The Number of Involutions with $r$ Fixed Points and a Long Increasing Subsequence

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A new summation theorem for Schur symmetric functions is given, over partitions with an upper bound on the largest part, and with respect to the number of columns of odd length in the Ferrers diagram of the partition. This allows us to deduce expressions for the number of involutions of $n$ symbols with $r$ fixed points, the largest increasing subsequence of which has length $m$. The most compact of these expressions is a double alternating summation which is valid for $(n - 1)/2 \leq m \leq n - 1$.

Let $c(n, m, r)$ be the number of involutions in $S_n$, the symmetric group on $\mathcal{S}_n = \{1, \ldots, n\}$, with $r$ fixed points, the longest increasing subsequence of which has length $m$. For example, when $n = 9$, $m = 4$, $r = 3$, one such involution is 321497685; there are six increasing subsequences of length 4, namely 3478, 2478, 1478, 3468, 2468, 1468, but none of length 5 or more.

Previously, Goulden [2] has evaluated the exponential generating function of $c(n, m, r)$ in $n$, for fixed $r$, as a determinant of order $[m/2]$, for arbitrary $m$. In this paper a simple expression is given for $c(n, m, r)$ when $(n - 1)/2 \leq m \leq n - 1$, analogous to the expression for $\sum_r c(n, m, r)$ given in Goulden [3].

We use the following result, in which $c(n, m, r)$ is identified as a sum over partitions $\lambda$ of $n$, written $\lambda \vdash n$. The largest part of $\lambda$ is denoted by $\lambda_1$, the number of (non-zero) parts is denoted by $l(\lambda)$, and the number of columns of odd length in the Ferrers diagram of $\lambda$ is denoted by $c(\lambda)$ (see, e.g., [1], [5], [7] and [8] for details).

**Proposition 1.** Let $f^\lambda$ be the degree of the $\lambda$th irreducible representation of $S_n$. Then

$$c(n, m, r) = \sum_{\lambda \vdash n} f^\lambda.$$

Gouyou-Beauchamps [4] has evaluated the sum of $f^\lambda$ over $\lambda \vdash n$, $c(\lambda) = r$ and $l(\lambda) = m$ in the cases $m = 4, 5$. This sum is related to $c(n, m, r)$, since it can be interpreted as the number of involutions in $S_n$ with $r$ fixed points, the longest decreasing subsequence of which has length $\leq m$.

We consider symmetric functions in the variables $\mathbf{x} = (x_1, \ldots, x_n)$ (see [6, Chapter I] for a complete treatment). Let $\delta = (n - 1, n - 2, \ldots, 0)$. If $\mathbf{b} = (b_1, \ldots, b_n)$ consists of non-negative integers, then the Schur symmetric function $s_{\mathbf{b}}(\mathbf{x})$ is given by

$$s_{\mathbf{b}}(\mathbf{x}) = a_{\mathbf{b}}(\mathbf{x})^{-1} a_{\mathbf{b} + \delta}(\mathbf{x}),$$

where, for $\mathbf{c} = (c_1, \ldots, c_n)$, $a_{\mathbf{c}}(\mathbf{x})$ is the alternant

$$a_{\mathbf{c}}(\mathbf{x}) = \det(x_{ij}^c)_{n \times n}.$$

Thus $a_{\delta}(\mathbf{x})$ is the Vandermonde determinant, and we denote it by $V(x)$. We need the
following facts about Schur functions:

\[ f^\lambda = [x_1 \cdots x_n] s_\lambda(x) \quad (1) \]

\[ \psi(t, x) = \sum_\lambda s_\lambda(x) t^{\ell(\lambda)} = \prod_{i=1}^{n} (1 - tx_i)^{-1} \prod_{j<k \leq n} (1 - x_jx_k)^{-1} \quad (2) \]

where \([x_1 \cdots x_n]\) denotes the coefficient of \(x_1 \cdots x_n\) in the expression to the right.

First, we give a new Schur function summation, which contains information about \(c(n, m, r)\) by means of (1) and Proposition 1. The proof makes use of (2). We use the following notation: if \(\alpha = (\alpha_1, \ldots, \alpha_k) \subseteq \mathcal{N}_n\) with \(\alpha_1 < \cdots < \alpha_k\), then \(\overline{\alpha} = \mathcal{N}_n - \alpha\), \(x_\alpha = (x_{\alpha_1}, \ldots, x_{\alpha_k})\), \(x_\alpha^i = x_{\alpha_1}^i \cdots x_{\alpha_k}^i\) and \(I(\alpha) = \{(i, j) \in \alpha \times \overline{\alpha} \text{ and } i > j\}\).

**Theorem 1.** For \(n \geq m \geq 1\),

\[
\sum_\lambda s_\lambda(x) t^{\ell(\lambda)} = \sum_{\text{even } |\alpha| = k} t^{-k} (-1)^{\ell(\overline{\alpha}) + k} \frac{V(x_\alpha)V(x_\overline{\alpha})}{V(x)} \psi(t^{-1}, x_\alpha) \psi(t, x_\overline{\alpha}) + \sum_{\text{odd } |\alpha| = k} t^{m+k} (-1)^{\ell(\overline{\alpha}) + k} \frac{V(x_\alpha)V(x_\overline{\alpha})}{V(x)} \psi(t, x_\alpha) \psi(t^{-1}, x_\overline{\alpha}).
\]

**Proof.** Let

\[ \Psi(u, t, x) = \sum_{j \geq 0} \sum_{\lambda \lambda_1 < j} s_\lambda(x) t^{\ell(\lambda)} u^j, \]

so

\[ \text{LHS} = [u^m] \Psi(u, t, x). \]

For \(n\) even, writing the Schur functions as a ratio of alternants, we obtain

\[ \Psi(u, t, x) = \frac{1}{V(x)} \sum_{j \geq 1} \sum_{1 \leq \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n} \text{sgn}(\sigma) x_{\overline{\lambda_n}}^{j_1+n-1} \cdots x_{\overline{\lambda_1}}^{j_n}, \]

Thus

\[
\Psi(u, t, x) = \frac{1}{u^n t^{n/2} V(x)} \sum_{\sigma \in S_n} \text{sgn}(\sigma) u^{d_0 + \cdots + d_n} t^{d_1 + \cdots + d_n} x_{\overline{\alpha_1}}^{d_1} \cdots x_{\overline{\alpha_n}}^{d_n} \]

\[ = \frac{1}{u^n t^{n/2} V(x)} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \frac{utx_{\alpha_1} \cdots x_{\alpha_n}}{1 - utx_{\sigma_1} \cdots x_{\sigma_n}} \frac{ux_{\sigma_1} \cdots x_{\sigma_{n-1}}}{1 - ux_{\sigma_1} \cdots x_{\sigma_{n-1}}} \quad (3) \]

since the summations over the \(d_i\)'s are all geometric series. In particular, note that

\[
\psi(t, x) = \{(1-u) \Psi(u, t, x)\}_{u=1}
\]

\[ = \frac{1}{t^{n/2} V(x)} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \frac{tx_{\sigma_1} \cdots x_{\sigma_n}}{1 - tx_{\sigma_1} \cdots x_{\sigma_n}} \]

\[ \times \frac{tx_{\sigma_1} \cdots x_{\sigma_{n-1}}}{1 - tx_{\sigma_1} \cdots x_{\sigma_{n-1}}} \quad (4) \]
But an alternant with an equal pair of entries is equal to zero, so we could also have started with the expression

\[ \Psi(u, t, x) = \frac{1}{V(x)} \sum_{\mu_0, \ldots, \mu_n} u^{\mu_0-(n-1)\mu_1-\mu_2+\cdots+\mu_{n-1}-\mu_n-n/2} \sum_{\sigma \in S_n} \text{sgn}(\sigma)x_{\sigma_1}^{\mu_1} \cdots x_{\sigma_n}^{\mu_n} \]

which, modifying the above argument, leads to

\[
\psi(t, x) = \frac{1}{t^{n/2}V(x)} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \frac{1}{1-tx_{\sigma_1}} \frac{1}{1-tx_{\sigma_1}x_{\sigma_2}} \cdots \\
\times \frac{1}{1-tx_{\sigma_1} \cdots x_{\sigma_{n-1}}} \frac{1}{1-x_{\sigma_1} \cdots x_{\sigma_n}}.
\]

But, from (3), \( \Psi(u, t, x) \) is a rational function with denominator

\[ u^{n-1}t^{n/2}V(x) \prod_{|\alpha| \text{ even}} (1-ux_{\alpha}) \prod_{|\alpha| \text{ odd}} (1-utx_{\alpha}), \]

so it has the partial fraction expansion

\[
\Psi(u, t, x) = \sum_{|\alpha| \text{ even}} \frac{A_{\alpha}}{u^{n-1}t^{n/2}(1-ux_{\alpha})} + \sum_{|\alpha| \text{ odd}} \frac{B_{\alpha}}{u^{n-1}t^{n/2}(1-utx_{\alpha})},
\]

where, for \(|\alpha| \) even,

\[ A_{\alpha} = \{u^{n-1}t^{n/2}(1-ux_{\alpha}) \Psi(u, t, x)\} \big|_{u=x_{\alpha}^{-1}}, \]

and, for \(|\alpha| \) odd,

\[ B_{\alpha} = \{u^{n-1}t^{n/2}(1-utx_{\alpha}) \Psi(u, t, x)\} \big|_{u=t^{-1}x_{\alpha}}. \]

Now let \(|\alpha| = k\), \( \omega = \omega_1 \cdots \omega_k = \sigma_k \cdots \sigma_1 \), \( \rho = \rho_1 \cdots \rho_{n-k} = \sigma_{k+1} \cdots \sigma_n \). Then, from (3),

\[
A_{\alpha} = \frac{1}{(1-x_{\alpha}^{-1})V(x)} \sum_{\omega \in S_n, \rho \in S_k} \text{sgn}(\omega) \frac{t(x_{\omega_1} \cdots x_{\omega_k})^{-1}}{1-t(x_{\omega_1} \cdots x_{\omega_k})^{-1}} \frac{(x_{\omega_1} \cdots x_{\omega_k})^{-1}}{1-(x_{\omega_1} \cdots x_{\omega_k})^{-1}} \cdots \frac{t(x_{\omega_1})^{-1}}{1-t(x_{\omega_1})^{-1}} \\
\times \frac{x_{\rho_1}}{1-tx_{\rho_1}} \frac{x_{\rho_1}x_{\rho_2}}{1-tx_{\rho_1}x_{\rho_2}} \cdots \frac{x_{\rho_k}}{1-tx_{\rho_1} \cdots x_{\rho_{k-1}}} \frac{x_{\rho_1}x_{\rho_2} \cdots x_{\rho_{k-1}}}{1-tx_{\rho_1} \cdots x_{\rho_{k-1}}x_{\rho_k}}. 
\]

But \( \text{sgn}(\sigma) = \text{sgn}(\omega) \text{sgn}(\rho)(-1)^{t \alpha(t+1)} \), so

\[
A_{\alpha} = \frac{x_{\alpha}^{-1}(-1)^{t \alpha(t+1)}}{V(x)} \sum_{\omega \in S_n} \text{sgn}(\omega) \frac{-1}{1-tx_{\omega_1}^{-1}} \frac{-1}{1-x_{\omega_1}x_{\omega_2}} \cdots \\
\times \frac{-1}{1-t^{-1}x_{\omega_1} \cdots x_{\omega_k}} \frac{-1}{1-x_{\omega_1} \cdots x_{\omega_k}} \\
\times \sum_{\rho \in S_k} \text{sgn}(\rho) \frac{x_{\rho_1}}{1-tx_{\rho_1}} \frac{x_{\rho_1}x_{\rho_2}}{1-tx_{\rho_1}x_{\rho_2}} \cdots \frac{x_{\rho_k}}{1-tx_{\rho_1} \cdots x_{\rho_{k-1}}} \frac{x_{\rho_1}x_{\rho_2} \cdots x_{\rho_{k-1}}}{1-tx_{\rho_1} \cdots x_{\rho_{k-1}}x_{\rho_k}}. 
\]

The summation in \( \omega \) can be evaluated by (5) and the summation in \( \rho \) can be evaluated by (4). This yields

\[
A_{\alpha} = \frac{x_{\alpha}^{-1}(-1)^{t \alpha(t+1)+k}}{V(x)} t^{-k/2} \psi(t^{-1}, x_{\alpha}) V(x_{\alpha}) t^{(n-k)/2} \psi(t, x_{\alpha}) V(x_{\alpha}).
\]
To calculate \( B_\alpha \), we need the analogous forms of (4) and (5) for \( n \) odd. These are not given, but yield

\[
B_\alpha = \frac{t^x \psi(t, x_\alpha) V(x_\alpha) t^{-(n-k-1)/2} \psi(t^{-1}, x_\alpha) V(x_\alpha)}{V(x)}
\]

The result follows, for \( n \) even, immediately from

\[
\text{LHS} = [u^m] \psi(u, t, x) = \sum_{k \text{ even}} \sum_{|\alpha|=k} t^{-n/2} x^{n+m-1} A_{\alpha}
\]

\[
+ \sum_{k \text{ odd}} \sum_{|\alpha|=k} t^{(n/2)+m-1} x^{n+m-1} B_{\alpha}.
\]

The details for \( n \) odd are similar.

If Theorem 1 is to be useful, we must be able to carry out the division by the Vandermonde determinant \( V(x) \). This can be done by means of the following result, which is proved in [3].

**Proposition 2.** Let \( F(z_1, \ldots, z_k) = \sum a_{\alpha} z_1^{a_1} \cdots z_k^{a_k} \) be a symmetric function in \( z_1, \ldots, z_k \). Then

\[
\sum_{\alpha \in \Lambda_k} (-1)^{l(\alpha)} \frac{V(x_\alpha)}{V(x)} x^{n-k} F(x_\alpha) = \sum c(a_1, \ldots, a_k) s(a_1, \ldots, a_k)(x).
\]

We now apply Proposition 2 to Theorem 1, yielding our main result, a compact expression for \( c(n, m, r) \) for certain values of \( m \).

**Theorem 2.** For \( (n-1)/2 \leq m \leq n-1 \),

\[
c(n, m, r) = \frac{n!}{(m-r)!} \sum_{i+j+l=\frac{1}{2}(n+r)-m} \frac{(-1)^{2i-l} (j + 2l + r + 1)}{i! j! (j + 2l + m + 1)!}.
\]

**Proof.** From Proposition 1,

\[
c(n, m, r) = \sum_{\lambda, m \vdash n-r} f_\lambda = [t^r] \left( \sum_{\lambda, m \vdash n-r} s_\lambda(x) t^{E(\lambda)} - \sum_{\lambda, m \vdash n-r} s_\lambda(x) t^{E(\lambda)} \right),
\]

from (1). These Schur function summations can be evaluated by means of Theorem 1, yielding

\[
c(n, m, r) = [t^r] \sum_k (F_{m,k} - F_{m-1,k}),
\]

where \( F_{m,k} \) is the \( k \)th summand on the RHS of Theorem 1.

Now consider applying Proposition 2 to evaluate \( F_{m,k} \), which yields a power series in \( x \) in which every monomial has total degree at least \( k(m+k) \). Thus, for \( m \geq (n-3)/2 \), \( i \geq 2 \), we have

\[
[x^1] F_{m,i} = 0.
\]

Accordingly, for \( (n-1)/2 \leq m \leq n-1 \),

\[
c(n, m, r) = [t^r] (F_{m,1} - F_{m-1,1})
\]

and, from Proposition 2,

\[
F_{m,1} - F_{m-1,1} = \sum_{s \geq 0} g_s h_s(x)
\]

where

\[
\sum_{s \geq 0} g_s z^s = \psi(t^{-1}, x) t^m z^m \frac{1-t^{-1} z}{1-z^2} \prod_{i=1}^n (1-zx_i).
\]
Now, for a symmetric function \( H(x) \),
\[
[x^1]H(x) = \sum_{\genfrac{}{}{0pt}{}{p_i(x) = x}{p_i(x) = 0, i \geq 2}} \frac{x^n}{n!}
\]
and the above power sum symmetric function substitutions yield \( h_i(x) = e_i(x) = x^i/i! \) for the complete and elementary symmetric functions. Thus, using (2),
\[
c(n, m, r) = \left[ x^n \right] \sum_{s \geq 0} g_s \frac{x^s}{s!},
\]
where
\[
\sum_{s \geq 0} g_s z^s = t^m z^m \frac{1 - t^{-1} z}{1 - z^2} \exp \left( t^{-1} x + \frac{x^2}{2} \right) \left( \sum_{j \geq 0} (-1)^j z^{j^2} \frac{x^j}{j!} \right),
\]
and the result follows. \( \square \)

The degree argument used in Theorem 2 means that expressions of increasing complexity can be obtained for \( c(n, m, r) \) as \( n/m \) grows. Note the curious duality in these expressions: they provide an identity between a degree sum (obtained via Proposition 1) over partitions with \( \lambda_1 = m \) and a degree sum (obtained via Proposition 2 applied to Theorem 1) over partitions with \( I(\lambda) = \lfloor n/m \rfloor \) or thereabouts. This is not an obvious relationship; it would be interesting to understand this by more elementary means.

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**REFERENCES**


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