb{R}$ *are measurable and there exists a constant* $C > 0$ *such that*

$$\int_S \left|g\big(\boldsymbol{U}_m(z)\big)\right| \left|h\big(\boldsymbol{U}_m(z)\big)\right| dz \leqslant C.$$

(iii) $g(\boldsymbol{U}_m) \to v$ *a.e. on* S.

Then the function $v \in L^1(S)$ *and* $g(\boldsymbol{U}_m) \to v$ *strongly in* $L^1(S)$.

Definition 2.2. A pair of functions $(\boldsymbol{U}(x,t), \boldsymbol{Z}(x,t))$, where $\boldsymbol{U} = (u_1, \dots, u_k) : Q \to \mathbb{R}^k$ and $\boldsymbol{Z} = (z_1, \dots, z_k) : \Sigma_1 \to \mathbb{R}^k$, is a strong solution to (2.1)–(2.5) if $(\boldsymbol{U}, \boldsymbol{Z})$ satisfies

$$\boldsymbol{U}, \boldsymbol{U}' \in L^\infty(0, T; \boldsymbol{\mathcal{H}}), \quad \boldsymbol{U}'' \in L^\infty\big(0, T; \boldsymbol{L}^2(\boldsymbol{\Omega})\big), \quad \boldsymbol{U}(t) \in \boldsymbol{H}(\boldsymbol{\Delta}, \boldsymbol{\Omega}) \text{ a.e. on } [0, T], \tag{2.10}$$

$$\boldsymbol{Z}, \boldsymbol{Z}' \in L^\infty\big(0, T; \boldsymbol{L}^2(\boldsymbol{\Gamma}_1)\big) \quad \text{and} \quad \boldsymbol{Z}'' \in L^2\big(0, T; \boldsymbol{L}^2(\boldsymbol{\Gamma}_1)\big), \tag{2.11}$$

$$\boldsymbol{U}'' - \boldsymbol{\Delta U} + \boldsymbol{G}(\boldsymbol{U}) = \boldsymbol{F} \quad \text{a.e. on } Q, \tag{2.12}$$

$$\boldsymbol{\gamma}_0(\boldsymbol{U}') + \boldsymbol{P Z}'' + \boldsymbol{L Z}' + \boldsymbol{R Z} = \boldsymbol{0} \quad \text{a.e. on } \Sigma_1, \tag{2.13}$$

$$\big\langle \boldsymbol{\gamma}_1\big(\boldsymbol{U}(t)\big), \boldsymbol{\gamma}_0(\boldsymbol{V})\big\rangle_{\boldsymbol{H}^{-1/2}(\boldsymbol{\Gamma}) \times \boldsymbol{H}^{1/2}(\boldsymbol{\Gamma})} = \big(\boldsymbol{Z}'(t), \boldsymbol{\gamma}_0(\boldsymbol{V})\big)_{\Gamma_1} \text{ for all } \boldsymbol{V} \in \boldsymbol{H} \text{ a.e. on } [0, T], \tag{2.14}$$

$$\boldsymbol{U}(0) = \boldsymbol{\Phi}, \quad \boldsymbol{U}'(0) = \boldsymbol{\Psi} \quad \text{a.e. on } \Omega. \tag{2.15}$$

3. Existence results

In this section we prove global solvability of the problem (2.1)–(2.5). First we prove the existence of weak solutions.

Theorem 3.1. *Let* $p_i, l_i, r_i \in C(\bar{\Gamma}_1)$, $1 \leqslant i \leqslant k$, *be given such that*

$$p_i(x) \geqslant 0, \qquad l_i(x) > 0, \quad \textit{and} \quad r_i(x) \geqslant 0 \quad \textit{for all } x \in \bar{\Gamma}_1. \tag{3.1}$$

If $\boldsymbol{\Phi} \in \boldsymbol{\mathcal{H}} \cap \boldsymbol{L}^4(\boldsymbol{\Omega})$, $\boldsymbol{\Psi} \in \boldsymbol{L}^2(\boldsymbol{\Omega})$ *and* $\boldsymbol{F} \in L^2(0, T; \boldsymbol{L}^2(\boldsymbol{\Omega}))$, *then there exists a pair of functions* $(\boldsymbol{U}, \boldsymbol{Z})$ *which is a weak solution to* (2.1)–(2.5).

Proof. Let $(\boldsymbol{W}_j)_{j \in \mathbb{N}}$, $(\boldsymbol{E}_j)_{j \in \mathbb{N}}$ be orthonormal bases in $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{L}^2(\boldsymbol{\Gamma}_1)$. Since the boundary Γ is sufficiently smooth, we have that $\boldsymbol{W}_j \in \boldsymbol{\mathcal{H}} \cap \boldsymbol{L}^\infty(\boldsymbol{\Omega})$ for all $j \in \mathbb{N}$. For each $m \in \mathbb{N}$ we consider $\boldsymbol{U}_m : \Omega \times [0, T_m] \to \mathbb{R}^k$ and $\boldsymbol{Z}_m : \Gamma_1 \times [0, T_m] \to \mathbb{R}^k$ defined by

$$\boldsymbol{U}_m(x,t) = \big(u_{1m}(x,t), \dots, u_{km}(x,t)\big) = \sum_{j=1}^m \beta_{jm}(t) \boldsymbol{W}_j(x),$$

$$\boldsymbol{Z}_m(x,t) = \big(z_{1m}(x,t), \dots, z_{km}(x,t)\big) = \sum_{j=1}^m \eta_{jm}(t) \boldsymbol{E}_j(x),$$

which are solutions to the approximate problem

$$\big(\boldsymbol{U}_m''(t)+\boldsymbol{G}\big(\boldsymbol{U}_m(t)\big)-\boldsymbol{F}(t),\boldsymbol{W}_j\big)+\big(\big(\boldsymbol{U}_m(t),\boldsymbol{W}_j\big)\big)-\big(\boldsymbol{Z}_m'(t),\boldsymbol{\gamma}_0(\boldsymbol{W}_j)\big)_{\Gamma_1}=0, \tag{3.2}$$

$$\big(\boldsymbol{\gamma}_0\big(\boldsymbol{U}_m'(t)\big)+\boldsymbol{P}\boldsymbol{Z}_m''(t)+\boldsymbol{L}\boldsymbol{Z}_m'(t)+\boldsymbol{R}\boldsymbol{Z}_m(t),\boldsymbol{E}_j\big)_{\Gamma_1}=0,\quad 1\leqslant j\leqslant m, \tag{3.3}$$

$$\boldsymbol{U}_m(0)=\boldsymbol{\Phi}_m=\sum_{j=1}^{m}(\boldsymbol{\Phi},\boldsymbol{W}_j)\boldsymbol{W}_j,\qquad \boldsymbol{U}_m'(0)=\boldsymbol{\Psi}_m=\sum_{j=1}^{m}(\boldsymbol{\Psi},\boldsymbol{W}_j)\boldsymbol{W}_j, \tag{3.4}$$

$$\boldsymbol{Z}_m(0)=\boldsymbol{Z}_0,\qquad \boldsymbol{Z}_m'(0)=\boldsymbol{\gamma}_1(\boldsymbol{\Phi}_m). \tag{3.5}$$

Here $\boldsymbol{Z}_0\in \boldsymbol{L}^2(\boldsymbol{\Gamma}_1)$ is an arbitrary fixed vector. The local existence of such solutions $(\boldsymbol{U}_m,\boldsymbol{Z}_m)_{m\in\mathbb{N}}$ is obvious. From (3.2) and (3.3) we have the approximate equations

$$\begin{aligned}&\big(\boldsymbol{U}_m''(t),\boldsymbol{W}\big)+\big(\big(\boldsymbol{U}_m(t),\boldsymbol{W}\big)\big)-\big(\boldsymbol{Z}_m'(t),\boldsymbol{\gamma}_0(\boldsymbol{W})\big)_{\Gamma_1}+\big(\boldsymbol{G}\big(\boldsymbol{U}_m(t)\big),\boldsymbol{W}\big)\\&\quad=\big(\boldsymbol{F}(t),\boldsymbol{W}\big),\end{aligned} \tag{3.6}$$

$$\big(\boldsymbol{\gamma}_0\big(\boldsymbol{U}_m'(t)\big),\boldsymbol{E}\big)_{\Gamma_1}+\big(\boldsymbol{P}\boldsymbol{Z}_m''(t),\boldsymbol{E}\big)_{\Gamma_1}+\big(\boldsymbol{L}\boldsymbol{Z}_m'(t),\boldsymbol{E}\big)_{\Gamma_1}+\big(\boldsymbol{R}\boldsymbol{Z}_m(t),\boldsymbol{E}\big)_{\Gamma_1}=0 \tag{3.7}$$

for all $\boldsymbol{W}\in\mathrm{Span}\{\boldsymbol{W}_1,\dots,\boldsymbol{W}_m\}$ and $\boldsymbol{E}\in\mathrm{Span}\{\boldsymbol{E}_1,\dots,\boldsymbol{E}_m\}$.

Estimate 1. Taking $\boldsymbol{W}=2\boldsymbol{U}_m'(t)$ in (3.6), $\boldsymbol{E}=2\boldsymbol{Z}_m'(t)$ in (3.7) and using the definition of $\boldsymbol{G}$, the symmetric property of a_{ij} $(a_{ij}=a_{ji})$ and $a_{ii}=0$, we find

$$\begin{aligned}&\frac{d}{dt}\Bigg\{\big|\boldsymbol{U}_m'(t)\big|^2+\big\|\boldsymbol{U}_m(t)\big\|^2+\sum_{i=1}^{k}\int_{\Gamma_1}\big[p_i(z_{im}')^2+r_i(z_{im})^2\big]\,d\Gamma_1\\&\qquad+\sum_{i=1}^{k}\alpha_i\int_{\Omega}(u_{im})^2\,dx+\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}a_{ij}\int_{\Omega}(u_{im})^2(u_{jm})^2\,dx\Bigg\}\\&\qquad+2\sum_{i=1}^{k}\int_{\Gamma_1}l_i(z_{im}')^2\,d\Gamma_1\\&\quad=2\big(\boldsymbol{F}(t),\boldsymbol{U}_m'(t)\big).\end{aligned} \tag{3.8}$$

Here and everywhere in the proof of Theorem 3.1 we omit the variables x and t of the functions under the integrals. Integrating this from 0 to $t\leqslant T_m$, we get

$$\begin{aligned}&\big|\boldsymbol{U}_m'(t)\big|^2+\big\|\boldsymbol{U}_m(t)\big\|^2+\sum_{i=1}^{k}\int_{\Gamma_1}\big[p_i(z_{im}')^2+r_i(z_{im})^2\big]\,d\Gamma_1+\sum_{i=1}^{k}\alpha_i\int_{\Omega}(u_{im})^2\,dx\\&\qquad+\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}a_{ij}\int_{\Omega}(u_{im})^2(u_{jm})^2\,dx+2\sum_{i=1}^{k}\int_0^t\int_{\Gamma_1}l_i(z_{im}')^2\,d\Gamma_1\,d\tau\\&\quad\leqslant C_1+\int_0^t\big|\boldsymbol{U}_m'(\tau)\big|^2\,d\tau,\end{aligned}$$

where $C_1 > 0$ does not depend on m and $t \in [0, T]$. From this and Gronwall's inequality we obtain

$$\begin{aligned}
&\left|\boldsymbol{U}'_m(t)\right|^2 + \left\|\boldsymbol{U}_m(t)\right\|^2 + \sum_{i=1}^{k} \int_{\Gamma_1} \left[p_i (z'_{im})^2 + r_i (z_{im})^2\right] d\Gamma_1 + \sum_{i=1}^{k} \alpha_i \int_{\Omega} (u_{im})^2\, dx \\
&\quad + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} a_{ij} \int_{\Omega} (u_{im})^2 (u_{jm})^2\, dx + 2\sum_{i=1}^{k} \int_0^t \int_{\Gamma_1} l_i (z'_{im})^2\, d\Gamma_1\, d\tau \\
&\leqslant C_2, \qquad (3.9)
\end{aligned}$$

which is Estimate 1.

By (3.9), we can extend the approximate solutions $\boldsymbol{U}_m$ and $\boldsymbol{Z}_m$ to the whole interval $[0, T]$. Since $l_i \in C(\bar{\Gamma}_1)$ and $l_i(x) > 0$ for all $x \in \bar{\Gamma}_1$, $1 \leqslant i \leqslant k$, we find

$$\int_0^T \left|\boldsymbol{Z}'_m(t)\right|^2_{\Gamma_1} dt \leqslant C_3. \qquad (3.10)$$

Estimate 2. Taking $\boldsymbol{W} = \boldsymbol{U}_m(t)$ in (3.6), we obtain

$$\begin{aligned}
\left(\boldsymbol{G}\left(\boldsymbol{U}_m(t)\right), \boldsymbol{U}_m(t)\right) &\leqslant \left(\boldsymbol{G}\left(\boldsymbol{U}_m(t)\right), \boldsymbol{U}_m(t)\right) + \left\|\boldsymbol{U}_m(t)\right\|^2 \\
&= -\frac{d}{dt}\left(\boldsymbol{U}'_m(t), \boldsymbol{U}_m(t)\right) + \left|\boldsymbol{U}'_m(t)\right|^2 + \left(\boldsymbol{Z}'_m(t), \boldsymbol{\gamma}_0\left(\boldsymbol{U}_m(t)\right)\right)_{\Gamma_1} + \left(\boldsymbol{F}(t), \boldsymbol{U}_m(t)\right).
\end{aligned}$$

Integrating this from 0 to T, we get

$$\begin{aligned}
&\int_0^T \left(\boldsymbol{G}\left(\boldsymbol{U}_m(t)\right), \boldsymbol{U}_m(t)\right) dt \\
&\quad \leqslant |\boldsymbol{\Psi}_m|^2 + |\boldsymbol{\Phi}_m|^2 + \left|\boldsymbol{U}'_m(T)\right|^2 + \left|\boldsymbol{U}_m(T)\right|^2 \\
&\qquad + \int_0^T \left[\left|\boldsymbol{U}'_m(t)\right|^2 + \left|\boldsymbol{Z}'_m(t)\right|^2_{\Gamma_1} + \left|\boldsymbol{\gamma}_0\left(\boldsymbol{U}_m(t)\right)\right|^2_{\Gamma_1} + \left|\boldsymbol{F}(t)\right|^2 + \left|\boldsymbol{U}_m(t)\right|^2\right] dt.
\end{aligned}$$

From the above inequality, (3.9), (3.10), the continuity of the operator $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ and the Poincaré inequality, we find $C_4 > 0$, independent of m and t, such that

$$\int_0^T \left(\boldsymbol{G}\left(\boldsymbol{U}_m(t)\right), \boldsymbol{U}_m(t)\right) dt \leqslant C_4. \qquad (3.11)$$

It follows from the definition of function $\boldsymbol{G}$ that $g_i(Y) y_i \geqslant 0$. Therefore (3.11) yields

$$\sum_{i=1}^{k} \int_Q \left|g_i\left(\boldsymbol{U}_m(x,t)\right)\right| \left|u_{im}(x,t)\right| dx\, dt \leqslant C_4, \qquad (3.12)$$

which is Estimate 2.

Using Estimate 1 and compactness argument, we can see that there exist a subsequence of $(\boldsymbol{U}_m)_{m\in\mathbb{N}}$ and a subsequence of $(\boldsymbol{Z}_m)_{m\in\mathbb{N}}$, which we still denote by the same notations, and functions $\boldsymbol{U}$, $\boldsymbol{Z}$ such that

$$\boldsymbol{U}_m \overset{\star}{\rightharpoonup} \boldsymbol{U} \quad \text{in } L^\infty(0,T;\mathcal{H}), \qquad \boldsymbol{U}'_m \overset{\star}{\rightharpoonup} \boldsymbol{U}' \quad \text{in } L^\infty\big(0,T;\boldsymbol{L}^2(\boldsymbol{\Omega})\big), \tag{3.13}$$

$$\boldsymbol{U}_m \to \boldsymbol{U} \quad \text{in } L^2\big(0,T;\boldsymbol{L}^2(\boldsymbol{\Omega})\big), \qquad \boldsymbol{Z}_m \overset{\star}{\rightharpoonup} \boldsymbol{Z} \quad \text{in } L^\infty\big(0,T;\boldsymbol{L}^2(\boldsymbol{\Gamma}_1)\big), \tag{3.14}$$

$$\boldsymbol{Z}'_m \overset{\star}{\rightharpoonup} \boldsymbol{Z}' \quad \text{in } L^\infty\big(0,T;\boldsymbol{L}^2(\boldsymbol{\Gamma}_1)\big),$$

$$g_i(\boldsymbol{U}_m) \to g_i(\boldsymbol{U}) \quad \text{a.e. on } Q,\ i=1,\dots,k. \tag{3.15}$$

Lemma 2.1 yield the following convergence:

$$\boldsymbol{G}(\boldsymbol{U}_m) \to \boldsymbol{G}(\boldsymbol{U}) \quad \text{in } \boldsymbol{L}^1(\boldsymbol{Q}). \tag{3.16}$$

Multiplying (3.6) by $\theta \in \mathcal{D}'(0,T)$, integrating from 0 to T and using (3.13)–(3.16), we prove by a straightforward computation that $\boldsymbol{U}$ and $\boldsymbol{Z}$ satisfy (2.6)–(2.9) of Definition 2.1. Whence, $(\boldsymbol{U},\boldsymbol{Z})$ is a weak solution to (2.1)–(2.5) and Theorem 3.1 is proved. □

Theorem 3.2. *Let p_i, l_i and r_i, $1 \leqslant i \leqslant k$, be as in Theorem* 3.1. *If $n \leqslant 3$, $\boldsymbol{\Phi} \in (\mathcal{H} \cap \boldsymbol{H}^2(\boldsymbol{\Omega}) \cap \boldsymbol{L}^4(\boldsymbol{\Omega}))$, $\boldsymbol{\Psi} \in \mathcal{H}$ and $\boldsymbol{F} \in L^2(0,T;\boldsymbol{L}^2(\boldsymbol{\Omega}))$ with $\boldsymbol{F}' \in L^2(0,T;\boldsymbol{L}^2(\boldsymbol{\Omega}))$, then there exists a unique pair of functions $(\boldsymbol{U},\boldsymbol{Z})$ which is a strong solution to* (2.1)–(2.5).

Proof. In this case we can get one more estimate for the approximate solutions.

Estimate 3. Since $\boldsymbol{F}, \boldsymbol{F}' \in L^2(0,T;\boldsymbol{L}^2(\boldsymbol{\Omega}))$, then $\boldsymbol{F} \in C([0,T];\boldsymbol{L}^2(\boldsymbol{\Omega}))$. From (3.6) and (3.7), we get

$$\big(\boldsymbol{U}''_m(0) - \boldsymbol{\Delta\Phi}_m + \boldsymbol{G}(\boldsymbol{\Phi}_m), \boldsymbol{U}''_m(0)\big) = \big(\boldsymbol{F}(0), \boldsymbol{U}''_m(0)\big), \tag{3.17}$$

$$\big(\boldsymbol{\gamma}_0(\boldsymbol{\Psi}_m) + \boldsymbol{P}\boldsymbol{Z}''_m(0) + \boldsymbol{L}\boldsymbol{Z}'_m(0) + \boldsymbol{R}\boldsymbol{Z}_m(0), \boldsymbol{Z}''_m(0)\big)_{\Gamma_1} = 0. \tag{3.18}$$

From here

$$\big|\boldsymbol{U}''_m(0)\big|^2 \leqslant \big(|\boldsymbol{\Delta\Phi}_m| + \big|\boldsymbol{G}(\boldsymbol{\Phi}_m)\big| + \big|\boldsymbol{F}(0)\big|\big)\big|\boldsymbol{U}''_m(0)\big|,$$

$$\big|\boldsymbol{Z}''_m(0)\big|^2_{\Gamma_1} \leqslant C_5\big(\big|\boldsymbol{\gamma}_0(\boldsymbol{\Psi}_m)\big|_{\Gamma_1} + |\boldsymbol{\gamma}_1(\boldsymbol{\Phi}_m)\big|_{\Gamma_1} + |\boldsymbol{Z}_0|_{\Gamma_1}|\big)\big|\boldsymbol{Z}''_m(0)\big|_{\Gamma_1},$$

hence

$$\big|\boldsymbol{U}''_m(0)\big|^2 + \big|\boldsymbol{Z}''_m(0)\big|^2_{\Gamma_1} \leqslant C_5, \tag{3.19}$$

where $C_5 > 0$ is independent of m and t.

Differentiating (3.6) and (3.7), after standard calculations we obtain

$$\begin{aligned}
&\frac{d}{dt}\Bigg[\big|\boldsymbol{U}''_m(t)\big|^2 + \big\|\boldsymbol{U}'_m(t)\big\|^2 + \sum_{i=1}^k \int_{\Gamma_1} \big[p_i(z''_{im})^2 + r_i(z'_{im})^2\big]\,d\Gamma_1 + \sum_{i=1}^k \alpha_i \int_{\Omega} (u'_{im})^2\,dx\Bigg] \\
&\quad + \sum_{i=1}^k \int_{\Gamma_1} l_i(z''_{im})^2\,d\Gamma_1 + \sum_{i=1}^k\sum_{j=1}^k a_{ij} \int_{\Omega} \big[2u'_{im}u''_{im}(u_{jm})^2 + 4u_{im}u_{jm}u'_{jm}u''_{im}\big]\,dx \\
&= 2\big(\boldsymbol{F}'(t), \boldsymbol{U}''_m(t)\big).
\end{aligned}$$

Consequently,

$$\frac{d}{dt}\left[\left|U''_m(t)\right|^2+\left\|U'_m(t)\right\|^2+\sum_{i=1}^{k}\int_{\Gamma_1}\left[p_i(z''_{im})^2+r_i(z'_{im})^2\right]d\Gamma_1+\sum_{i=1}^{k}\alpha_i\int_{\Omega}(u'_{im})^2\,dx\right]$$
$$+\sum_{i=1}^{k}\int_{\Gamma_1}l_i(z''_{im})^2\,d\Gamma_1$$
$$\leqslant\left|F'(t)\right|^2+\left|U''_m(t)\right|^2+2\Big(\max_{\substack{1\leqslant i\leqslant k\\ i\leqslant j\leqslant k}}a_{ij}\Big)\sum_{i=1}^{k}\sum_{j=1}^{k}\int_{\Omega}u_{jm}^2|u'_{im}|\,|u''_{im}|\,dx$$
$$+4\Big(\max_{\substack{1\leqslant i\leqslant k\\ i\leqslant j\leqslant k}}a_{ij}\Big)\sum_{i=1}^{k}\sum_{j=1}^{k}\int_{\Omega}|u_{im}|\,|u_{jm}|\,|u'_{jm}|\,|u''_{im}|\,dx. \tag{3.20}$$

Since $n\leqslant 3$, then $H^1(\Omega)\hookrightarrow L^6(\Omega)$. Using Estimate 1, we find

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\int_{\Omega}(u_{jm})^2|u'_{im}|\,|u''_{im}|\,dx\leqslant C_7\left(\left\|U'_m(t)\right\|^2+\left|U''_m(t)\right|^2\right),$$
$$\sum_{i=1}^{k}\sum_{j=1}^{k}\int_{\Omega}|u_{im}|\,|u_{jm}|\,|u'_{jm}|\,|u''_{im}|\,dx\leqslant C_8\left(\left\|U'_m(t)\right\|^2+\left|U''_m(t)\right|^2\right).$$

From these inequalities and (3.20) we get

$$\frac{d}{dt}\left\{\left|U''_m(t)\right|^2+\left\|U'_m(t)\right\|^2+\sum_{i=1}^{k}\int_{\Gamma_1}\left[p_i(z''_{im})^2+r_i(z'_{im})^2\right]d\Gamma_1+\sum_{i=1}^{k}\alpha_i\int_{\Omega}(u'_{im})^2\,dx\right\}$$
$$+\sum_{i=1}^{k}\int_{\Gamma_1}l_i(z''_{im})^2\,d\Gamma_1$$
$$\leqslant C_{10}\left(\left|F'(t)\right|^2+\left|U''_m(t)\right|^2+\left\|U'_m(t)\right\|^2\right). \tag{3.21}$$

Integrating this from 0 to $t\leqslant T$ we can complete Estimate 3 which asserts that there exists a constant $C_6>0$, independent of m and t, such that

$$\left|U''_m(t)\right|^2+\left\|U'_m(t)\right\|^2+\sum_{i=1}^{k}\int_{\Gamma_1}\left[p_i(z''_{im})^2+r_i(z'_{im})^2\right]d\Gamma_1$$
$$+\sum_{i=1}^{k}\alpha_i\int_{\Omega}(u'_{im})^2\,dx+\sum_{i=1}^{k}\int_0^t\int_{\Gamma_1}l_i(z''_{im})^2\,d\Gamma_1\,d\tau$$
$$\leqslant C_6. \tag{3.22}$$

Estimates 1–3 assure that we can pass to the limit as $m \to \infty$ in (3.6), (3.7) and find a pair of functions $(\boldsymbol{U}, \boldsymbol{Z})$ which satisfies (2.11)–(2.13) and (2.15). Employing standard arguments of elliptic problems, we can see that $\boldsymbol{U}(t) \in \boldsymbol{H}(\boldsymbol{\Delta}, \boldsymbol{\Omega})$. Thus, the pair $(\boldsymbol{U}, \boldsymbol{Z})$ is a strong solution to (2.1)–(2.5) according to Definition 2.2. □

4. Uniform decay

In this section we prove the exponential decay of strong solutions to (2.1)–(2.5) when $\boldsymbol{F} = \boldsymbol{0}$ in Q, $\boldsymbol{P} = \boldsymbol{0}$ on Σ_1, the coefficients α_i are sufficiently small, $r_i(x) > 0$ for all $x \in \Gamma_1$ and $i = 1, \dots, k$; and the partition $\Gamma = \Gamma_0 \cup \Gamma_1$ satisfies a geometrical restriction.

Throughout this section let $x^0 \in \mathbb{R}^n$ and let m, Γ_0 and Γ_1 be such that $m(x) = (m_1(x), \dots, m_n(x)) = x - x^0$, $\Gamma_0 = \{x \in \Gamma;\ \langle m(x) \bullet \nu(x)\rangle \leqslant 0\}$, $\Gamma_1 = \{x \in \Gamma;\ \langle m(x) \bullet \nu(x)\rangle > 0\}$, and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$. Here $\langle m(x) \bullet \nu(x)\rangle = \sum_{j=1}^n m_j(x)\nu_j(x)$ is the inner product in $\mathbb{R}^n$. Denoting

$$
\begin{aligned}
&M = \max_{1 \leqslant j \leqslant n}\Bigl(\max_{x \in \bar{\Omega}}\bigl|m_j(x)\bigr|\Bigr),\\
&0 < B_0 = \min_{x \in \Gamma_1}\bigl\langle m(x) \bullet \nu(x)\bigr\rangle \leqslant \bigl\langle m(x) \bullet \nu(x)\bigr\rangle \leqslant \max_{x \in \Gamma_1}\bigl\langle m(x) \bullet \nu(x)\bigr\rangle = B_1,\\
&0 < l_0 = \min_{1 \leqslant i \leqslant k}\Bigl(\min_{x \in \Gamma_1} l_i(x)\Bigr) \leqslant l_i(x) \leqslant \max_{1 \leqslant i \leqslant k}\Bigl(\max_{x \in \Gamma_1} l_i(x)\Bigr) = \bar{l},\\
&0 < r_0 = \min_{1 \leqslant i \leqslant k}\Bigl(\min_{x \in \Gamma_1} r_i(x)\Bigr) \leqslant r_i(x) \leqslant \max_{1 \leqslant i \leqslant k}\Bigl(\max_{x \in \Gamma_1} r_i(x)\Bigr) = \bar{r},
\end{aligned}
$$

from the geometrical restriction on Γ_0 and Γ_1, one can see that Γ_0, Γ_1 are compact sets. This assures the existence of the above numbers B_0, B_1, l_0, $\bar{l}$, r_0 and $\bar{r}$. We also note that the geometrical restriction excludes domains Ω having a smooth connected boundary.

Assuming that the assumptions of Theorem 3.2 hold, there exists a unique pair of functions $(\boldsymbol{U}, \boldsymbol{Z})$ in the class

$$
\begin{aligned}
&\boldsymbol{U}, \boldsymbol{U}' \in L^\infty_{\mathrm{loc}}(0, \infty; \boldsymbol{\mathcal{H}}), \quad \boldsymbol{U}'' \in L^\infty_{\mathrm{loc}}\bigl(0, \infty; \boldsymbol{L}^2(\boldsymbol{\Omega})\bigr),\\
&\quad \boldsymbol{U}(t) \in \boldsymbol{H}(\boldsymbol{\Delta}, \boldsymbol{\Omega}) \quad \text{a.e. on } [0, \infty),\\
&\boldsymbol{Z}, \boldsymbol{Z}' \in L^\infty_{\mathrm{loc}}\bigl(0, \infty; \boldsymbol{L}^2(\boldsymbol{\Gamma}_1)\bigr) \quad \text{and} \quad \boldsymbol{Z}'' \in L^2_{\mathrm{loc}}\bigl(0, \infty; \boldsymbol{L}^2(\boldsymbol{\Gamma}_1)\bigr),
\end{aligned}
$$

which is a solution to the problem

$$\boldsymbol{U}'' - \boldsymbol{\Delta U} + \boldsymbol{G}(\boldsymbol{U}) = \boldsymbol{0} \quad \text{a.e. on } \Omega \times (0, \infty), \tag{4.1}$$

$$\boldsymbol{U} = \boldsymbol{0} \quad \text{a.e. on } \Gamma_0 \times (0, \infty), \tag{4.2}$$

$$\boldsymbol{U}' + \boldsymbol{L}\boldsymbol{Z}' + \boldsymbol{R}\boldsymbol{Z} = \boldsymbol{0} \quad \text{a.e. on } \Sigma_1 \times (0, \infty), \tag{4.3}$$

$$\frac{\boldsymbol{\partial U}}{\boldsymbol{\partial \nu}} = \boldsymbol{Z}' \quad \text{a.e. on } \Gamma_1 \times (0, \infty), \tag{4.4}$$

$$\boldsymbol{U}(0) = \boldsymbol{\Phi}, \quad \boldsymbol{U}'(0) = \boldsymbol{\Psi} \quad \text{a.e. on } \Omega. \tag{4.5}$$

We define the energy $E = E(t)$,

$$E(t) = \left|U'(t)\right|^2 + \left\|U(t)\right\|^2 + \sum_{i=1}^{k} \int_{\Gamma_1} r_i(x)\big(z_i(x,t)\big)^2\, d\Gamma_1 + \sum_{i=1}^{k} \alpha_i \int_{\Omega} \big(u_i(x,t)\big)^2\, dx$$
$$+ \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} a_{ij} \int_{\Omega} \big(u_i(x,t)\big)^2 \big(u_j(x,t)\big)^2\, dx, \quad t \geqslant 0. \tag{4.6}$$

It follows from (4.1)–(4.5) that

$$E'(t) = -2 \sum_{i=1}^{k} \int_{\Gamma_1} l_i(x)\big(z_i'(x,t)\big)^2\, d\Gamma_1 < 0. \tag{4.7}$$

Theorem 4.1. *Let* $(\boldsymbol{U}, \boldsymbol{Z})$ *be a global strong solution to* (4.1)–(4.5). *If* $r_i(x) > 0$ *for all* $x \in \Gamma_1$, $1 \leqslant i \leqslant k$, *and*

$$\bar{\alpha} = \max_{1 \leqslant i \leqslant k} \alpha_i < \frac{2(2+\theta-n)}{3C(n-\theta)}, \tag{4.8}$$

where

$$\frac{n}{2} < \theta < n \quad \text{and} \quad |\boldsymbol{W}|^2 \leqslant C\|\boldsymbol{W}\|^2 \quad \text{for all } \boldsymbol{W} \in \mathcal{H}, \tag{4.9}$$

then there exist positive constants C_0 *and* C_1 *such that*

$$E(t) \leqslant C_0 E(0) e^{-C_1 t}, \quad t \geqslant 0. \tag{4.10}$$

Proof. For $\epsilon > 0$, let E_ϵ be the perturbed energy defined by

$$E_\epsilon(t) = E(t) + \epsilon\rho(t), \quad t \geqslant 0, \tag{4.11}$$

where

$$\rho(t) = 2\sum_{i=1}^{k} \int_{\Omega} \big\langle m(x) \bullet \nabla u_i(x,t)\big\rangle u_i'(x,t)\, dx + \theta\big(\boldsymbol{U}(t), \boldsymbol{U}'(t)\big)$$
$$+ (2B_1\bar{r}+1) \sum_{i=1}^{k} \int_{\Gamma_1} u_i(x,t) z_i(x,t)\, d\Gamma_1$$
$$+ \left(\frac{2B_1\bar{r}+1}{2}\right) \sum_{i=1}^{k} \int_{\Gamma_1} l_i(x)\big(z_i(x,t)\big)^2\, d\Gamma_1.$$

Omitting the variables x and t of the functions under the integrals, we calculate

$$\frac{1}{\epsilon}\left|E_\epsilon(t) - E(t)\right|$$
$$\leqslant \left(M + \frac{\theta}{2}\right)\left|\boldsymbol{U}'(t)\right|^2 + M\left\|\boldsymbol{U}(t)\right\|^2 + \frac{\theta}{2}\left|\boldsymbol{U}(t)\right|^2 + (2B_1\bar{r}+1)\left|\boldsymbol{U}(t)\right|_{\Gamma_1}^2$$
$$+ \frac{1}{r_0}\left[(2B_1\bar{r}+1) + \left(\frac{2B_1\bar{r}+1}{2}\right)\bar{l}\right] \sum_{i=1}^{k} \int_{\Gamma_1} r_i(z_i)^2\, dx.$$

By Poincaré's inequality, there exist positive constants C and C' such that

$$\left|\boldsymbol{U}(t)\right|^2 \leqslant C\left\|\boldsymbol{U}(t)\right\|^2 \quad \text{and} \quad \left|\boldsymbol{U}(t)\right|_{\Gamma_1}^2 \leqslant C'\left\|\boldsymbol{U}(t)\right\|^2. \tag{4.12}$$

Using (4.12), we get

$$\left|E_\epsilon(t) - E(t)\right| \leqslant \epsilon C_2 E(t) \quad \text{for all } t \geqslant 0, \tag{4.13}$$

where

$$C_2 = \max\left\{\left(M + \frac{\theta}{2}\right), \left[M + \frac{\theta C}{2} + (2B_1\bar{r} + 1)C'\right], \frac{1}{r_0}\left[(2B_1\bar{r} + 1) + \left(\frac{2B_1\bar{r} + 1}{2}\right)\bar{l}\right]\right\}.$$

From (4.13), there exist positive constants C_3 and C_4 such that

$$E(t) \leqslant C_3 E_\epsilon(t) \leqslant C_4 E(t) \quad \text{for all } t \geqslant 0 \text{ and } \epsilon \in \left(0, \frac{1}{C_2}\right). \tag{4.14}$$

Differentiating E_ϵ we find

$$E_\epsilon'(t) = E'(t) + \epsilon\rho'(t), \quad t > 0. \tag{4.15}$$

From the definition of $\rho(t)$, we have

$$\begin{aligned}
\rho'(t) = {} & 2\sum_{i=1}^k \int_\Omega \langle m \bullet \nabla u_i'\rangle u_i' \, dx + 2\sum_{i=1}^k \int_\Omega \langle m \bullet \nabla u_i\rangle \Delta u_i \, dx \\
& - 2\sum_{i=1}^k \int_\Omega \langle m \bullet \nabla u_i\rangle g_i(\boldsymbol{U}) \, dx - \theta\big(\boldsymbol{U}(t), \boldsymbol{G}\big(\boldsymbol{U}(t)\big)\big) \\
& + \theta\big(\boldsymbol{U}(t), \boldsymbol{\Delta U}(t)\big) + \theta\left|\boldsymbol{U}'(t)\right|^2 \\
& + \frac{d}{dt}\left[(2B_1\bar{r} + 1)\sum_{i=1}^k \int_{\Gamma_1} u_i z_i \, d\Gamma_1 + \left(\frac{2B_1\bar{r} + 1}{2}\right)\sum_{i=1}^k \int_{\Gamma_1} l_i(z_i)^2 \, d\Gamma_1\right].
\end{aligned} \tag{4.16}$$

Next we analyze the terms on the right-hand side of (4.16) (see [7,17]),

$$I_1 = 2\sum_{i=1}^k \int_\Omega \langle m \bullet \nabla u_i'\rangle u_i' \, dx = -n\left|\boldsymbol{U}'(t)\right|^2 + \sum_{i=1}^k \int_{\Gamma_1} \langle m \bullet \nu\rangle (u_i')^2 \, d\Gamma_1, \tag{4.17}$$

$$\begin{aligned}
I_2 = {} & +2\sum_{i=1}^k \int_\Omega \langle m \bullet \nabla u_i\rangle \Delta u_i \, dx = (n-2)\left\|\boldsymbol{U}(t)\right\|^2 + \sum_{i=1}^k \int_{\Gamma_0} \langle m \bullet \nu\rangle \left(\frac{\partial u_i}{\partial \nu}\right)^2 d\Gamma_0 \\
& - \sum_{i=1}^k \int_{\Gamma_1} \langle m \bullet \nu\rangle \|\nabla u_i\|_{\mathbb{R}^n}^2 \, d\Gamma_1 + 2\sum_{i=1}^k \int_{\Gamma_1} \langle m \bullet \nabla u_i\rangle z_i' \, d\Gamma_1.
\end{aligned}$$

Taking into account (4.2)–(4.4), we get

$$I_2 \leqslant (n-2)\|\boldsymbol{U}(t)\|^2 + 2\sum_{i=1}^{k}\int_{\Gamma_1}\langle m \bullet \nabla u_i\rangle z_i'\,d\Gamma_1 - \sum_{i=1}^{k}\int_{\Gamma_1}\langle m \bullet \nu\rangle \|\nabla u_i\|_{\mathbb{R}^n}^2\,d\Gamma_1, \tag{4.18}$$

$$\begin{aligned} I_3 &= -2\sum_{i=1}^{k}\int_{\Omega}\langle m \bullet \nabla u_i\rangle g_i(\boldsymbol{U})\,dx - \theta\big(\boldsymbol{U}(t), \boldsymbol{G}(\boldsymbol{U}(t))\big) \\ &= -2\sum_{i=1}^{k}\int_{\Omega}\alpha_i\langle m \bullet \nabla u_i\rangle u_i\,dx - 2\sum_{i=1}^{k}\sum_{j=1}^{k}a_{ij}\int_{\Omega}\langle m \bullet \nabla u_i\rangle u_i(u_j)^2\,dx \\ &\quad - \theta\sum_{i=1}^{k}\alpha_i\int_{\Omega}(u_i)^2\,dx - \theta\sum_{i=1}^{k}\sum_{j=1}^{k}a_{ij}\int_{\Omega}(u_i)^2(u_j)^2\,dx. \end{aligned}$$

To complete the analysis of this term, we observe that

$$\begin{aligned} -2\sum_{i=1}^{k}\int_{\Omega}\alpha_i\langle m \bullet \nabla u_i\rangle u_i\,dx &= \alpha_i\sum_{i=1}^{k}\left(n\int_{\Omega}u_i^2\,dx - \int_{\Gamma_1}\langle m \bullet \nu\rangle u_i^2\,d\Gamma_1\right) \\ &\leqslant n\sum_{i=1}^{k}\alpha_i\int_{\Omega}u_i^2\,dx \end{aligned}$$

and

$$\begin{aligned} &-2\sum_{i=1}^{k}\sum_{j=1}^{k}a_{ij}\int_{\Omega}\langle m \bullet \nabla u_i\rangle u_i(u_j)^2\,dx \\ &\quad= -\sum_{\lambda=1}^{n}\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}a_{ij}\left[-\int_{\Omega}\frac{\partial m_\lambda}{\partial x_\lambda}(u_i)^2(u_j)^2\,dx + \int_{\Gamma}m_\lambda(u_i)^2(u_j)^2\langle \nu \bullet e_\lambda\rangle\,d\Gamma\right] \\ &\quad\leqslant n\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}a_{ij}\int_{\Omega}(u_i)^2(u_j)^2\,dx. \end{aligned}$$

Combining these inequalities, we obtain

$$I_3 \leqslant (n-\theta)\sum_{i=1}^{k}\alpha_i\int_{\Omega}(u_i)^2\,dx + (n-2\theta)\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}a_{ij}\int_{\Omega}(u_i)^2(u_j)^2\,dx, \tag{4.19}$$

$$I_4 = \theta\big(\boldsymbol{U}(t), \boldsymbol{\Delta U}(t)\big) = -\theta\|\boldsymbol{U}(t)\|^2 + \theta\big(\boldsymbol{Z}'(t), \boldsymbol{U}(t)\big)_{\Gamma_1}. \tag{4.20}$$

We deduce from (4.7) and (4.15)–(4.20) that

$$\begin{aligned}
E_\epsilon'(t) \leqslant {} & -\epsilon \Biggl\{ (n-\theta)\bigl|\boldsymbol{U}'(t)\bigr|^2 + (2+\theta-n)\bigl\|\boldsymbol{U}(t)\bigr\|^2 + (\theta-n)\sum_{i=1}^{k}\alpha_i \int_\Omega (u_i)^2\,dx \\
& + (2\theta-n)\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} a_{ij}\int_\Omega (u_i)^2(u_j)^2\,dx + \sum_{i=1}^{k}\int_{\Gamma_1} r_i(z_i)^2\,d\Gamma_1 \Biggr\} \\
& - 2\sum_{i=1}^{k}\int_{\Gamma_1} l_i(z_i')^2\,d\Gamma_1 \\
& + \epsilon \Biggl\{ \sum_{i=1}^{k}\int_{\Gamma_1} \langle m\bullet\nu\rangle (u_i')^2\,d\Gamma_1 - \sum_{i=1}^{k}\int_{\Gamma_1} \langle m\bullet\nu\rangle \|\nabla u_i\|_{\mathbb{R}^n}^2\,d\Gamma_1 \\
& + 2\sum_{i=1}^{k}\int_{\Gamma_1} \langle m\bullet\nabla u_i\rangle z_i'\,d\Gamma_1 + \theta\bigl(\boldsymbol{U}(t),\boldsymbol{Z}'(t)\bigr)_{\Gamma_1} + \sum_{i=1}^{k}\int_{\Gamma_1} r_i(z_i)^2\,d\Gamma_1 \\
& + \frac{d}{dt}\Biggl[(2B_1\bar r+1)\sum_{i=1}^{k}\int_{\Gamma_1} u_i z_i\,d\Gamma_1 + \biggl(\frac{2B_1\bar r+1}{2}\biggr)\sum_{i=1}^{k}\int_{\Gamma_1} l_i(z_i)^2\,d\Gamma_1 \Biggr]\Biggr\}.
\end{aligned} \tag{4.21}$$

We can see that $(n-\theta) > 0$, $(2+\theta-n) > 0$, and $(2\theta-n) > 0$, because θ satisfies (4.9). However, we still have to compensate terms in (4.21). Since $(\theta-n) < 0$, then

$$\begin{aligned}
-\epsilon(\theta-n)\sum_{i=1}^{k}\alpha_i\int_\Omega (u_i)^2\,dx \leqslant {} & -\epsilon\frac{3(\theta-n)}{2}\bar\alpha C\bigl\|\boldsymbol{U}(t)\bigr\|^2 \\
& - \epsilon\frac{(\theta-n)}{2}\sum_{i=1}^{k}\alpha_i\int_\Omega (u_i)^2\,dx.
\end{aligned}$$

Combining this and (4.21), we get

$$\begin{aligned}
E_\epsilon'(t) \leqslant {} & -C_5\epsilon E(t) - 2\sum_{i=1}^{k}\int_{\Gamma_1} l_i(z_i')^2\,d\Gamma_1 \\
& + \epsilon \Biggl\{ \sum_{i=1}^{k}\int_{\Gamma_1} \langle m\bullet\nu\rangle (u_i')^2\,d\Gamma_1 - \sum_{i=1}^{k}\int_{\Gamma_1} \langle m\bullet\nu\rangle \|\nabla u_i\|_{\mathbb{R}^n}^2\,d\Gamma_1 \\
& + 2\sum_{i=1}^{k}\int_{\Gamma_1} \langle m\bullet\nabla u_i\rangle z_i'\,d\Gamma_1 + \theta\bigl(\boldsymbol{U}(t),\boldsymbol{Z}'(t)\bigr)_{\Gamma_1} + \sum_{i=1}^{k}\int_{\Gamma_1} r_i(z_i)^2\,d\Gamma_1 \\
& + \frac{d}{dt}\Biggl[(2B_1\bar r+1)\sum_{i=1}^{k}\int_{\Gamma_1} u_i z_i\,d\Gamma_1 + \biggl(\frac{2B_1\bar r+1}{2}\biggr)\sum_{i=1}^{k}\int_{\Gamma_1} l_i(z_i)^2\,d\Gamma_1 \Biggr]\Biggr\},
\end{aligned} \tag{4.22}$$

where

$$C_5 = \min\left\{(n-\theta), \left[(2+\theta-n) + \frac{3(\theta-n)C\bar{\alpha}}{2}\right], (2\theta-n), \frac{(n-\theta)}{2}, 1\right\} > 0.$$

To estimate the right-hand side of (4.22), we note that

$$\begin{aligned}
I_5 &= \sum_{i=1}^{k}\int_{\Gamma_1}\langle m\bullet\nu\rangle(u_i')^2\,d\Gamma_1 + \sum_{i=1}^{k}\int_{\Gamma_1} r_i(z_i)^2\,d\Gamma_1\\
&\leqslant 2B_1\bar{l}\sum_{i=1}^{k}\int_{\Gamma_1} l_i(z_i')^2\,d\Gamma_1 + (2B_1\bar{r}+1)\sum_{i=1}^{k}\int_{\Gamma_1} r_i(z_i)^2\,d\Gamma_1\\
&= 2B_1\bar{l}\sum_{i=1}^{k}\int_{\Gamma_1} l_i(z_i')^2\,d\Gamma_1 + (2B_1\bar{r}+1)\sum_{i=1}^{k}\int_{\Gamma_1}[-u_i'z_i - l_iz_iz_i']\,d\Gamma_1\\
&= -\frac{d}{dt}\left[(2B_1\bar{r}+1)\sum_{i=1}^{k}\int_{\Gamma_1} u_iz_i\,d\Gamma_1 + \left(\frac{2B_1\bar{r}+1}{2}\right)\sum_{i=1}^{k}\int_{\Gamma_1} l_i(z_i)^2\,d\Gamma_1\right]\\
&\quad + (2B_1\bar{r}+1)\sum_{i=1}^{k}\int_{\Gamma_1} u_iz_i'\,d\Gamma_1 + 2B_1\bar{l}\sum_{i=1}^{k}\int_{\Gamma_1} l_i(z_i')^2\,d\Gamma_1.
\end{aligned}$$

Therefore, for an arbitrary $\eta > 0$ we have

$$\begin{aligned}
I_5 &\leqslant -\frac{d}{dt}\left[(2B_1\bar{r}+1)\sum_{i=1}^{k}\int_{\Gamma_1} u_iz_i\,d\Gamma_1 + \left(\frac{2B_1\bar{r}+1}{2}\right)\sum_{i=1}^{k}\int_{\Gamma_1} l_i(z_i)^2\,d\Gamma_1\right]\\
&\quad + 2B_1\bar{l}\sum_{i=1}^{k}\int_{\Gamma_1} l_i(z_i')^2\,d\Gamma_1 + \frac{\eta}{2}\left\|\boldsymbol{U}(t)\right\|^2 + \frac{2C'}{\eta}(2B_1\bar{r}+1)^2\sum_{i=1}^{k}\int_{\Gamma_1}(z_i')^2\,d\Gamma_1.
\end{aligned} \tag{4.23}$$

Moreover,

$$I_6 = -\sum_{i=1}^{k}\int_{\Gamma_1}\langle m\bullet\nu\rangle\|\nabla u_i\|_{\mathbb{R}^n}^2\,d\Gamma_1 \leqslant -B_0\sum_{i=1}^{k}\int_{\Gamma_1}\|\nabla u_i\|_{\mathbb{R}^n}^2\,d\Gamma_1, \tag{4.24}$$

$$I_7 = 2\sum_{i=1}^{k}\int_{\Gamma_1}\langle m\bullet\nabla u_i\rangle z_i'\,d\Gamma_1 \leqslant B_0\sum_{i=1}^{k}\int_{\Gamma_1}\|\nabla u_i\|_{\mathbb{R}^n}^2\,d\Gamma_1 + \frac{nM^2}{B_0}\sum_{i=1}^{k}\int_{\Gamma_1}(z_i')^2\,d\Gamma_1, \tag{4.25}$$

$$I_8 = \theta\big(\boldsymbol{U}(t), \boldsymbol{Z}'(t)\big)_{\Gamma_1} \leqslant \frac{\eta}{2}\left\|\boldsymbol{U}(t)\right\|^2 + \frac{C'\theta^2}{2\eta}\sum_{i=1}^{k}\int_{\Gamma_1}(z_i')^2\,d\Gamma_1. \tag{4.26}$$

Substituting I_5–I_8 into (4.22), we obtain

$$\begin{aligned}
E'_\epsilon(t) &\leqslant -C_5\epsilon E(t) - 2\sum_{i=1}^{k}\int_{\Gamma_1} l_i (z'_i)^2\, d\Gamma_1 + \epsilon\Bigg[2B_1\bar{l}\sum_{i=1}^{k}\int_{\Gamma_1} l_i (z'_i)^2\, d\Gamma_1 \\
&\quad + \left(\frac{2C'}{\eta}(2B_1\bar{r}+1)^2 + \frac{nM^2}{B_0} + \frac{C'\theta^2}{2\eta}\right)\sum_{i=1}^{k}\int_{\Gamma_1}(z'_i)^2\, d\Gamma_1 + \eta\big\|\boldsymbol{U}(t)\big\|^2\Bigg] \\
&\leqslant -\epsilon(C_5-\eta)E(t) \\
&\quad - \left\{2 - \epsilon\left[2B_1\bar{l} + \frac{1}{l_0}\left(\frac{2C'}{\eta}(2B_1\bar{r}+1)^2 + \frac{nM^2}{B_0} + \frac{C'\theta^2}{2\eta}\right)\right]\right\} \\
&\quad \times \sum_{i=1}^{k}\int_{\Gamma_1} l_i (z'_i)^2\, d\Gamma_1.
\end{aligned} \tag{4.27}$$

We choose $\eta > 0$ such that $(C_5 - \eta) > 0$, and then $\epsilon \in (0, C_2^{-1})$ such that

$$\left\{2 - \epsilon\left[2B_1\bar{l} + \frac{1}{l_0}\left(\frac{2C'}{\eta}(2B_1\bar{r}+1)^2 + \frac{nM^2}{B_0} + \frac{C'\theta^2}{2\eta}\right)\right]\right\} > 0.$$

Hence, there exists a constant $C_6 > 0$ independent of t such that

$$E'_\epsilon(t) \leqslant -\epsilon C_6 E(t) \quad \text{for all } t \geqslant 0. \tag{4.28}$$

Using (4.14) and (4.28), one can easily see that (4.10) holds. This completes the proof of Theorem 4.1. □

References

[1] H.D. Alber, J. Cooper, Quasilinear hyperbolic 2 × 2 systems with a free damping boundary condition, J. Reine Angew. Math. 406 (1990) 10–43.

[2] J.T. Beale, Spectral properties of an acoustic boundary condition, Indiana Univ. Math. J. 25 (1976) 895–917.

[3] J.T. Beale, Acoustic scattering from locally reacting surfaces, Indiana Univ. Math. J. 26 (1977) 199–222.

[4] J.T. Beale, S.I. Rosencrans, Acoustic boundary conditions, Bull. Amer. Math. Soc. 80 (1974) 1276–1278.

[5] A.T. Cousin, C.L. Frota, N.A. Larkin, Global solvability and asymptotic behaviour of hyperbolic problem with acoustic boundary conditions, Funkcial. Ekvac. 44 (2001) 471–485.

[6] C.L. Frota, J.A. Goldstein, Some nonlinear wave equations with acoustic boundary conditions, J. Differential Equations 164 (2000) 92–109.

[7] V. Komornik, E. Zuazua, A direct method for boundary stabilization of the wave equation, J. Math. Pures Appl. 69 (1990) 33–54.

[8] I. Lasiecka, Stabilization of the wave equations with nonlinear dissipative damping on the boundary, in: Proc. 26th IEEE Conference on Decision and Control, Los Angeles, CA, 1987, pp. 2348–2349.

[9] J.L. Lions, Quelques methodes de resolution des problemes aux limites non lineaires, Dunod, Paris, 1969.

[10] J.L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Springer-Verlag, New York, 1972.

[11] L.A. Medeiros, G. Perla Menzala, On a mixed problem for a class of nonlinear Klein–Gordon equations, Acta Math. Hungar. 52 (1988) 61–69.

[12] I. Segal, Nonlinear partial differential equations in quantum field theory, Proc. Sympos. Appl. Math. 17 (1965) 210–226.

[13] B. Sakita, Quantum Theory of Many Variable Systems and Fields, in: World Scientific Lecture Notes in Physics, vol. 1, World Scientific, 1985.

[14] W.A. Strauss, On weak solutions of semilinear hyperbolic equations, An. Acad. Brasil. Ciênc. 42 (1970) 645–651.
[15] R. Temam, Navier–Stokes Equations, North-Holland, Amsterdam, 1979.
[16] A.M. Tsvelik, Quantum Field Theory in Condensed Matter Physics, Cambridge Univ. Press, Cambridge, 1995.
[17] E. Zuazua, Uniform stabilization of the wave equation by nonlinear boundary feedback, SIAM J. Control Optim. 28 (1990) 466–477.