On J-Selfadjoint Extensions of J-Symmetric Ordinary Differential Operators*

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1. INTRODUCTION

The study of boundary value problems involving linear differential equations is becoming a well-established area of analysis. Applying the extension theory of symmetric operators to concrete differential operators, we obtained a general characterization of selfadjoint extensions of symmetric differential operators [1, 10, 12, 13]; the domain of any selfadjoint extension of the minimal operator $T_0$ generated by the symmetric differential expression

$$\tau(y) = \sum_{k=0}^{N} p_k(x) D^{N-k} y, \quad a < x < b$$

can be described completely by means of a set of linearly independent boundary conditions which are separated at the endpoints of the interval $(a, b)$, where $p_0^{-1}(x), p_1(x), \ldots, p_N(x)$ are complex-valued and sufficiently smooth. For the general symmetric quasi-differential operators, the same results were obtained [2].

To study a similar problem in the case of non-symmetric differential expressions, Glazman introduced in [6] the concepts of the J-symmetric operator and the J-selfadjoint operator.

Let $J$ be a conjugate-linear involution from complex Hilbert space $\mathcal{H}$ into itself, i.e., $J$ is surjective and satisfies

$$J(x + y) = Jx + Jy$$

$$J(\lambda x) = \lambda Jx$$

$$J(xy) = JyJx$$

$$J^2x = x$$

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and

$$(Jx, y) = (y, x)$$

for all $x$ and $y$ in $A$. A closed, densely defined linear operator $A$ in $A$ is said to be $J$-symmetric if

$$(Jx, Ay) = (JAx, y) \quad (1.1)$$

for all $x$ and $y$ in $\mathcal{D}(A)$, the domain of $A$. It is clear that $A$ is $J$-symmetric if and only if

$$A \subset JA^*J \quad (1.2)$$

where $A^*$ is the adjoint of $A$. If

$$A = JA^*J \quad (1.3)$$

then $A$ is said to be $J$-selfadjoint.

Concerning the general theory of $J$-selfadjoint extensions of $J$-symmetric operators, by using different methods, Galindo [5] and Knowles [8] proved that every $J$-symmetric operator has a $J$-selfadjoint extension. For the problem of concretely describing all, the $J$-selfadjoint extensions of a given $J$-symmetric operator $A$, Zhikhar [15], in the late 1950s, derived partial results, based upon a work of Vishik; he gave a characterization of all the so-called "well-posed" $J$-selfadjoint extensions of $A$ when the regularity field, $\Pi(A)$, of $A$ was non-empty ($\tilde{A}$ is said to be a well-posed $J$-selfadjoint extension of $A$ relative to a fixed value $\lambda_0$ in $\Pi(A)$ if $A$ is a $J$-selfadjoint extension of $A$ and $\lambda_0$ is in $\Pi(\tilde{A})$). Later, Knowles [8] developed his more general theory describing all the $J$-selfadjoint extensions of $A$ provided the regularity field $\Pi(A)$ was non-empty, and deleted the restriction that the extensions were well-posed. Recently, Race [11] further removed the restriction on the regularity field, and gave a complete solution to the problem of describing all the $J$-selfadjoint extensions of a $J$-symmetric operator.

Applying this theory to the $J$-symmetric differential operator $T_0$ with only one singular endpoint, Knowles and Race gave a characterization of the boundary conditions which determine the domain of any $J$-selfadjoint extension of $T_0$ with minimal deficiency index (i.e., in the limit-point case), respectively, in the case when the regularity field $\Pi(T_0)$ was non-empty and in the general case. These boundary conditions are only restricted at the regular endpoint. In this paper we use Race's theory and Cao's and Sun's methods [1, 12] to remove the restriction that $T_0$ have a minimal deficiency index. We see that the boundary conditions which determine the
domain of any \( J \)-selfadjoint extension of \( T_0 \) not only have the restriction at the regular endpoint, but also have the restriction at the singular endpoint; however, these boundary conditions are separated.

2. Preliminaries

Most of the contents in this section are selected from \[11\].

**Lemma 2.1 [11].** Every \( J \)-symmetric operator has a \( J \)-selfadjoint extension.

From the definition of \( J \)-symmetric operator and the properties of conjugate-linear involution, we obtain that if \( \tilde{A} \) is a \( J \)-symmetric extension of \( A \), a \( J \)-symmetric operator, then \( JA^*J \) is a retraction of \( JA^*J \), i.e.,

\[
A \subseteq \tilde{A} \subseteq JA^*J \subseteq JA^*J.
\]  

(2.1)

Consequently any \( J \)-symmetric extension of \( A \) must be a restriction of \( JA^*J \) to a linear manifold of \( \mathcal{D}(JA^*J) \) containing \( \mathcal{D}(A) \). Since \( A \) is closable, and since the \( J \)-selfadjoint operator is closed, we need only search among the closed \( J \)-symmetric extensions of \( \tilde{A} \) to find all the \( J \)-selfadjoint extensions of \( A \). From (2.1) and Lemma 2.1, we know that any \( J \)-selfadjoint extension \( A' \) of \( A \) satisfies

\[
A \subseteq A' \subseteq JA^*J.
\]  

(2.2)

We define an inner product on \( \mathcal{D}(JA^*J) \) by

\[
(x, y)^* = (Jx, Jy) + (A^*Jx, A^*Jy),
\]  

(2.3)

i.e.,

\[
(x, y)^* = (y, x) + (JA^*Jy, JA^*Jx).
\]  

(2.4)

With this inner product, \( \mathcal{D}(JA^*J) \) becomes a Hilbert space \([3]\). Using \( \oplus \) and \( \ominus \) to denote the orthogonal sum and the orthogonal complement with respect to this inner product, respectively, it is then clear that the quotient space \( \mathcal{D}(JA^*J)/\mathcal{D}(A) \) is isomorphic to \( \mathcal{D}(JA^*J) \ominus \mathcal{D}(A) \), i.e.,

\[
\mathcal{D}(JA^*J)/\mathcal{D}(A) \cong \mathcal{D}(JA^*J) \ominus \mathcal{D}(A).
\]  

(2.5)

**Definition 2.2 [11].** We define the defect number of \( A \), written \( \text{def} \ A \), to be one-half of the dimension of the quotient space \( \mathcal{D}(JA^*J)/\mathcal{D}(A) \), i.e.,

\[
\text{def} \ A = \frac{1}{2} \dim \mathcal{D}(JA^*J)/\mathcal{D}(A).
\]  

(2.6)
It follows from (2.5) and (2.6) that
\[
def A = - \frac{1}{2} \dim [\mathcal{D}(JA^*J) \ominus \mathcal{D}(A)]. \tag{2.7}
\]

**Lemma 2.3 [11].** Let \( A \) be a closed J-symmetric operator; then
\[
\mathcal{D}(JA^*J) = \mathcal{D}(A) \oplus \{ y \in \mathcal{D}(A^*JA^*J): A^*JA^*Jy = -y \}.
\]

**Corollary 2.4 [11].** Let \( A \) be a closed J-symmetric operator; then the defect number of \( A \), \( \text{def } A \), is precisely one-half of the number of linearly independent solutions of \( A^*JA^*Jy = -y \).

Now, we consider the following differential expression over the interval \((a, b) = \mathbb{R}\),
\[
\tau(y) = \sum_{k=0}^{n} (-1)^{n-k} D^{n-k} p_k(x) D^{-k} y, \quad a < x < b, \tag{2.8}
\]
where the functions
\[
p_0^{-1}, p_1, \ldots, p_n \tag{2.9}
\]
are complex-valued, measurable over \((a, b)\) and Lebesgue integrable on all compact subsets of \((a, b)\), and \(p_0\) is non-vanishing.

The endpoint \(a\) is said to be regular if \(a < -\infty\), and each of the functions in (2.9) is integrable in every interval \([a, \beta]\), \(\beta < b\); otherwise \(a\) is said to be singular. Similar definitions apply to \(b\).

The expression \(\tau\) is said to be regular if it is regular at both endpoints; otherwise \(\tau\) is said to be singular.

We say
\[
\tau^+(y) = \sum_{k=0}^{n} (-1)^{n-k} D^{n-k} \overline{p_k(x)} D^{-k} y, \quad a < x < b \tag{2.10}
\]
to be the formal adjoint of \(\tau\).

We introduce the quasi-derivatives of a function \(y\), \(y^{[k]}\), \(k = 0, 1, \ldots, 2n\) as follows:
\[
y^{[0]} = D^k y, \quad k = 0, 1, \ldots, n - 1
\]
\[
y^{[n]} = p_0 D^n y
\]
\[
y^{[n+k]} = p_k D^{n-k} y - D y^{[n+k-1]}, \quad k = 1, \ldots, n; \tag{2.11}
\]
then \(\tau\) in (2.8) may be simply written by
\[
\tau(y) = y^{[2n]}. \tag{2.12}
\]
We define the maximal operator $T_1(\tau)$ generated by $\tau$ in $L^2(a, b)$ as follows:

$$\mathcal{D}(T_1(\tau)) = \{ y \in L^2(a, b): y^{[k]} \text{ is locally absolutely continuous for } 0 \leq k \leq 2n - 1 \text{ and } \tau(y) \text{ is in } L^2(a, b) \},$$

$$T_1(\tau) = \tau(y), \quad \text{for } y \in \mathcal{D}(T_1(\tau)).$$

We then define the minimal operator $T_0(\tau)$ generated by $\tau$ in $L^2(a, b)$ to be the closure of $\tau$ restricted to $C^\infty_0(a, b)$.

An important relation associated with $\tau$ is

$$T_0(\tau)^* = T_1(\tau^+).$$

**Lemma 2.5 (Lagrange’s identity) [11].**

$$\tau(y)\tau - y\tau(z) = D[y, z]$$

for any $y, z$ in $\mathcal{D}(T_1(\tau))$, where

$$[y, z] = \sum_{k=1}^{n} \left( y^{[k-1]} z^{[2n-k]} - y^{[2n-k]} z^{[k-1]} \right)$$

$$= r(\bar{z}) QC(y)$$

is called the Lagrange bilinear form corresponding to $\tau$. The matrix $Q$, the row vector $r(\bar{z})$, and the column vector $C(y)$ in (2.16) are given by

$$Q = \begin{pmatrix}
-1 & & \\
& \ddots & \\
& & -1
\end{pmatrix},
\begin{pmatrix}
\circ \\
& 1 \\
& & \circ
\end{pmatrix},
\begin{pmatrix}
y^{[0]} \bar{z}^{[1]} \cdots \bar{z}^{[2n-1]}
\end{pmatrix},
\begin{pmatrix}
y^{[0]} y^{[1]} \cdots y^{[2n-1]}
\end{pmatrix}^T.$$ (2.17) (2.18) (2.19)

Clearly $Q$ has the property

$$Q^T = Q^{-1}.$$ (2.17)

Let $J$ denote the usual operation of complex conjugation in $L^2(a, b)$; then $J$ is a conjugate-linear involution. We have
Lemma 2.6 [11]. 1. $T_0(\tau)$ is a closed, densely defined, $J$-symmetric operator, and the following is true:

$$T_1(\tau) = JT_0(\tau)^* J = JT_1(\tau^+) J.$$  

2. For any $y$ and $z$ in $D(T_1(\tau))$, both the limits $[y, \bar{z}]_a = \lim_{x \to a} [y, \bar{z}]$ and $[y, \bar{z}]_b = \lim_{x \to b} [y, \bar{z}]$ exist and we have

$$\int_a^b \tau(y) z dx = [y, \bar{z}]_b^a - \int_a^b y \tau(z) dx,$$  \hspace{1cm} (2.20)

where $[y, \bar{z}]_b^a = [y, \bar{z}]_b - [y, \bar{z}]_a$.

3. $D(T_0(\tau)) = \{ y \in D(T_1(\tau)) : [y, \bar{z}]_a^b = 0 \text{ for all } z \in D(T_1(\tau)) \}.$

It is not difficult to see that if $a$ is regular and $b$ is singular, then

$$D(T_0(\tau)) = \{ y \in D(T_1(\tau)) : y^{(k)}(a) = 0 \text{ for } 0 \leq k \leq 2n - 1, \text{ and } [y, \bar{z}]_a^b = 0 \text{ for all } z \in D(T_1(\tau)) \}.$$  \hspace{1cm} (2.21)

Lemma 2.7 [11].

$$D(T_1(\tau)) = D(T_0(\tau)) \oplus \{ y \in D(JT_1(\tau)JT_1(\tau)) : JT_1(\tau)JT_1(\tau) y = -y \}.$$  \hspace{1cm} (2.22)

Obviously,

$$JT_1(\tau) J = T_1(\tau^+).$$  \hspace{1cm} (2.23)

Corollary 2.8 [11]. 1. The defect number of $T_0(\tau)$, $\text{def } T_0$, is equal to one-half of the number of linearly independent solutions of the equation $\tau^+\tau(y) = -y$ for which both $y$ and $\tau(y)$ are in $L^2(a, b)$.

2. $0 \leq \text{def } T_0(\tau) \leq 2n.$

3. If $\tau$ is regular at $a$, then $n \leq \text{def } T_0(\tau) \leq 2n.$

4. If $d(\tau)$ is the mean deficiency index of $\tau$ defined by Kauffman in [7], i.e.,

$$d(\tau) = \frac{1}{2} \dim D(T_1(\tau))/D(T_0(\tau)),$$

then we have

$$\text{def } T_0(\tau) = d(\tau).$$  \hspace{1cm} (2.24)

Lemma 2.9 [11]. Let $m = \text{def } T_0(\tau)$; then the linear manifold $D$ in
\( \mathcal{D}(\mathcal{T}_1(\tau)) \) is the domain of a J-selfadjoint extension of \( T_0(\tau) \) if and only if there exist functions \( w_1, \ldots, w_m \) in \( \mathcal{D}(\mathcal{T}_1(\tau)) \) which satisfy

1. \( w_1, \ldots, w_m \) are linearly independent modulo \( \mathcal{D}(T_0(\tau)) \),

2. \( [w_r, w_j]_a^b = 0, \quad r, j = 1, \ldots, m \) (2.25)

and

3. \( \mathcal{D} = \{ y \in \mathcal{D}(\mathcal{T}_1(\tau)) : [y, w_j]_a^b = 0, j = 1, \ldots, m \} \) (2.26)

### 3. Some Auxiliary Lemmas

In this section we prove several lemmas which will be needed in the next section. Assume that \( a \) is regular and \( b \) is singular.

Let \( m = \text{def } T_0(\tau) \); it follows from Corollary 2.8 that there exist \( 2m \) linearly independent solutions, \( \theta_1, \ldots, \theta_{2m} \) of \( \tau^+ \tau(y) = -y \), such that

\[
\theta_r, \tau(\theta_r) \in L^2(a, b), \quad r = 1, \ldots, 2m.
\] (3.1)

We may assume that \( \theta_1, \ldots, \theta_{2m} \) are normally orthogonal with respect to the inner product (2.4) in which \( A \) is replaced by \( T_0(\tau) \). By Lemma 2.7 we have

\[
\mathcal{D}(\mathcal{T}_1(\tau)) = \mathcal{D}(T_0(\tau)) \oplus L(\theta_1, \ldots, \theta_{2m})
\] (3.2)

where \( L(\theta_1, \ldots, \theta_{2m}) \) denotes the linear span of \( \theta_1, \ldots, \theta_{2m} \).

**Lemma 3.1.** For \( 1 \leq k, l \leq 2m \),

\[
[\theta_k, \tau(\theta_l)]_a^b = (\theta_l, \theta_k)^* = \delta_{kl}.
\]

**Proof.**

\[
[\theta_k, \tau(\theta_l)]_a^b = \int_a^b \tau(\theta_k) \overline{\tau(\theta_l)} \, dx - \int_a^b \overline{\theta_k} \tau(\overline{\tau(\theta_l)}) \, dx
\]

\[
= \int_a^b \tau(\theta_k) \overline{\tau(\theta_l)} \, dx - \int_a^b \overline{\theta_k} \tau^+ \tau(\theta_l) \, dx
\]

\[
= \int_a^b \tau(\theta_k) \overline{\tau(\theta_l)} \, dx + \int_a^b \theta_k \bar{\theta}_l \, dx
\]

\[
= (\theta_l, \theta_k)^* = \delta_{kl}.
\]
**Lemma 3.2.** Both \( \{\theta_1, ..., \theta_{2m}\} \) and \( \{\tau(\theta_1), ..., \tau(\theta_{2m})\} \) are linearly independent sets and

\[
\mathcal{D}(JT_1(\tau) J) = \mathcal{D}(JT_0(\tau) J) \oplus L(\theta_1, ..., \theta_{2m})
\]

\[
= \mathcal{D}(JT_0(\tau) J) \oplus L(\tau(\theta_1), ..., \tau(\theta_{2m})).
\]

**Proof.** It is obvious that \( \{\theta_1, ..., \theta_{2m}\} \) is linearly independent. Suppose that there exist constants \( \alpha_1, ..., \alpha_{2m} \) such that

\[
\alpha_1 \tau(\theta_1) + \cdots + \alpha_{2m} \tau(\theta_{2m}) = 0;
\]

then

\[
\tau^+ \tau(\alpha_1 \theta_1 + \cdots + \alpha_{2m} \theta_{2m}) = 0.
\]

Hence

\[
\alpha_1 \theta_1 + \cdots + \alpha_{2m} \theta_{2m} = 0.
\]

so

\[
\alpha_1 = \cdots = \alpha_{2m} = 0
\]

and \( \{\tau(\theta_1), ..., \tau(\theta_{2m})\} \) is linearly independent.

Since \( JT_0(\tau) J \) is the minimal operator associated with \( \tau^+ \), it is \( J \)-symmetric. But since \( JT_1(\tau) J \) is the maximal operator associated with \( \tau^+ \), it follows from the definition of the defect number that

\[
def JT_0(\tau) J = \frac{1}{2} \dim \mathcal{D}(JT_1(\tau) J)/\mathcal{D}(JT_0(\tau) J)
\]

\[
= \frac{1}{2} \dim \mathcal{D}(T_1(\tau))/\mathcal{D}(T_0(\tau))
\]

\[
= \text{def } T_0(\tau)
\]

\[
= m.
\]

It is clear that both \( \{\theta_1, ..., \theta_{2m}\} \) and \( \{\tau(\theta_1), ..., \tau(\theta_{2m})\} \) are linearly independent solutions of \( T_1(\tau) JT_1(\tau) y = -y \). So, from Lemma 2.3, we have

\[
\mathcal{D}(JT_1(\tau) J) = \mathcal{D}(JT_0(\tau) J) \oplus L(\theta_1, ..., \theta_{2m})
\]

\[
= \mathcal{D}(JT_0(\tau) J) \oplus L(\tau(\theta_1), ..., \tau(\theta_{2m})).
\]

**Lemma 3.3.** \( \text{rank}([\theta_k, \theta_s]_{b})_{1 \leq k, s \leq 2m} = 2m - 2n. \)

**Proof.** \( (1) \) \( \text{rank}([\theta_k, \theta_s]_{b})_{1 \leq k, s \leq 2m} \leq 2m - 2n. \)
By [10, Lemma 17.2], there exists a set of functions $z_l$, $l = 1, ..., 2n$ in $\mathcal{D}(T_1(\tau))$ which satisfy the conditions
\[
z^{[j-1]}(a) = \delta_{lj}, \quad z^{[j-1]}(a') = 0, \quad j = 1, ..., 2n
\]
and
\[
z_l(x) = 0, \quad a' \leq x < b,
\]
where $a'$ is given between $a$ and $b$. By (3.2),
\[
z_l = z_{l0} + \sum_{k=1}^{2n} a_{lk} \theta_k, \quad l = 1, ..., 2n, \quad (3.5)
\]
where $z_{l0} \in \mathcal{D}(T_0(\tau))$. Hence
\[
z^{[j-1]} = z_{l0}^{[j-1]} + \sum_{k=1}^{2m} a_{lk} \theta_k^{[j-1]}, \quad l, j = 1, ..., 2n.
\]
Let $x = a$, by (2.21) and (3.4) we have
\[
\begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}
= \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{12m} \\
a_{21} & a_{22} & \cdots & a_{22m} \\
\cdots & \cdots & \cdots & \cdots \\
a_{2n1} & a_{2n2} & \cdots & a_{2n2m}
\end{pmatrix}
\begin{pmatrix}
\theta_1(a) & \theta_1^{[1]}(a) & \cdots & \theta_1^{[2^{n-1}]}(a) \\
\theta_2(a) & \theta_2^{[1]}(a) & \cdots & \theta_2^{[2^{n-1}]}(a) \\
\vdots & \vdots & \cdots & \vdots \\
\theta_{2m}(a) & \theta_{2m}^{[1]}(a) & \cdots & \theta_{2m}^{[2^{n-1}]}(a)
\end{pmatrix}.
\]
So
\[
\text{rank}(a_{lk})_{2n \times 2m} = 2n, \quad \text{rank}(\theta_k^{[j-1]}(a))_{2m \times 2n} = 2n.
\]
By (3.5),
\[
[z_l, \theta_s]_b = [z_{l0}, \theta_s]_b + \sum_{k=1}^{2m} a_{lk} [\theta_k, \theta_s]_b, \quad l = 1, ..., 2n, \quad s = 1, ..., 2m.
\]
Note that $[z_l, \theta_s]_b = [z_{l0}, \theta_s]_b = 0$, we have
\[
0 = (a_{lk})_{2n \times 2m}([\theta_k, \theta_s]_b)_{2m \times 2m}. \quad (3.6)
\]
From this fact we obtain
\[ \text{rank}([\theta_k, \theta_s]_{1 \leq k, s \leq 2m}) \leq 2m - 2n. \] (2)
\[ \text{rank}([\theta_k, \theta_s]_{1 \leq k, s \leq 2m}) \geq 2m - 2n. \]

From Lemma 3.2 there exist constants \( c_{sl}, s, l = 1, \ldots, 2m, \) such that
\[ \theta_s = \sum_{l=1}^{2m} c_{sl}\tau(\theta_l) \]
and \( \text{rank}(c_{sl})_{2m \times 2m} = 2m. \) So by Lemma 3.1,
\[
\begin{align*}
[\theta_k, \theta_s]_{a}^b &= \sum_{l=1}^{2m} c_{sl}[\theta_k, \tau(\theta_l)]_{a}^b \\
&= \sum_{l=1}^{2m} c_{sl}\delta_{lk} = \bar{c}_{sk};
\end{align*}
\]
thus
\[ \text{rank}([\theta_k, \theta_s]_{a}^b)_{1 \leq k, s \leq 2m} = 2m. \]

However,
\[ \text{rank}([\theta_k, \theta_s]_{a}^b)_{1 \leq k, s \leq 2m} \leq \text{rank}([\theta_k, \theta_s]_{b}^b)_{1 \leq k, s \leq 2m} + \text{rank}([\theta_k, \theta_s]_{a}^a)_{1 \leq k, s \leq 2m}. \]

So
\[ \text{rank}([\theta_k, \theta_s]_{b})_{1 \leq k, s \leq 2m} \geq \text{rank}([\theta_k, \theta_s]_{a}^a)_{1 \leq k, s \leq 2m} - \text{rank}([\theta_k, \theta_s]_{a}^a)_{1 \leq k, s \leq 2m} \geq 2m - 2n. \]

The conclusion follows from these two inequalities.

Without loss of generality, we may assume that the first \( 2m - 2n \) rows of \( E = ([\theta_k, \theta_s]_{b})_{1 \leq k, s \leq 2m} \) are linearly independent. Let
\[ E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \] (3.7)
where \( E_1 \) is a \( (2m - 2n) \times 2m \) matrix and \( E_2 \) is a \( 2n \times 2m \) matrix; then
\[ \text{rank} E_1 = 2m - 2n. \] (3.8)
Assume in (3.6) that
\[(a_{jk})_{2n \times 2m} = (C_{2n \times (2m - 2n)} D_{2n \times 2n});\]
then we have

**Lemma 3.4.** rank \(D_{2n \times 2n} = 2n.\)

**Proof.** Equation (3.6) can be written as
\[CE_1 + DE_2 = 0. \tag{3.9}\]
If rank \(D < 2n,\) then there exists a non-singular matrix of order \(2n,\) say \(G,\) such that
\[GD = \begin{pmatrix} \bar{D}_{(2n - 1) \times 2n} \\ 0_{1 \times 2n} \end{pmatrix}.\]
Denote that
\[GC = \begin{pmatrix} \bar{C}_{(2n - 1) \times (2m - 2n)} \\ a_1 \times (2m - 2n) \end{pmatrix}.\]
Since \(GCE_1 + GDE_2 = 0,\) we have
\[\alpha E_1 = 0_{1 \times 2m}.\]
But rank \(E_1 = 2m - 2n,\) so
\[\alpha = 0_{1 \times (2m - 2n)}.\]
This contradicts the fact that rank \((a_{jk})_{2n \times 2m} = 2n.\)

**Lemma 3.5.** Suppose \(\{\theta_1, ..., \theta_{2m}\}\) are the functions defined before which satisfy (3.8); then each \(\theta_k, k = 2m - 2n + 1, ..., 2m,\) has a unique representation
\[\theta_k = \theta_{k0} + \sum_{l=1}^{2n} c_{kl} z_l + \sum_{e=1}^{2m-2n} b_{ke} \theta_e, \tag{3.10}\]
where \(\theta_{k0} \in \mathcal{D}(T_0(\tau))\) and \(z_l, l = 1, ..., 2n,\) satisfy (3.4).

**Proof.** From (3.5), we have
\[z_l = z_0 + \sum_{s=1}^{2m-2n} a_{ls} \theta_s + \sum_{k=2m-2n+1}^{2m} a_{lk} \theta_k, \quad l = 1, ..., 2n.\]
By Lemma 3.4,

\[ \text{rank}(a_{jk})_{1 \leq l \leq 2m, 1 \leq k \leq 2m} = \text{rank} D = 2n, \]

so we can solve \( \theta_k, k = 2m - 2n + 1, \ldots, 2m \), by Cramer's rule from the above equations and obtain the unique representation

\[ \theta_k = \theta_{k0} + \sum_{l=1}^{2n} c_{kl} z_l + \sum_{s=1}^{2m-2n} b_k \theta_s, \]

where \( \theta_{k0} = -\sum_{l=1}^{2n} c_{kl} z_{k0} \in D(T_0(\tau)) \). The lemma is proved.

**Lemma 3.6.** Suppose \( \{\theta_1, \ldots, \theta_{2m}\} \) and \( \{z_1, \ldots, z_{2n}\} \) are the same as before; then each \( y \) in \( D(T_1(\tau)) \) can be uniquely written as

\[ y = y_0 + \sum_{l=1}^{2n} d_l z_l + \sum_{s=1}^{2m-2n} c_s \theta_s, \quad (3.11) \]

where \( y_0 \in D(T_0(\tau)) \).

Clearly, Lemma 3.6 is equivalent to

\[ D(T_1(\tau)) = D(T_0(\tau)) + L(z_1, \ldots, z_{2n}) \]

\[ + L(\theta_1, \ldots, \theta_{2m-2n}). \quad (3.12) \]

**Proof of Lemma 3.6.** From (3.2), for each \( y \) in \( D(T_1(\tau)) \), we have a unique representation

\[ y = y_0' + \sum_{s=1}^{2m-2n} d_s' \theta_s + \sum_{j=2m-2n+1}^{2m} d_j' \theta_j, \quad (3.13) \]

where \( y_0' \in D(T_0(\tau)) \). By Lemma 3.5, each \( \theta_j, j = 2m - 2n + 1, \ldots, 2m \), can be uniquely written as

\[ \theta_j = \theta_{j0} + \sum_{l=1}^{2n} c_{jl} z_l + \sum_{s=1}^{2m-2n} b_j \theta_s. \quad (3.14) \]

Now substituting (3.14) into (3.13) we have (3.11), in which

\[ y_0 = y_0' + \sum_{j=2m-2n+1}^{2m} d_j' \theta_{j0}, \]

\[ d_l = \sum_{j=2m-2n+1}^{2m} d_j' c_{jl}, \]

\[ c_s = d_s' + \sum_{j=2m-2n+1}^{2m} d_j' b_j. \]
4. The Main Result

Suppose \( \{ \theta_1, ..., \theta_{2m} \} \) are the functions defined in Section 3 which satisfy (3.8); let

\[
B = ([\theta_k, \theta_s]_b)_{1 \leq k, s \leq 2m - 2n}.
\]

**Theorem 4.1.** Let \( T_0(\tau) \) be the minimal operator associated with \( \tau \) defined by (2.8) on the interval \([a, b)\). Let \( \tau \) be regular at \( a \) and singular at \( b \), and \( m = \text{def} \ T_0(\tau) \) \((n < m < 2n)\). Then the linear manifold \( \mathcal{D} \) in \( \mathcal{D}(T_1(\tau)) \) is the domain of a \( J \)-selfadjoint extension of \( T_0(\tau) \) if and only if there exist an \( m \times 2n \) matrix \( M \) and an \( m \times (2m - 2n) \) matrix \( G \) which satisfy

1. \( \text{rank}(MG) = m \),
2. \( MQM^T = GBG^T \),

and such that

\[
\mathcal{D} = \left\{ y \in \mathcal{D}(T_1(\tau)) : M \begin{pmatrix} y^{[0]}(a) \\ \vdots \\ y^{[2n-1]}(a) \end{pmatrix} - G \begin{pmatrix} [y, \theta_1]_b \\ \vdots \\ [y, \theta_{2m-2n}]_b \end{pmatrix} = 0 \right\}.
\]

**Proof of the Sufficiency.** Suppose that \( \mathcal{D}, M, \) and \( G \) satisfy conditions (1), (2) and (3). Let

\[
QM^T = (\xi_{ij})_{2n \times m}, \quad G = (\eta_{ij})_{m \times (2m - 2n)}
\]

and

\[
\nu_r = \sum_{j=1}^{2m-2n} \eta_{rj} \theta_j, \quad r = 1, ..., m.
\]

By [10, Lemma 17.2] there exist functions \( w_r, r = 1, ..., m, \) in \( \mathcal{D}(T_1(\tau)) \) which satisfy the conditions

\[
\begin{align*}
w_r^{[j-1]}(a) &= \xi_{jr}, \\
w_r^{[j-1]}(a') &= w_r^{[j-1]}(a'), \\
w_r(x) &= v_r(x), \quad a' < x < b,
\end{align*}
\]

where \( a' \) is given between \( a \) and \( b \). It follows from (4.1), (4.3) and (2.16) that

\[
M \begin{pmatrix} y^{[0]}(a) \\ \vdots \\ y^{[2n-1]}(a) \end{pmatrix} = (\xi_{ij})^T QC(y)|_a = W(a)^T QC(y)|_a = \begin{pmatrix} [y, \bar{w}_1]_a \\ \vdots \\ [y, \bar{w}_m]_a \end{pmatrix}.
\]
where

\[
W(a) = \begin{pmatrix}
    w_{1}^{(0)}(a) & w_{2}^{(0)}(a) & \cdots & w_{m}^{(0)}(a) \\
    w_{1}^{(1)}(a) & w_{2}^{(1)}(a) & \cdots & w_{m}^{(1)}(a) \\
    \vdots & \vdots & \ddots & \vdots \\
    w_{1}^{(2n-1)}(a) & w_{2}^{(2n-1)}(a) & \cdots & w_{m}^{(2n-1)}(a)
\end{pmatrix}.
\]

But from (4.2) and (4.3), we have

\[
G\begin{pmatrix}
y, \tilde{\theta}_1 \\
y, \tilde{\theta}_{2m-2n}
\end{pmatrix}_b = \begin{pmatrix}
y, \sum_{j=1}^{2m-2n} \eta_1 \tilde{\theta}_j \\
y, \sum_{j=1}^{2m-2n} \eta_{m_j} \tilde{\theta}_j
\end{pmatrix}_b = \begin{pmatrix}
y, \tilde{w}_1 \\
y, \tilde{w}_m
\end{pmatrix}_b.
\]

So condition (3) in Theorem 4.1 becomes condition (iii) in Lemma 2.9. It remains to show that \(w_1, \ldots, w_m\) satisfy conditions (i) and (ii) in Lemma 2.9.

If condition (i) is not true, then there exist constants \(c_1, \ldots, c_m\), not all zero, such that

\[
u = \sum_{r=1}^{m} c_r u_r \in D(T_0(\tau)).
\]

Hence by (2.21),

\[
0 = \begin{pmatrix}
u^{(0)}(a) \\
\vdots \\
u^{(2n-1)}(a)
\end{pmatrix} = W(a) \begin{pmatrix}c_1 \\
\vdots \\
c_m\end{pmatrix} = (\xi_{ij})_{2n \times m} \begin{pmatrix}c_1 \\
\vdots \\
c_m\end{pmatrix}
\]

i.e.,

\[
0 = (c_1 \cdots c_m)(\xi_{ij})_{2n \times m} = (c_1 \cdots c_m) MQ^T.
\]

Since rank \(Q^T = 2n\),

\[
(c_1 \cdots c_m) M = 0. \tag{4.4}
\]

When \(a' \leq x < b\), \(w_r = v_r = \sum_{j=1}^{2m-2n} \eta_j \theta_j\), so \(u = \sum_{r=1}^{m} c_r \sum_{j=1}^{2m-2n} \eta_j \theta_j\), \(a' \leq x < b\). Since \(u \in D(T_0(\tau))\), from (2.21) we have

\[
0 = ([u, \tilde{\theta}_1]_b \cdots [u, \tilde{\theta}_{2m-2n}]_b) = (c_1 \cdots c_m) Ge_1.
\]

From (3.8), \(\text{rank } E_1 = 2m - 2n\), so we have

\[
(c_1 \cdots c_m) G = 0. \tag{4.5}
\]
It follows from (4.4) and (4.5) that
\[(c_1 \cdots c_m)(MG) = 0.\]

This contradicts the fact that \(\text{rank}(MG) = m\).

To show that condition (ii) of Lemma 2.9 is true: According to (4.2) and (4.3), we have
\[
([w_r, \tilde{w}_j])_{1 \leq r, j \leq m} = \left(\sum_{k=1}^{2m-2n} \eta_{rk} \theta_k, \sum_{s=1}^{2m-2n} \tilde{\eta}_{js} \tilde{\theta}_s\right)_{1 \leq r, j \leq m}
\]
\[= \sum_{k=1}^{2m-2n} \sum_{s=1}^{2m-2n} \eta_{rk} [\theta_k, \tilde{\theta}_s] b_{ij} \eta_{js} = GBG^T.
\]

and
\[
([w_r, \tilde{w}_j])_{a} |_{1 \leq r, j \leq m} = W(a)^T Q W(a) = (\xi_{1})_{2m \times m} Q(\xi_{1})_{2m \times m}
\]
\[= MQM^T.
\]

So, by condition (2),
\[
([w_r, \tilde{w}_j])_{b} |_{1 \leq r, j \leq m} = GBG^T - MQM^T - 0.
\]

Thus the sufficiency of Theorem 4.1 is proved.

**Proof of the Necessity.** Suppose that \(\mathcal{D}\) is the domain of a \(J\)-selfadjoint extension of \(T_0(\tau)\). It then follows from Lemma 2.9 that there exist functions \(w_1, \ldots, w_m \in \mathcal{D}(T_1(\tau))\) satisfying conditions (i) and (ii) of Lemma 2.9, and \(\mathcal{D}\) is the domain determined by condition (iii). By Lemma 3.6, \(w_r, r = 1, \ldots, m\), can be uniquely written as
\[
w_r = w_{r0} + \sum_{j=1}^{2n} d_{rj} z_j + \sum_{s=1}^{2m-2n} \eta_{rs} \theta_s,
\]
where \(w_{r0} \in \mathcal{D}(T_0(\tau))\). Let
\[M = W(a)^T Q, \quad G = (\eta_{rs})_{m \times (2m - 2n)}.
\]

From (2.16)
\[
\left(\begin{array}{c}
[y, \tilde{w}_1]_a \\
\vdots \\
[y, \tilde{w}_m]_a
\end{array}\right) = W(a)^T QC(y) |_a = M \left(\begin{array}{c}
y^{[0]}(a) \\
\vdots \\
y^{[2m-1]}(a)
\end{array}\right).
\]
But \([y, \bar{w}_0]_b = 0\) and \([y, \bar{z}]_b = 0\), from (4.6),

\[
\begin{pmatrix}
[y, \bar{w}_1]_b \\
\vdots \\
[y, \bar{w}_m]_b
\end{pmatrix} = \begin{pmatrix}
[y, \sum_{s=1}^{2m-2n} \bar{\eta}_s \theta_s]_b \\
\vdots \\
[y, \sum_{s=1}^{2m-2n} \bar{\eta}_s \theta_s]_b
\end{pmatrix} = G \begin{pmatrix}
[y, \theta_1]_b \\
\vdots \\
[y, \theta_{2m-2n}]_b
\end{pmatrix}
\]

and hence the boundary condition (iii) of Lemma 2.9 becomes condition (3) of Theorem 4.1. To complete the proof, it suffices to show that \(M\) and \(G\) satisfy conditions (1) and (2).

1) If \(\text{rank } (MG)<m\), then there exist constants \(c_1, ..., c_m\), not all zero, such that

\[(c_1 \cdots c_m)(MG) = 0.\]  \hspace{1cm} (4.7)

So \((c_1 \cdots c_m) W(a)^T Q = (c_1 \cdots c_m) M = 0\). Since \(Q\) is non-singular,

\[W(a) \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = 0.\]

Let

\[u = \sum_{r=1}^m c_r w_r;\]

then

\[
\begin{pmatrix}
u^{(0)}(a) \\
\vdots \\
u^{(2n-1)}(a)
\end{pmatrix} = W(a) \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = 0.\]  \hspace{1cm} (4.8)

By (4.6), we have

\[u = \sum_{r=1}^m c_r z_{r0} + \sum_{j=1}^{2n} \sum_{r=1}^m c_r d_{rj} z_j + \sum_{r=1}^m \sum_{s=1}^{2m-2n} c_r \eta_{rs} \theta_s,\]

but from (4.7), \((c_1 \cdots c_m) G = 0\), so

\[u = \sum_{r=1}^m c_r w_{r0} + \sum_{j=1}^{2n} \sum_{r=1}^m c_r d_{rj} z_j.\]
Consequently for any \( y \in \mathcal{D}(T_1(\tau)) \), we get

\[
[u, y]_b = 0.
\] (4.9)

Then \( u \in \mathcal{D}(T_0(\tau)) \) follows from (4.8), (4.9) and (2.21). This contradicts the fact that \( w_1, ..., w_m \) are linearly independent modulo \( \mathcal{D}(T_0(\tau)) \).

(2) From (2.16) and the definition of \( M \), we have

\[
([w_r, \bar{w}_j]_b)_{1 \leq r, j \leq m} = W(a)^T QW(a) = MQM^T.
\]

But \( w_{r_0} \in \mathcal{D}(T_0(\tau)) \) and \( z_j, j = 1, ..., 2n \), are vanishing on \([a', b)\); from (4.6) we get

\[
([w_r, \bar{w}_j]_b)_{1 \leq r, j \leq m} = \left( \sum_{k=1}^{2m-2n} \eta_{rk} \theta_k, \sum_{s=1}^{2m-2n} \bar{\eta}_{js} \theta_s \right)_{1 \leq r, j \leq m} = GBG^T.
\]

Hence

\[
MQM^T - GBG^T = ( - [w_r, \bar{w}_j]_b)_{1 \leq r, j \leq m} = 0.
\]

The proof is completed.

5. The Case with Two Singular Endpoints

It is similar to the case of symmetric differential operators in [13] that the result of Theorem 4.1 can be generalized to the case when \( \tau \) is singular both at \( a \) and at \( b \). For this we need to prove Kodaira's deficiency index formula for \( J \)-symmetric differential operators.

Let \( \tau(y) \) be the differential expression defined by (2.8) which is singular both at \( a \) and at \( b \). \( T_0 \) is the minimal operator corresponding to \( \tau \) and \( \mathcal{D}(T_0) \) is the domain of \( T_0 \). Choose \( c \) to be a fixed point between \( a \) and \( b \) and write \( T_0^- \) and \( T_0^+ \) as the minimal operators generated in \( L^2(a, c) \) and \( L^2[c, b) \), respectively, by \( \tau; \mathcal{D}(T_0^-) \) and \( \mathcal{D}(T_0^+) \) are the domains associated with them. We use \( T_1, T_1^- \) and \( T_1^+ \) to denote the maximal operators generated in \( L^2(a, b) \), \( L^2(a, c) \) and \( L^2[c, b) \) by \( \tau \), respectively; \( \mathcal{D}(T_1), \mathcal{D}(T_1^-) \) and \( \mathcal{D}(T_1^+) \) are the domains associated with them. Let \( T \) be the restriction of \( \tau \) to the following linear manifold:

\[
\mathcal{D}(T) = \{ y \in \mathcal{D}(T_0); y^{[k-1]}(c) = 0, k = 1, ..., 2n \};
\] (5.1)

then \( T \subset T_0 \).
Lemma 5.1. T is a closed, densely defined J-symmetric operator in $L^2(a, b)$ and

$$\mathcal{D}(JT^*) = \left\{ z \in L^2(a, b) : z = \begin{cases} z^-, & a < x \leq c \\ z^+, & c \leq x < b, \end{cases} \right\},$$

where $z^- \in \mathcal{D}(T^-_0)$ and $z^+ \in \mathcal{D}(T^+_0)$. \hfill (5.2)

$$JT^*(z) = \begin{cases} \tau(z^-), & a < x \leq c, \\ \tau(z^+), & c \leq x < b, \end{cases} z \in \mathcal{D}(JT^*).$$ \hfill (5.3)

Proof. From (2.21) we see that $y \in \mathcal{D}(T)$ if and only if

$$y = \begin{cases} y^-, & a < x \leq c \\ y^+, & c \leq x < b, \end{cases}$$

where $y^- \in \mathcal{D}(T^-_0)$ and $y^+ \in \mathcal{D}(T^+_0)$. Let $h$ be a function in $L^2(a, b)$ such that $(y, h) = 0$ for all $y$ in $\mathcal{D}(T)$; then we have

$$(y, h) = \int_a^b yh \, dx = \int_a^c y^- h \, dx + \int_c^b y^+ h \, dx = 0. \hfill (5.5)$$

Let $y^- = 0$ and $y^+$ be an arbitrary function in $\mathcal{D}(T^+_0)$; then from the denseness of $\mathcal{D}(T^+_0)$ in $L^2[c, b]$ we obtain that $h = 0$ on $[c, b)$. For the same reason it follows that $h = 0$ on $(a, c]$. So $h = 0$ over $(a, b)$; this shows that $\mathcal{D}(T)$ is dense in $L^2(a, b)$.

We now consider $JT^*$. Let $z$ and $z^*$ be in $L^2(a, b)$ such that

$$(Ty, z) = (y, z^*) \hfill (5.6)$$

for all $y$ in $\mathcal{D}(T)$. If we use $z^-$ and $z^+$ to denote the restrictions of $z$ to $(a, c]$ and $[c, b)$, respectively, then we have that the left side of (5.6) equals

$$\int_a^b \tau(y) \bar{z} \, dx = \int_a^c \tau(y^-) \bar{z}^- \, dx + \int_c^b \tau(y^+) \bar{z}^+ \, dx$$

$$= (T^-_0(y^-), z^-) + (T^+_0(y^+), z^+) \hfill (5.7)$$

and the right side of (5.6) equals

$$\int_a^b yz^* \, dx = \int_a^c y^- z^*^- \, dx + \int_c^b y^+ z^*+ \, dx$$

$$- (y^-, z^-) + (y^+, z^+) \hfill (5.8)$$
where $z^*$ and $z^{**}$ denote the restrictions of $z^*$ to $(a, c]$ and $[c, b)$, respectively.

Let $y^- = 0$ and $y^+$ in $\mathcal{D}(T^+_0)$ be free; then by (5.7) and (5.8),

\[(T^+_0(y^+), z^+) = (y^+, z^{**}).\]  

(5.9)

This shows that $z^+ \in \mathcal{D}(T^+_0)$ and $z^{**} = T^+_0(z^+) = JT^+_1 J(z^+)$. For the same reason, it follows that $z^- \in \mathcal{D}(T^-_0)$ and $z^{*--} = T^-_0(z^-) = JT^-_1 J(z^-)$. Consequently, $z \in \mathcal{D}(JT^*J)$ if and only if $z^- \in \mathcal{D}(T^-_0)$, $z^+ \in \mathcal{D}(T^+_0)$ and

\[JT^*J(z) = \begin{cases} \tau(z^-), & a < x \leq c \\ \tau(z^+), & c \leq x < b. \end{cases}\]

From the above analysis we see that $T$ is in fact "the direct sum" of $T^-_0$ and $T^+_0$. Since $T^-_0$ and $T^+_0$ are both closed $J$-symmetric operators, $T$ is also a closed $J$-symmetric operator.

**Lemma 5.2.** def $T = def T_0 + 2n.$

**Proof.** By [10, Lemma 17.2], there exist $\tilde{u}_j$, $j = 1, \ldots, 2n$, in $\mathcal{D}(T^-_1)$ and $\tilde{v}_j$, $j = 1, \ldots, 2n$, in $\mathcal{D}(T^+_1)$ satisfying the conditions

\[\tilde{u}_j^{[k-1]}(c) = \delta_{jk}, \quad \tilde{u}_j^{[k-1]}(a') = 0, \quad k = 1, \ldots, 2n\]

\[\tilde{u}_j(x) = 0, \quad a < x \leq a'\]  

(5.10)

\[\tilde{v}_j^{[k-1]}(c) = \delta_{jk}, \quad \tilde{v}_j^{[k-1]}(b') = 0, \quad k = 1, \ldots, 2n\]

\[\tilde{v}_j(x) = 0, \quad b' < x < b,\]  

(5.11)

where $a'$ is between $a$ and $c$ and $b'$ is between $c$ and $b$. For $j = 1, \ldots, 2n$ let

\[u_j = \begin{cases} \tilde{u}_j, & a < x \leq c \\ 0, & c \leq x < b \end{cases}, \quad v_j = \begin{cases} 0, & a < x \leq c \\ \tilde{v}_j, & c \leq x < b \end{cases}, \quad \phi_j = u_j + v_j,\]  

(5.12)

then $u_j, v_j \in \mathcal{D}(JT^*J)$ and $\phi_j \in \mathcal{D}(T_0)$. It is obvious that sets $\{u_1, \ldots, u_{2n}\}$, $\{v_1, \ldots, v_{2n}\}$ and $\{\phi_1, \ldots, \phi_{2n}\}$ are all linearly independent modulo the manifold

\[\mathcal{D}_0 = \{y \in \mathcal{D}(T_1): y^{[k-1]}(c) = 0, k = 1, \ldots, 2n\}.\]  

(5.13)

It is easy to prove that

\[\mathcal{D}(T_0) = \mathcal{D}(T) \oplus L(\phi_1, \ldots, \phi_{2n}),\]  

(5.14)

\[\mathcal{D}(JT^*J) = \mathcal{D}_0 \oplus L(u_1, \ldots, u_{2n}) \oplus L(v_1, \ldots, v_{2n}),\]  

(5.15)
and

$$\mathcal{D}(T_1) = \mathcal{D}_0 + \mathcal{L}(\phi_1, ..., \phi_{2n}). \quad (5.16)$$

Since

$$\mathcal{D}(T) \subset \mathcal{D}(T_0) \subset \mathcal{D}(T_1) \subset \mathcal{D}(JT^*J), \quad (5.17)$$

we have

$$\text{def } T = \frac{1}{2} \dim \mathcal{D}(JT^*J)/\mathcal{D}(T)$$

$$= \frac{1}{2} \left[ \dim (T_1)/\mathcal{D}(T_0) + \dim \mathcal{D}(T_0)/\mathcal{D}(T) + \dim \mathcal{D}(JT^*J)/\mathcal{D}(T_1) \right]$$

$$= \text{def } T_0 + 2n.$$

**Lemma 5.3.** \( \text{def } T = \text{def } T_0^- + \text{def } T_0^+ \).

**Proof.** By Corollary 2.4, \( \text{def } T \) is equal to one-half of the number of linearly independent solutions of \( T^*JT^*Jy = -y \). From Lemma 5.1 we know that \( y \) is a solution of \( T^*JT^*Jy = -y \) if and only if \( y^- \) and \( y^+ \) are the solutions of \( \tau^+\tau(y) = -y \) for which \( y^- \) and \( \tau(y^-) \) belong to \( L^2(a, c] \) but \( y^+ \) and \( \tau(y^+) \) belong to \( L^2[c, b) \), where \( y^- \) and \( y^+ \) are the restrictions of \( y \) to \( (a, c] \) and \( [c, b) \), respectively. Let \( s = \text{def } T_0^- \) and \( l = \text{def } T_0^+ \); it follows from Corollary 2.4 that there exist \( 2s \) solutions, say \( \psi_1, ..., \psi_{2s} \), of \( \tau^+\tau(y) = -y \) such that \( \psi_k \) and \( \tau(\psi_k) \), \( k = 1, ..., 2s \), are in \( L^2(a, c] \). Also, there exist \( 2l \) solutions, say \( \psi_{2s+1}, ..., \psi_{2s+2l} \), of \( \tau^+\tau(y) = -y \) such that \( \psi_{2s+k} \) and \( \tau(\psi_{2s+k}) \), \( k = 1, ..., 2l \), are in \( L^2[c, b) \). Let

$$\theta_r = \begin{cases} \psi_r, & a < x \leq c, \\ 0, & c \leq x < b, \end{cases} \quad r = 1, ..., 2s,$$

$$\theta_{2s+j} = \begin{cases} 0, & a < x \leq c, \\ \psi_{2s+j}, & c \leq x < b, \end{cases} \quad j = 1, ..., 2l,$$

then \( \theta_1, ..., \theta_{2s+2l} \) are linearly independent solutions of the equation \( T^*JT^*Jy = -y \). This shows that \( \text{def } T = s + l = \text{def } T_0^- + \text{def } T_0^+ \).

From Lemmas 5.2 and 5.3, we obtain

**Theorem 5.4** (Kodaira's formula). \( \text{def } T_0 = \text{def } T_0^- + \text{def } T_0^+ - 2n. \)

Assume \( s = \text{def } T_0^- \) and \( l = \text{def } T_0^+ \), \( n \leq s, l \leq 2n \). From Corollary 2.4, the equation \( \tau^+\tau(y) = -y \) has \( 2s \) linearly independent solutions \( \theta_1^-, ..., \theta_{2s}^- \) in \( L^2(a, c] \) and \( 2l \) linearly independent solutions \( \theta_1^+, ..., \theta_{2l}^+ \) in \( L^2(c, b) \) such that

$$\text{rank}([\theta_k^-, \theta_r^-]_{1 \leq k \leq 2s, 1 \leq r \leq 2s}) = 2s - 2n.$$  \quad (5.18)
and

\[
\text{rank}(\begin{bmatrix} \theta_k^+, \theta_r^- \end{bmatrix})_{1 \leq k, r \leq 2l-2n} = 2l-2n - 2l - 2n
\]  

(5.19)

Let

\[
B^- = \begin{bmatrix} \theta_k^-, \theta_r^- \end{bmatrix}_{1 \leq k, r \leq 2s-2n}, \\
B^+ = \begin{bmatrix} \theta_k^+, \theta_r^+ \end{bmatrix}_{1 \leq k, r \leq 2l-2n}.
\]  

(5.20)

(5.21)

We have

Theorem 5.5. Let \( T \) be singular at \( a \) and \( b \), \( s = \text{def } T_0^- \) and \( l = \text{def } T_0^+ \); then \( m = \text{def } T_0 = s + l - 2n \). A linear manifold \( \mathcal{D} \) in \( \mathcal{D}(T) \) is the domain of a \( J \)-selfadjoint extension if and only if there exist an \( m \times (2s-2n) \) matrix \( M \) and an \( m \times (2l-2n) \) matrix \( G \) such that the following conditions are satisfied:

1. \( \text{rank}(MG) = m \),
2. \( MB^- M^T = GB^+ G^T \),
3. \( \mathcal{D} = \left\{ y \mathcal{D}(T_1): M \left( \begin{bmatrix} y, \theta_1^- \end{bmatrix}_a \right) = G \left( \begin{bmatrix} y, \theta_1^+ \end{bmatrix}_b \right) \right\} \).

The same proof for Theorem 4.1 will work here.

6. \( J \)-Symmetric Quasi-Differential Operators

In this section we generalize our result to a very general class of \( J \)-symmetric quasi-differential operators. This class includes the classical \( J \)-symmetric differential operators [4. 14].

Let \( I \) be an interval of the real line; its endpoints may be \( \pm \infty \). Let \( Z_m(I) \) denote the collection of all square matrices \( F = (f_{kr}) \) of order \( N \) which satisfy the following conditions:

1. each \( f_{kr} \) is a complex-valued function on \( I \) and \( f_{kr} \in L_{\text{loc}}(I) \), \( k, r = 1, \ldots, N \),
2. \( f_{k,k+1} \neq 0 \), \( k = 1, \ldots, N-1 \),
3. \( f_{kr} = 0 \), \( r \geq k + 2 \),

where \( L_{\text{loc}}(I) \) is the linear space of all locally Lebesgue integrable functions.
Given $F \in Z_n(I)$, define the quasi-derivatives $y^{[k]}$, $k = 0, 1, \ldots, N$, of function $y$ by

$$y^{[0]} = y,$$

$$y^{[k]} = (f_{k,k+1})^{-1} \left[ D y^{[k-1]} - \sum_{r=1}^{k} f_{kr} y^{[r-1]} \right], \quad k = 1, \ldots, N-1,$$

$$y^{[N]} = D y^{[N-1]} - \sum_{r=1}^{N} f_{Nr} y^{[r-1]}.$$

(6.4)

Define the quasi-differential expression of order $N$ associated with $F$ by

$$\tau_F(y) = (-1)^{\left(\frac{N+1}{2}\right)} i^N y^{[N]},$$

where $i = \sqrt{-1}$. (6.5)

If $N = 2n$, then $\tau_F(y) = y^{[2n]}$. It has the same form which we considered in Section 2.

Let $L_N = ((-1)^k \delta_{k,N+1-r})_{1 \leq k, r \leq N}$. For any $F \in Z_n(I)$, we define its adjoint by

$$F^+ = -L_N^{-1} F^* L_N,$$

(6.6)

where $F^* = F^T$. It is easy to verify that $F^+ \in Z_n(I)$ and $(F^+)^+ = F$. For the differential expression $\tau_F(y)$, we define its formal adjoint by

$$\tau_F^+(y) = \tau_{F^*}(y).$$

(6.7)

If $F^+ = F$, then $\tau_F^+(y) = \tau_F(y)$. In this case we call $\tau_F$ symmetric [14]. If $F^+ = F$, then $J \tau_F^+ J(y) = \tau_F(y)$, where $J$ is the complex conjugation. In this case, $\tau_F$ is called $J$-symmetric.

Now we consider differential operators arising from $J$-symmetric quasi-differential expressions. For simplicity, we omit the subscript $F$. Let $I = [a, b)$ and $\tau$ be regular at $a$ and singular at $b$. Define the maximal operator $T_1$ associated with $\tau$ as follows:

$$T_1 = \{ y \in L^2(a, b) : y^{[k-1]} \in AC_{loc}[a, b], k = 1, \ldots, N, \tau(y) \in L^2(a, b) \},$$

$$T_1 y = \tau(y), \quad y \in \mathcal{D}(T_1).$$

We can prove the following results:

**Lemma 6.1.** For any $\alpha$ and $\beta$ in $(a, b)$, $\alpha < \beta$, and a set of complex constants $\alpha_1, \ldots, \alpha_N$, $\beta_1, \ldots, \beta_N$, there exists a function $z \in D(T_1)$ satisfying

$$z^{[j-1]}(\alpha) = \alpha_j, \quad z^{[j-1]}(\beta) = \beta_j, \quad j = 1, \ldots, N.$$
**Lemma 6.2.** For any $y$ and $z$ in $\mathcal{D}(T_1)$ and $[\alpha, \beta] \subset [a, b)$,

\[
\int_a^\alpha \tau(y) z \, dx - \int_a^\beta \tau(z) y \, dx = [y, z]_{|\alpha}^{\beta},
\]

where $[y, z]_{|\alpha}^{\beta} = [y, z]_{\beta} - [y, z]_{\alpha}$ and

\[
[y, z] = (-1)^{[(N+1)/2]} i^N \sum_{k=1}^{N} (-1)^{k-1} y^{[N-k]} z^{[k-1]}.
\]

**Lemma 6.3.** For any $y$ and $z$ in $\mathcal{D}(T_1)$ the limit $[y, z]_b = \lim_{x \to b^-} [y, z]$ exists and we have

\[
\int_a^b \tau(y) z \, dx - \int_a^b \tau(z) y \, dx = [y, z]_{|a}^{b}.
\]

Denote that

\[
R = (-1)^{[(N+1)/2]} i^N L_N;
\]

then $R$ is a skew-Hermitian matrix, i.e.,

\[
R^* = R^{-1} = -R.
\]

The Lagrange bilinear form (6.9) can be written as

\[
[y, z] = r(z) RC(y).
\]

The minimal operator $T_0$ associated with $T$ is defined by

\[
\mathcal{D}(T_0) = \{ y \in \mathcal{D}(T_1) : y^{[k]}(a) = 0, k = 0, 1, \ldots, N-1 \}
\]

and $[y, z]_b = 0$ for all $z \in \mathcal{D}(T_1)$,

\[
T_0 y = \tau(y), \quad y \in \mathcal{D}(T_0).
\]

**Lemma 6.4.** Both $T_0$ and $T_1$ are closed densely defined operators in $L^2(a, b)$. $T_0$ is $J$-symmetric and

\[
JT_0^* J = T_1, \quad JT_1^* J = T_0.
\]

**Lemma 6.5.** (1) def $T_0$ is equal to one half of the number of linearly independent solutions of $\tau^+ \tau(y) = -y$ for which both $y$ and $\tau(y)$ are in $L^2(a, b)$.

(2) $[(N+1)/2] \leq \text{def } T_0 \leq N$.

**Lemma 6.6.** Let $m = \text{def } T_0([[(N+1)/2] \leq m \leq N)$; then the linear
manifold $\mathcal{D}$ in $\mathcal{D}(T_1)$ is the domain of a $J$-selfadjoint extension of $T_0$ if and only if there exist $m$ linearly independent functions $w_1, \ldots, w_m$ satisfying

1. $w_1, \ldots, w_m$ are linearly independent modulo $\mathcal{D}(T_0)$,
2. $[w_r, \tilde{w}_j]_a^b = 0, \ r, j = 1, \ldots, m,$
3. $\mathcal{D} = \{ y \in \mathcal{D}(T_1): [y, \tilde{w}_j]_a^b = 0, \ j = 1, \ldots, m \}.$

Let $m = \text{def } T_0$ and $\theta_1, \ldots, \theta_{2m}$ be a set of linearly independent solutions of $\tau^+(\tau(y)) = -y$ satisfying

$$\theta_k, \tau(\theta_k) \in L^2(a, b), \quad k = 1, \ldots, 2m.$$  \hbox{(6.14)}

We may let $\{\theta_1, \ldots, \theta_{2m}\}$ be normally orthogonal under the inner product (2.4), where $A = T_0$.

**Lemma 6.7.** $\text{rank}( [\theta_k, \tilde{\theta}_r]_{b})_{1 \leq k, r \leq 2m} = 2m - N.$

Assume that

$$([\theta_k, \tilde{\theta}_r]_{b})_{1 \leq k \leq 2m-N} = 2m - N;$$  \hbox{(6.15)}

then we have

**Lemma 6.8.**

$$\mathcal{D}(T_1) = \mathcal{D}(T_0) + L(z_1, \ldots, z_N) + L(\theta_1, \ldots, \theta_{2m-N}),$$  \hbox{(6.16)}

where $z_j, j = 1, \ldots, N$, are the functions in $\mathcal{D}(T_1)$ satisfying the conditions

$$z_j^{[k-1]}(a) = \delta_{jk}, \quad z_j^{[k-1]}(a') = 0, \quad k = 1, \ldots, N,$$

$$z_j(x) = 0, \quad a' \leq x < b.$$  \hbox{(6.17)}

Here $a'$ is a fixed point between $a$ and $b$.

Let

$$B = ([\theta_k, \tilde{\theta}_r]_{b})_{1 \leq k, r \leq 2m-N}.$$  \hbox{(6.18)}

**Theorem 6.9.** Let $\tau$ be a formally $J$-symmetric quasi-differential expression of order $N$ on $[a, b]$ defined by (6.5) which is regular at $a$ and singular at $b$. $T_0$ is the minimal operator and $T_1$ is the maximal operator associated with $\tau$. Let $m = \text{def } T_0$. Then a linear manifold $\mathcal{D}$ in $\mathcal{D}(T_1)$ is the domain of a $J$-selfadjoint extension of $T_0$ if and only if there exist an $m \times N$ matrix $M$ and an $m \times (2m-N)$ matrix $G$ satisfying

1. $\text{rank}(MG) = m,$
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(2) $M RM^T = GBG^T$,

(3) $\mathcal{D} = \left\{ y \in \mathcal{D}(T_1): M \left( y^{[1]}(a) \right) - G \left( \left[ y, \theta b \right]_b \right) = 0 \right\}.$

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