MATHEMATICS

ADDENDUM TO "ON THE MEASURABILITY OF A FUNCTION WHICH OCCURS IN A PAPER BY A. C. ZAANEN"

BY

W. A. J. LUXEMBURG [1])

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1. *Introduction.* Let μ_1 and μ_2 be σ-finite measures defined in the point sets X_1 and X_2 respectively. We assume that on the set $P=P(X_2)$ of all μ_2-measurable non-negative real functions v defined on X_2 a function ϱ is defined which has the following properties: (i) $0 \leqslant \varrho(v) \leqslant \infty$ for all $v \in P$, if $v=0$ (μ_2-almost everywhere), then $\varrho(v)=0$, $\varrho(av)=a\varrho(v)$ for every $v \in P$ and every finite constant $a \geqslant 0$, $\varrho(v_1+v_2) \leqslant \varrho(v_1)+\varrho(v_2)$ for all v_1, $v_2 \in P$ and (ii) $v_1, v_2 \in P$ and $v_1 \leqslant v_2$ implies that $\varrho(v_1) \leqslant \varrho(v_2)$, i.e., ϱ is a *function seminorm.*

If f is a $\mu_1 \times \mu_2$-measurable non-negative real function defined on $X_1 \times X_2$, then, as is well-known, for μ_1-almost every $x \in X_1$, $v_x(y)=f(x,y)$, $y \in X_2$, is a μ_2-measurable function of y. Hence, the function $u(x)=\varrho(v_x)$ is defined for μ_1-almost all $x \in X_1$. In [1], the following problem was discussed: *Under which condition is the function $u(x)=\varrho(v_x)$, $x \in X_1$ except on a set of μ_1-measure zero, a μ_1-measurable function of x for every $\mu_1 \times \mu_2$-measurable non-negative real function f defined on $X_1 \times X_2$?* It was shown in [1] that the answer to this question is *affirmative* if ϱ has the *Fatou property,* i.e., if $v_k \in P$ $(k=1, 2, \ldots)$ and $v_k \uparrow v$ on X_2, then $\varrho(v_k) \uparrow \varrho(v)$. (This is property ($e$) of λ in [1]). At the end of [1] the question was raised whether the Fatou property of ϱ is essential for the validity of this theorem. The object of this note is to show by means of a counterexample that the theorem may be false if ϱ does not have the Fatou property.

2. *Construction of non-Lebesgue measurable functions.* It is well-known that there exist real functions of a real variable which are not measurable in the sense of Lebesgue. The purpose of this section is to show that in some cases these functions can be generated in a certain sense which will be made precise in due course.

The non-Lebesgue-measurable functions on which we shall focus our attention belong to the class of the *non-Lebesgue-measurable almost periodic functions of a real variable.* We shall therefore use in this section some results from the theory of almost periodic functions on groups. For these

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results and other results about topological groups we refer the reader to [2] and [3].

Let Z denote as usual the additive group of integers endowed with the discrete topology. The Banach algebra of all bounded complex functions defined on Z will be denoted by $B=B(Z)$ (the operations of addition and multiplication are defined pointwise and the norm is the sup norm).

A complex function f on Z is called *almost periodic* if it is the uniform limit of a sequence of trigonometric polynomials on Z (A function on Z of the form $\sum_{l=1}^{m} a_l e^{ix_l n}$, $n \in Z$, where $a_1, \ldots, a_m$ are complex numbers and $x_1, \ldots, x_m$ are real numbers reduced mod 2π, is called a trigonometric polynomial on Z). This is not the usual definition of almost periodicity but it is equivalent to it and particularly suitable for our purposes. The set of all almost periodic functions on Z will be denoted by $A=A(Z)$. As is well-known, A is a self-adjoint proper closed subalgebra of B with a unit. Furthermore, A is separating on Z, i.e., if $n, m \in Z$ and $n \neq m$, then there exists a function $f \in A$ such that $f(n) \neq f(m)$. Indeed, for some real x, reduced mod 2π, $e^{inx} \neq e^{imx}$.

We shall consider the *Bohr topology* on Z which is the weakest topology on Z which makes all the functions of A continuous. The group Z with its Bohr topology will be denoted by Z_b. The Bohr topology is the topology of the uniform convergence on the finite subsets of the additive group T of real numbers mod 2π; and the Bohr topology is Hausdorff. Furthermore, the Bohr topology is strictly weaker than the discrete topology and a bounded complex function on Z is uniformly continuous in the Bohr topology if and only if it is almost periodic.

The *Bohr compactification* of the discrete group Z, and the *Gelfand representation space* of the commutative Banach algebra A are all one and the same compact space and will be denoted by $\tilde{Z}_b$. Every element $f \in A$ has a unique extension over $\tilde{Z}_b$ which we shall denote by $\tilde{f}$. Then the following result is well-known (compare Satz 5 on p. 71 of [2]).

Theorem. *A character γ of the group T is discontinuous and hence non-measurable in the sense of Lebesgue if and only if there exists an element $a \in \tilde{Z}_b - Z_b$ such that $\gamma(x) = (x, \gamma) = \tilde{e}^{ixa}$ for all $x \in T$.*

Since $\widetilde{\sin}\, xa = (\tilde{e}^{ixa} - \tilde{e}^{-ixa})/2i$ and $\widetilde{\cos}\, xa = (\tilde{e}^{ixa} + \tilde{e}^{-ixa})/2$, $x \in T$, we obtain immediately that if $a \in \tilde{Z}_b - Z_b$, then these functions are *real almost periodic functions which are not measurable in the sense of Lebesgue.*

The extension $\tilde{f}$ of $f \in A$ over $\tilde{Z}_b$ can be given in terms of a generalized limit which we shall now describe. If $a \in \tilde{Z}_b - Z_b$, then the set $M_a = \{f : f \in A$ and $\tilde{f}(a) = 0\}$ is a free maximal ideal of A. To M_a there corresponds a filter $\mathfrak{F}_a$ on Z defined in the following way: $E \in \mathfrak{F}_a$ if and only if there exist elements $f_1, \ldots, f_m$ in M_a and positive numbers $\varepsilon_1, \ldots, \varepsilon_m$ such that the set $\bigcap_{i=1}^{m} F(f_i; \varepsilon_i)$, where $F(f_i; \varepsilon_i) = \{n : n \in Z$ and $|f_i(n)| \leqslant \varepsilon_i\}$, is contained in E. In order to prove that $\mathfrak{F}_a$ is a filter we have to show that the family $\mathfrak{B}_a$ of all subsets of the form $F(f; \varepsilon)$, where $f \in M_a$ and $\varepsilon > 0$, is a filter base on Z. It is evident that $\phi \notin \mathfrak{B}_a$ since otherwise M_a would contain a unit. If $F_1(f_1; \varepsilon_1)$ and $F_2(f_2; \varepsilon_2)$ belong to $\mathfrak{B}_a$, then $F(|f_1|^2 + |f_2|^2; \min^2(\varepsilon_1, \varepsilon_2)) \subset F_1 \cap F_2$. Furthermore, $\mathfrak{F}_a$ is a free filter, i.e., $\cap \mathfrak{F}_a = \phi$. Indeed, if there exists an integer n_0 such that $n_0 \in F$ for all $F \in \mathfrak{F}_a$, then $M_a = \{f : f \in A \text{ and } f(n_0) = 0\}$. Since the algebra $\widetilde{A}$ of extended functions is the algebra of all complex continuous functions on $\widetilde{Z}_b$ and $\widetilde{Z}_b$ is a Hausdorff space, we have that for every $a \in \widetilde{Z}_b - Z_b$ there exists a function $f \in M_a$ such that $f(n_0) \neq 0$. We shall now show that every $f \in A$ is convergent relative to the filter $\mathfrak{F}_a$ and that $\lim_{\mathfrak{F}_a} \tilde{f} = f(a)$ for all $f \in A$. Indeed, by definition of $\mathfrak{F}_a$, $f \in M_a$ implies that $\lim_{\mathfrak{F}_a} f = 0$. Since M_a is a maximal ideal it follows that for every $f \in A$ there exists uniquely an element $f' \in M_a$ such that $f = \tilde{f}(a) + f'$. Hence, $\lim_{\mathfrak{F}_a} f = \tilde{f}(a)$ for all $f \in A$. In view of the latter result the following lemma is now evident.

Lemma. *For every $a \in \widetilde{Z}_b - Z_b$ there exists a free ultrafilter $\mathfrak{U}_a$ on Z such that $f(a) = \lim_{\mathfrak{U}_a} \tilde{f}$ for all $f \in A$.*

Proof. Since $\mathfrak{F}_a$ is a free filter and since every $f \in A$ is convergent relative to $\mathfrak{F}_a$ the lemma holds for every ultrafilter containing $\mathfrak{F}_a$.

Combining this Lemma with the Theorem of this section we obtain the interesting result that *there exists a free ultrafilter $\mathfrak{U}$ on Z such that the real periodic and almost periodic functions* $\lim_{n,\mathfrak{U}} \sin xn$ *and* $\lim_{n,\mathfrak{U}} \cos xn$, *$x$ real, are not measurable in the sense of Lebesgue.*

Remarks 1. It is of importance to observe that not for every free ultrafilter $\mathfrak{U}$ on Z, $\lim_{n,\mathfrak{U}} \sin xn$, x real, is a function which is not measurable in the sense of Lebesgue. Indeed, if $\mathfrak{U}$ is a free ultrafilter on Z which is finer than the neighborhood filter of a point $n_0 \in Z$ in the Bohr topology, then $\lim_{\mathfrak{U}} \sin xn = \sin xn_0$, x real. Furthermore, for different ultrafilters $\mathfrak{U}_1$ and $\mathfrak{U}_2$ on Z it may occur that $\lim_{\mathfrak{U}_1} f = \lim_{\mathfrak{U}_2} f$ for all $f \in A$. If we introduce on the Čech-Stone compactification space βZ of the discrete space Z the equivalence relation: $p \cong q$ if and only if $p, q \in \beta Z$ and $f^0(p) = f^0(q)$ for all $f \in A$, where f^0 denotes the unique extension of f over βZ, then the set of equivalence classes can be identified with $\widetilde{Z}_b$. The topology of $\widetilde{Z}_b$ is not the quotient topology.

2. The reader who is familiar with non-standard analysis may translate the above Theorem and Lemma into the following statement: There exists a non-standard model R^* of R, the system of real numbers, and there exists an infinitely large integer $\omega \in Z^*$ such that $st(e^{*ix\omega})$, $x \in R$, is a periodic and almost periodic function which is not measurable in the sense of Lebesgue.

3. The Lemma of this section can also be interpreted in the following way: Every continuous homomorphism of A into the complex numbers can be extended to a continuous homomorphism of B into the complex numbers. Needless to say that the extension is not unique.

4. Although we did not mention it explicitly, it is evident that the proofs of the results of this section require the use of some form of the axiom of choice. In fact, it is easy to see that all we need to appeal to is the prime ideal theorem for Boolean algebras which is weaker than the axiom of choice (see [4]).

3. *A counterexample.* Let $X_1 = T$, the additive group of real numbers mod 2π, μ_1 Lebesgue measure, $X_2 = Z$ and μ_2 the discrete measure respectively.

We shall now define a functionnorm ϱ on the set $P = P(Z)$ of all non-negative real sequences on Z. For this purpose, let $a \in \widetilde{Z}_b - Z_b$ and let $\mathfrak{U}_a$ be a free ultrafilter on Z which contains the free filter $\mathfrak{F}_a$ (for notation see the preceeding section). Then we set

$$\varrho(v) = \sup\,(v(n) : n \in Z) + \lim_{\mathfrak{U}_a} v, \text{ if } v \text{ is bounded, and}$$
$$\varrho(v) = \infty \text{ otherwise.}$$

It is easy to verify that ϱ is a *functionnorm* ($\varrho(v) = 0$ if and only if $v(n) = 0$ for all $n \in Z$) which does *not have the Fatou property.*

Consider now the following non-negative real function $f = f(x, n) = 1 + \sin xn$, $x \in T$, $n \in Z$. Then f is continuous on $T \times Z$ and hence is $\mu_1 \times \mu_2$ measurable. But, according to the Lemma of the preceeding section, $u(x) = \varrho(u_x) = \varrho(1 + \sin xn) = 2 + \sup\,(\sin xn : n \in Z) + \lim_{n, \mathfrak{U}_a} \sin xn = 3 + \widetilde{\sin}\, xa$, $x \in T$, is *not measurable in the sense of Lebesgue*, which shows that the result of [1] quoted in the introduction of this paper may fail to hold if ϱ does not have the Fatou property.

Remark. Although ϱ does not have the Fatou property it has a property which is very close to it and which is called the *weak Fatou property*, namely, if $v_k \in P$ $(k = 1, 2, \ldots)$, $v_k \uparrow v$ on Z and $\sup\,(\varrho(v_k) : k = 1, 2, \ldots)$ is finite, then $\varrho(v) < \infty$. This suggests that it may not be impossible that the Fatou property is also sufficient for the result to hold.

California Institute of Technology

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