On the Semantics of Fuzzy Logic*

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ABSTRACT

This paper presents a formal characterization of the major concepts and constructs of fuzzy logic in terms of notions of distance, closeness, and similarity between pairs of possible worlds. The formalism is a direct extension (by recognition of multiple degrees of accessibility, conceivability, or reachability) of the major modal logic concepts of possible and necessary truth.

Given a function that maps pairs of possible worlds into a number between 0 and 1, generalizing the conventional concept of an equivalence relation, the major constructs of fuzzy logic (conditional and unconditioned possibility distributions) are defined in terms of this similarity relation using familiar concepts from the mathematical theory of metric spaces. This interpretation is different in nature and character from the typical, chance-oriented, meanings associated with probabilistic concepts, which are grounded on the mathematical notion of set measure. The similarity structure defines a topological notion of continuity in the space of possible worlds (and in that of its subsets, i.e., propositions) that allows a form of logical "extrapolation" between possible worlds.

This logical extrapolation operation corresponds to the major deductive rule of fuzzy logic – the compositional rule of inference or generalized modus ponens of Zadeh – an inferential operation that generalizes its classical counterpart by virtue of its ability to be utilized when propositions representing available evidence match only approximately the antecedents of conditional propositions. The relations between the similarity-based interpretation of the role of conditional possibility distributions and the approximate inferential procedures of Baldwin are also discussed.

A straightforward extension of the theory to the case where the similarity scale is symbolic rather than numeric is described. The problem of generating similarity functions from a given set of possibility distributions, with the latter interpreted as defining a number of (graded) discernibility relations and the former as the result

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of combining them into a joint measure of distinguishability between possible worlds, is briefly discussed.

KEYWORDS: ***fuzzy logic, semantics, modal logics, possible worlds, generalized modus ponens***

INTRODUCTION

This paper presents a semantic characterization of the major concepts and constructs of fuzzy logic in terms of notions of similarity, closeness, and proximity between possible states of a system that is being reasoned about. Informally, a "possible state" (to be formalized later using the notion of "possible world") is an assignment of a well-defined truth value (i.e., either *true* or *false*) to all relevant declarative knowledge statements about that system.

The primary goal that guided the research leading to the results presented in this work was one of conceptual clarification. A great deal of energy has been directed in the past few years to debating the methodological necessity and relative merits of various approximate reasoning methodologies. As a result of these exchanges, the need to consider certain nonclassical approaches has been questioned on a variety of bases.

Recognizing the need for the development of sound semantic formalisms that shed light on the nature of different approaches, I have pursued, in the past few years, a line of theoretical research seeking to describe various approximate reasoning methodologies using a common framework. These investigations have recently shown the close connection between the Dempster-Shafer [38] calculus of evidence [35] and epistemic logics. This relationship was elucidated by straightforward application of conventional probabilistic concepts to models of knowledge states that distinguish between the true of a proposition and knowledge (by rational agents) of that truth. Central to this development is the notion of "possible world" used by Carnap [6] to develop logical bases for probability theory.

The central notion of possible state of affairs is also the conceptual basis of the results presented in this paper, which is aimed at establishing the semantic bases of possibilistic logic with emphasis on the study of its possible relations and differences, if any, with probabilistic reasoning.

The results of this investigation clearly show that possibilistic logic can be interpreted in terms of nonprobabilistic concepts that are related to the notions of continuity and proximity. The major functional structures of fuzzy logic, possibility and necessity distributions,[1] may be defined in terms of the more primitive notion of similarity between possible states of a system using constructs that are the direct extension of well-known concepts in the theory of

[1] It is important to remark that the scope of this work is limited to the most fundamental concepts and constructs of fuzzy logic without examining related notions such as generalized quantifiers.

metric spaces. The topological metric structure that is so defined may be used to derive a sound inferential rule that is a form of logical "extrapolation." This rule is also shown to be the compositional rule of inference or generalized modus ponens proposed by Zadeh [53]. Conversely, possibility distributions—expressing resemblance in some specific regard—may be used to derive the actual similarity functions, discerning between possible worlds from the multiple points of view.

The constructs that are used to derive the interpretation presented in this paper are formally, structurally, and conceptually different from those that explain probabilistic reasoning, in either its objective or subjective interpretations, irrespective of methodological reliance on interval-based approaches to represent ignorance. The latter class of methods—measuring the relative proportion of the (either observed or believed) occurrence of some event—are based on the mathematical notion of set measure, while the former—seeking to establish similarities between situations that may be used for analogical reasoning—are related to the theory of distances and metric spaces.

This presentation of the relationships between similarity-based concepts and possibilistic notions, while grounded on a formal treatment that is based on rigorous logical and mathematical formalisms, will be kept at a level that is as informal as possible. The purpose of this presentation style is to facilitate comprehension of major ideas without the clutter that would otherwise need to be introduced keep matters strictly precise. For this reason, I will refrain from formal introduction of structures and axiom schemata, that, although correct and proper, may encumber understanding of the basic concepts.

Before we proceed to the detailed consideration of semantic models, I must briefly remark on the epistemological implication of these developments. The present interpretation is not the only that may be advanced to define the notion of possibility in terms of simpler concepts, nor do I claim that it may not be sometimes possible, even desirable, to model possibilistic structures from other bases. My intent is not to prove the conceptual superiority of one approach over another or to argue about the relative utility of different technologies. Rather, I hope that these results have contributed to establish the basic conceptual differences in the treatment of imprecise and uncertain information that are inherent in probabilistic and possibilistic methods—the former oriented toward quantifying believed or measured frequency of occurrence, and the latter seeking to determine propositions, implied by the evidence, that are similar in some sense to a hypothesis of interest. In other words, beyond accidental domain-specific relations, both types of methods are needed to analyze and clarify the significance of imprecise and uncertain information.

APPROXIMATE REASONING AND POSSIBLE WORLDS

Our point of departure is the model-theoretic formalisms of modal logics. Let us assume that declarative statements about the state, situation, or behavior of a real-world system under study are symbolically represented by the letters of some alphabet

$$\mathscr{A} = \{p, q, r, \ldots\}$$

which are combined in the customary way using the logical operators $\neg$, $\vee$, $\wedge$, $\rightarrow$, and $\leftrightarrow$ (to be interpreted with their usual meanings) to derive a language $\mathscr{L}$ (i.e., a collection of sentences). Furthermore, we augment this language by the use of two unary operators **N** and **Π**, called the *necessity* and *possibility* operators, respectively, having usage governed by the rule

If ϕ is a sentence, then $\mathbf{N}\phi$ and $\mathbf{\Pi}\phi$ are also sentences,

which introduces the ability to represent different modalities for the truth of propositions.

A *model* for this propositional system is a structure consisting of three components:

1. A nonempty set of possible worlds $\mathscr{U}$ introduced to represent states, situations, or behaviors of the system being modeled by our sentences. In what follows we will refer to this set as the *universe of discourse*, or *universe* for short.

   We will also need to consider a nonempty subset $\mathscr{E}$ of the universe $\mathscr{U}$, which is introduced to model the set of conceivable worlds that are consistent with observed evidence. This set (possibly equal to the whole universe $\mathscr{U}$) will be called the *evidential set*. Throughout this paper, we will assume that evidence about the world is always given by means of conventional propositions that allow us to determine, without ambiguity, whether a possible world either is or is not a member of the evidential set.

2. A function (called a *valuation*) that assigns one and only one of the truth values *true* or *false* to every possible world w in the universe $\mathscr{U}$ and every sentence ϕ in the language. Assignment of the truth value *true* to a pair (w, ϕ) will be denoted $w \vdash \phi$ (i.e., ϕ is true in the world w).

   In what follows, we will use the same symbols to describe subsets of possible worlds and the propositions that are true only in worlds that are members of such subsets. For example, the symbol $\mathscr{E}$ will be used to denote both the evidential set and the proposition that asserts the validity of the corresponding evidential observations. Using this notation, for example, we will write $w \vdash \mathscr{E}$ to indicate that the world w is compatible (i.e., logically consistent) with the evidence $\mathscr{E}$. Furthermore, we will use the symbol $\mathscr{L}$, introduced above as a set of well-formed sentences, to denote also the power set of the universe $\mathscr{U}$. Rigorously, subsets of $\mathscr{U}$ strictly correspond to the classes of equivalence of the sentence set $\mathscr{L}$ that are obtained by equating logically equivalent sentences. In the same simplifying vein, we will also drop the customary distinction between sentences—the linguistic expressions of something that may be true or false—and propositions—the actual things being asserted.

3. A binary relation R between possible worlds, called the *accessibility, conceivability, or reachability* relation, introduced to model the semantic of the modal operators **N** and **Π**.

It is not necessary to review here the well-known axioms (Hughes and Creswell [21]) that restrict the assignment of truth values to well-formed sentences according to the rules of propositional logic. To facilitate comprehension of our formalism, we need to recall solely the rules that constrain assignment of truth values to sentences formed by prefixing other valid expressions with the modal operators, that is,

1. The sentence ϕ is *necessarily true* in the possible world w (i.e., $w \vdash \mathbf{N}\phi$) if and only if it is true in every world w' that is related to the world w by the relation R.
2. The sentence ϕ is *possibly true* in the possible world w (i.e., $w \vdash \Pi\phi$) if and only if it is true in some world w' that is related to the world w by the relation R.

If, for example, the relation R relates worlds that share the same (possibly empty) subset of true sentences of the prespecified set of expressions

$$\mathscr{F} = \{\phi_1, \phi_2, \ldots\}$$

that is, if $R(w, w')$ if and only if any sentence ϕ in $\mathscr{F}$ is either true in both w and w' or false in both w and w', then the resulting system has an "epistemic" interpretation that regards related possible worlds as "being possible for all we know" (i.e., observed evidence, corresponding to a subset of $\mathscr{F}$, is the same for both worlds). In this case, the necessity operator **N** corresponds to the epistemic operator **K** of epistemic logics, with the corresponding system having the properties of the modal system S5, which was used in the context of probability theory as the semantic basis for the Dempster–Shafer [38] calculus of evidence (Ruspini [35]).

If, on the other hand, the original interpretation of logical necessity—corresponding to a relation R that is equal to $\mathscr{U} \times \mathscr{U}$, that is, that relates every pair of possible worlds—is given to the operator **N**, then a proposition is necessarily true if and only if it is true in every possible world.

If the relation R is chosen as

$$R = \mathscr{E} \times \mathscr{E}$$

then this interpretation may be used to characterize approximate reasoning problems as those where a hypothesis of interest is neither necessarily true nor necessarily false in worlds in the evidential set $\mathscr{E}$, reflecting the inability of conventional deductive techniques to unambiguously determine the truth value of the hypothesis.[2]

[2] The notion of approximate reasoning problem is often extended to encompass situations where deductive techniques cannot always be used because of practical limitations on computational resources.

In those problems, in spite of this fundamental impossibility, we may resort to approximate reasoning methods to describe various properties of the evidential set $\mathscr{E}$. For example, the probabilistic structures used by various probabilistic reasoning approaches typically characterize relations of the form

$$\mu(\mathscr{H} \wedge \mathscr{E}) : \mu(\neg \mathscr{H} \wedge \mathscr{E})$$

between the measures of the subsets of the evidential set $\mathscr{E}$ where a hypothesis $\mathscr{H}$ is true or false, respectively.

Our aim will be to study how other structures, defining a *metric* or *distance* in the universe $\mathscr{U}$, can be used to describe the nature of the evidential set. To do so, we will assign a different meaning to the accessibility relation, giving it an interpretation that regards related worlds as "similar" or "close" in some sense. We will require, however, a scheme that is richer than that provided by a single relation so that we can extend modal notions and derive semantics bases for fuzzy logic, which relies on concepts of degrees of matching or closeness expressed by real numbers between 0 and 1.

In what follows we will use the symbols $\Rightarrow$ and $\Leftrightarrow$ to denote strong implication and equivalence, respectively. A proposition q *strongly implies* p (denoted $q \Rightarrow p$) if and only if p is true in any world where q is. Similarly, p is *logically equivalent* to q (denoted $p \Leftrightarrow q$) if and only if p and q are true in the same subset of worlds of $\mathscr{U}$.

Following traditional terminology, we will say also that a proposition p is *satisfiable* if there exists a possible world p such that $w \vdash p$.

EXTENDED MODALITIES

We first turn our attention to the problem of generalizing modal logic formalisms to explain the structures and functions of fuzzy logic.

A number of authors have studied various relations between fuzzy and modal logics. Lakoff [24], Murai et al. [28], and Schocht [36] have proposed graded generalizations of basic modal constructs. Dubois and Prade [13, 14] have also explored analogies between these nonstandard logics. In a recent paper [12], they developed, in addition, a modal basis for possibility theory by introducing fuzzy structures into modal frameworks with the goal of deriving proof mechanisms that can be used in possibilistic reasoning.

The goal for the model presented in this paper is somewhat different from the objectives guiding those efforts. We will seek explanations for possibilistic constructs on the basis of previously existing notions rather than generalizations of modal frameworks by means of fuzzy constructs. The model presented here is not based on the use of graded notions of possibility and necessity as primitive—and, by implication, easy to understand—structures. The foundation for this model is provided by a generalization of the accessibility relation, which is given a simple interpretation as a measure of resemblance and proximity between possible worlds.

We will extend the notion of accessibility relation to encompass a family of nonempty binary relations R_α that are indexed by a numerical parameter α between 0 and 1. These relations, which are nested,

$$R_\alpha \subseteq R_\beta \qquad \text{whenever } \beta \leq \alpha$$

are introduced to represent different degrees of similarity, using a scheme that is akin to that used by Lewis in his study of counterfactuals [25]. The family of accessibility relations introduced here differs from that proposed by Lewis, however, in its use of numerical indexes[3] and in the nature of the overall modeling goals that, in Lewis's formalism, are intended to represent changes of scale induced by consideration of different restrictive statements.

Similarity Relations

To facilitate the definition of a family of accessibility relations, we introduce a *similarity function*

$$S : \mathscr{U} \times \mathscr{U} \mapsto [0, 1]$$

assigning to each pair of possible worlds (w, w') a unique *degree of similarity* between 0 (corresponding to maximum dissimilarity) and 1 (corresponding to maximum similarity).

With the help of this function, we will then say that w and w' are related to the degree α, denoted $R_\alpha(w, w')$, if and only if $S(w, w') \geq \alpha$. In this way, the relations R_α have the required nesting property with R_0 corresponding to the whole Cartesian product $\mathscr{U} \times \mathscr{U}$ (or, every possible world is at least similar in a degree zero to every other possible world).

Some properties are required to assure that the function S has the required semantics of a metric relationship capturing the intuitive notion of similarity or "proximity." It is first necessary to demand that the degree of similarity between any world and itself be as high as possible, that is,

$$S(w, w) = 1 \qquad \text{for all } w \text{ in } \mathscr{U}$$

This property assures that every one of the accessibility relations R_α will be *reflexive* and, following the nomenclature introduced by Zadeh for fuzzy relations [52], we will also say that the similarity relation is reflexive.

Next, we will call for the function S to be *symmetric*, that is,

$$S(w, w') = S(w', w) \qquad \text{for any worlds } w \text{ and } w' \text{ in } \mathscr{U}$$

[3] We will later see that similarities can be measured by using more general, nonnumeric, scales. For simplicity reasons, I will avoid at this point the introduction of more general schemes that unnecessarily complicate the exposition.

This is a very natural requirement of any relation intended to represent a relation of resemblance between objects.

Finally, and most important, we will impose a form of *transitivity* requirement upon the similarity function S that turns it into a generalized equivalence relation. The purpose of this restriction is to assure that S has reasonable behavior as a metric in the universe of possible worlds. It would certainly be surprising if, for some similarity S, we were to be told that w and w' are very similar and that w' and w'' are also very similar, but that w does not resemble w'' at all. Clearly, there should be a lower bound on the possible values of $S(w, w'')$ that can be expressed as a function of the values of $S(w, w')$ and $S(w', w'')$. We will express such a constraint using a numeric operation, denoted $\circledast$, that takes as arguments two real numbers between 0 and 1 and returns another number in the same range, that is,

$$\circledast : [0, 1] \times [0, 1] \mapsto [0, 1]$$

in the form of the inequality

$$S(w, w'') \geq S(w, w') \circledast S(w', w'')$$

assumed valid for any worlds w, w', and w'' in the universe $\mathcal{U}$. Reverting to a modal terminology, the above transitivity constraint, which will be called *$\circledast$-transitivity*, may be rewritten in relational form as

$$R_{\alpha \circledast \beta} \subseteq R_\alpha \circ R_\beta \qquad \text{for all } 0 \leq \alpha, \beta \leq 1$$

making obvious its generalization of the conventional definition of transitivity for ordinary binary relations, that is,

$$R \subseteq R \circ R$$

Since the role of $\circledast$, through recursive application, is that of providing a lower bound for the similarity between the two end members w_1 and w_n of a chain of possible worlds $[w_1, w_2, \ldots, w_n]$, it is obvious that the operation $\circledast$ should be commutative and associative. Furthermore, it should also be nondecreasing in each argument, as it is reasonable to ask that the desired lower bound be a monotonic function of its arguments. Finally, it is also desirable to ask that

$$\alpha \circledast 1 = 1 \circledast \alpha = \alpha$$

that is, that the values of the similarities of two indistinguishable objects to a third should be the same. These requirements are equivalent to demanding that the operation $\circledast$ be a *triangular norm* (Schweizer and Sklar [37]), or *T-norm*, for short.

Triangular norms, originally introduced in the theory of probabilistic metric spaces to treat certain statistical problems, play a distinguished role in [0, 1]-

multivalued logics (Alsina and Trillas [1], Dubois and Prade [11], Gaines [17], Rescher [31]) as the result of imposing reasonable requirements upon operations that produce the truth value of the conjunction of two expressions as a function of the truth values of the conjuncts. Furthermore, generalized similarity relations (called B-R relations by Zadeh [54]) also have an important function, to be examined further later in this paper, in the generalization of the inferential rule of modus ponens (Dubois and Prade [10], Trillas and Valverde [43]). Our axiomatic derivation for the requirement that $\circledast$ be a T-norm is based, however, solely on metric considerations, applied here to a space of possible worlds but valid in general metric spaces.

From the axioms of triangular norms, it is easy to see that

$$\alpha \circledast \beta \leq \min(\alpha, \beta)$$

which shows that the minimum function, itself a T-norm, is the largest element in this class of operations. Its minimal element, on the other hand, is the noncontinuous function $\circledast$ defined by

$$\alpha \circledast \beta = \begin{cases} \alpha & \text{if } \beta = 1 \\ \beta & \text{if } \alpha = 1 \\ 0 & \text{otherwise} \end{cases}$$

In what follows, we will also impose a most reasonable additional assumption of continuity of $\circledast$ with respect to its arguments (i.e., why should there be a jump in the value of a lower bound provided by $\circledast$ when the values of its arguments are slightly changed?). The class of continuous T-norms does not have a minimal element, although under certain additional assumptions (requiring T-norms to be also J-copulas [37]), the inequality

$$\max(\alpha + \beta - 1, 0) \leq \alpha \circledast \beta$$

also holds true, showing that certain important continuous T-norms lie between that of the $\aleph_1$-logic of Lukasiewicz (see [17]) and that of the original fuzzy logic proposed by Zadeh [53].

Continuous triangular norms play a significant part in the theories of pattern recognition and automatic classification (Ruspini [32]). In [33] I proposed the use of generalized similarity relations based on the T-norm of Lukasiewicz to generalize existing classification techniques—based on the mapping of a similarity function into a conventional equivalence relation—to the fuzzy domain by mapping these T-norms (which I called likeness relations) into generalized fuzzy partitions. Bezdek and Harris [3] independently studied axiomatic approaches to cluster analysis based on the use of continuous T-norms.

I have also studied [34] the possible relation between the multivalued logic and similarity related aspects of T-norms, and suggested that the degrees of similarity between two objects A and B may be regarded as the "degree of truth'' of the vague proposition

''A is similar to B.''

Having argued that S should have the structure of a generalized equivalence relation, we will assume, mainly for reasons of simplicity, that the function S is the dual of a ''true'' distance, that is, that

$$S(w, w') = 1 \text{ if and only if } w = w'$$

This restriction, which is not substantial, is introduced primarily to assure that different possible worlds may be distinguished by means of the function S. Otherwise, the equivalence relation that relates two worlds w and w' if and only if $S(w, w') = 1$ may be used to partition our universe $\mathcal{U}$ into ''indistinguishable'' nonintersecting classes, indicating that our metric cannot discriminate between significant differences in system state.

Before closing our presentation of generalized similarity relations, it is important to remark upon the close relation between the notion of similarity and that of distance. If a function δ is defined in terms of a similarity function S by the simple relation

$$\delta = 1 - S$$

then it is easy to see that the function δ has the properties of a metric or distance. This is evident if the operation $\circledast$ corresponds to the T-norm of Lukasiewicz, since the transitivity condition is equivalent to the well-known triangular inequality, that is,

$$\delta(w, w'') \leq \delta(w, w') + \delta(w', w'')$$

If other T-norms are used, even stronger inequalities hold, with the so-called ''ultrametric inequality''

$$\delta(w, w'') \leq \max[\delta(w, w'), \delta(w', w'')]$$

being valid for the T-norm of Zadeh. In this case, each of the relations in the family R_α (known in fuzzy set theory as the α-cut[4] of the similarity S) is a conventional equivalence relation. This fact was exploited, prior to the introduction of fuzzy set theory and fuzzy cluster analysis, by a variety of clustering procedures of the ''single-link'' type (Jardine and Sibson [22], Sokal and Sneath [40]).

Possible and Necessary Similarity

Our semantic formalization needs require the introduction of constructs to indicate the extent by which a concept exemplifies, illustrates, or is an

[4]The α-cut [46] of a fuzzy set μ: $\mathcal{U} \mapsto [0, 1]$ is the conventional set of all points w such that $\mu(w) \geq \alpha$. A similar concept is defined for relations as subsets of a product space $\mathcal{U} \times \mathcal{U}$.

adequate model of another concept. Our interpretations will therefore be oriented toward characterization of the degree to which a concept can be said to be a good example of another concept with the purpose of defining vague concepts by means of measures of proximity between defined and defining concepts. In our treatment, each of the multiple "definiens" will be a conventional proposition corresponding to a subset of possible worlds. It is conceivable, however, that new vague concepts might also be described metric relations to other vague concepts.

The required constructs are based on the idea that whenever p and q are propositions such that $p \Rightarrow q$, then any p-world is an "example" of a q-world. This basic notion will be generalized by the introduction of modal structures that define to what degree possible worlds that satisfy a certain proposition q fit a vague concept. Some of those possible worlds are "paradigmatic" of the vague concept, that is, they fit it to a degree equal to 1 in the same sense that we may say, for example, that somebody whose height is 7 ft is definitely "tall." If we use a notion of graded fitness, however, certain worlds will fit the concept to a degree, that is, they resemble (or are similar to) some paradigmatic example of the vague concept.

The conventional interpretation of possibility must be modified, therefore, to capture the idea that a particular possible world is similar in some degree to another world that satisfies a "reference" proposition.

More generally, however, we will be interested in relations of similarity between *pairs of subsets* of possible worlds rather than between pairs of possible worlds. This requirement complicates matters considerably, because we will be forced to consider both the "validity" of a proposition p in *some* world where another proposition q is true and its applicability in *every* world where q is true. In the former case, we will care about the existence of q-worlds that are similar to some degree to some p-world, whereas in the latter we will be concerned with the size of the minimum neighborhood of p (as a subset of the universe $\mathcal{U}$) that fully encloses the subset q.

This dual concern for what may possibly apply and what must necessarily hold—an essential aspect of modal logic—is typical of situations where relationships between ensembles of objects are described in terms of relations between their members. In the probability calculus, for example, knowledge of probabilities over certain families of subsets provides "sharp" upper and lower bounds (called *lower* and *upper probabilities*, respectively) for the probabilities of other subsets—an important fact in the extension of set measures to larger domains (Halmos [19]). The role and properties of these bounds in the Dempster–Shafer [38] calculus of evidence is well known, having been described in the original paper of Dempster [8], related to concepts of modal logic by Ruspini [35], and being also the subjects of considerable formal study (Choquet [7]) as mathematical structures.

Analogies between the role of probabilistic bounds (i.e., bounds for probability values) and possibility/necessity distributions, have been the source of much of the confusion about the need for possibilistic schemes. Each upper/lower-bound pair, however, leads to a substantially different description of the nature of a subset of possible worlds, being, in either case, measures that arise naturally when pointwise properties are extended to set partitions. General properties of these measures have been studied by Dubois and Prade [15] in the context of approximate reasoning and in other regards by Pavlak [30].

Our generalizations of the notions of possibility and necessity are related to the so-called *de re* (Hughes and Creswel [21]) interpretation of the statement "If q, then p is possible" as the modal propositional relation

$$q \Rightarrow \Pi p$$

We will say that the proposition q implies, or is a *necessary model* of, the proposition p to the degree α if and only if for every q-world w there exists a p-world w' that is at least α-similar to it [i.e., $S(w, w') \geq \alpha$] or, equivalently, whenever

$$q \Rightarrow \Pi_\alpha p$$

Similarly, we will say that the proposition q is *consistent with*, or is a *possible model* of, the proposition *p to the degree* α[5] if and only there exist a q-world w and a p-world w' that are at least α-similar or, equivalently, whenever

$$\neg(p \Rightarrow \neg\Pi_\alpha q)$$

The similarity function that we have introduced in the universe $\mathcal{U}$ provides us with a simple mechanism to quantify both the extent of "inclusion" and that of the "intersection" between pairs of subsets of possible worlds.[6]

Possibilistic Implication and Consistency

The notion of subset inclusion and its related concept of set identity are of central importance in deductive logic, since subsets of possible worlds are formally equivalent to propositions with subset inclusion and identity corre-

[5]Note that our characterizations of both possibility and necessity distributions are based in the modal possibility operators Π_α.

[6]For reasons that by now should be evident, we will not need to introduce a concept of "unconditioned possibility" although it would be easy to do so using $q = \mathcal{U}$. Being concerned with the power of certain propositions to exemplify other conditions, we will not have much occassion to deal with the strength of tautologies in that regard.

sponding to logical implication and equivalence, respectively. These propositional relationships are the basis of derivation rules such as the modus ponens. The notion of intersection plays a similar role in modal analyses because of its ability to express the potential validity of a statement.

Classical accounts, however, recognize only two "degrees" of inclusion corresponding to the cases when either a set q is a subset of another set p or it is not, with a similar dichotomy applying to degrees of intersection. Our generalization exploits the metric structures defined between sets of possible worlds by introducing measures that describe a subset as enclosed in a *neighborhood* (of some size) of another set while intersecting another of its neighborhoods (of "smaller" size).[7] The problem of measuring the "size" of those neighborhoods is the subject of our immediate considerations.

DEGREE OF IMPLICATION Our definition of partial implication between propositions was based on conditions that determine whether, given two propositions p and q, one of them implies the other to the same value α. In particular, since every world w is always similar in a degree that is at least equal to zero to any other world w', it is always true that any proposition q implies any other proposition p to the degree zero. It is often the case, however, that the degree of implication between p and q is at least equal to some certain positive value α.

If we want to generalize procedures based on inclusion relationships, such as the modus ponens, in an efficient fashion, we will need to measure the "optimal" (or maximum) value of the parameter α such that q implies p to the degree α. This value is a measure of the degree to which the set of all p-worlds must be "stretched" to encompass the set of all q-worlds. The least upper bound of the values of the similarities between any q-world w' and some p-world w is given by the *degree of implication* function:

DEFINITION 1 *The* degree of implication *of p by q is the value*

$$\mathbf{I}(p \mid q) = \inf_{w' \vdash q} \sup_{w \vdash p} S(w, w')$$

Defined in this way, the degree of implication $\mathbf{I}(p \mid q)$ is a measure of the "minimal amount" of stretching required to reach a p-world from any q-world, in the sense that if $\beta < \mathbf{I}(p \mid q)$, then

$$q \Rightarrow \Pi_\beta p$$

[7] It is important to recall that, owing to our reliance on similarity rather than on the dual notion of dissimilarity or distance, high values of α correspond to low values of "stretching" or to smaller set neighborhoods.

Furthermore, α is the largest real value for which the above statement may be made.

As the following theorem makes clearer, this function provides the basis for the generalization of the modus ponens. This truth-derivation procedure may be thought of as an expression of the nesting relationships that hold between the sizes of neighborhoods of such subsets.

THEOREM 1 *The degree of implication function,*

$$\mathbf{I}: \mathscr{L} \times \mathscr{L} \mapsto [0, 1]$$

has the following properties:

(i) *If $p \Rightarrow r$, then $I(p \mid q) \leq I(r \mid q)$*
(ii) *If $p \Rightarrow r$, then $I(p \mid q) \geq I(p \mid r)$*
(iii) *$I(p \mid q) \geq I(p \mid r) \circledast I(r \mid q)$*

where p, q, and r are any satisfiable propositions.

Proof The first two properties are an immediate consequence of the definition of degree of implication. To prove the third, observe that by definition of similarity,

$$S(w, w') \geq S(w, w'') \circledast S(w'', w')$$

for any worlds w, w', and w''.

Taking the supremum on both sides of this inequality with respect to all worlds $w \vdash p$, it follows, because $\circledast$ is continuous, that

$$\sup_{w \vdash p} S(w, w') \geq \left[\sup_{w \vdash p} S(w, w'') \right] \circledast S(w'', w')$$

Since this expression is true, in particular, for all worlds $w'' \vdash r$, it is true that

$$\sup_{w \vdash p} S(w, w') \geq \left[\inf_{w'' \vdash r} \sup_{w \vdash p} S(w, w'') \right] \circledast S(\hat{w}, w')$$
$$= \mathbf{I}(p \mid r) \circledast S(\hat{w}, w')$$

where $\hat{w}$ is any world such that $\hat{w} \vdash r$.

From this inequality, it follows, since $\circledast$ is continuous, that

$$\sup_{w \vdash p} S(w, w') \geq \mathbf{I}(p \mid r) \circledast \left[\sup_{\hat{w} \vdash r} S(\hat{w}, w') \right]$$

Taking now the infimum on both sides of this expression over all worlds w' such that $w' \vdash q$, it is easy to see, using again the continuity of $\circledast$, that

$$\inf_{w' \vdash q} \sup_{w \vdash p} S(w, w') \geq \mathbf{I}(p \mid r) \circledast \left[\inf_{w' \vdash q} \sup_{\hat{w} \vdash r} S(\hat{w}, w') \right]$$

proving the $\circledast$-transitivity of **I**. ∎

Note, that since $\mathbf{I}(q \mid q) = 1$ for any proposition q, the following statement is also true.

COROLLARY *If p and q are propositions in $\mathscr{L}$, then*

$$\mathbf{I}(p \mid q) = \sup_r \left[\mathbf{I}(p \mid r) \circledast \mathbf{I}(r \mid q)\right]$$

Notice also that if $\mathbf{I}(\mathrm{p} \mid \mathrm{q}) = 1$, then

$$\sup_{w \vdash p} S(w, w') = 1 \qquad \text{for all } w' \vdash q$$

Under minimal assumptions (assuring that the supremum operation is actually a maximization), this relation is equivalent to stating that q strongly implies p, or that any q-world is also a p-world.

The nonsymmetric function $\mathbf{I}$ measures the extent to which every world w' in a certain class resembles some world w (dependent on w') in a reference class, explicating the nature of the nonsymmetric assessments (Tversky [44]) found in psychological experimentation when subjects are asked to evaluate the degree to which an object "resembles" another. The results obtained in those experiments suggest that human beings, when assessing similarity between objects, use one of them (or a class of similar objects) as a reference landmark to describe the other. Such asymmetries might be explained by noticing that, in general, $\mathbf{I}(p \mid q) \neq \mathbf{I}(q \mid p)$, indicating that the stronger stimulus might generally be used to construct a reference class, which is then used to describe other stimuli.

The degree of implication of one proposition by another can be readily used to generate a measure of similarity between propositions that generalizes our original measure of similarity between possible worlds:

$$\hat{S}(p, q) = \min\left[\mathbf{I}(p \mid q), \mathbf{I}(q \mid p)\right]$$

quantifying the degree by which the propositions p and q are equivalent. It can be readily proved (Valverde [45]) from its definition and from the transitivity property of $\mathbf{I}$ that $\hat{S}$ is a reflexive, symmetric, and $\circledast$-transitive function between subsets of possible worlds. This similarity function is the dual of the well-known *Hausdorff distance*, defined between subsets of a metric as a function of the distance between pairs of their members (Dieudonné [9]), which is given by the expression

$$\hat{\delta}(A, B) = \max\left\{\left[\sup_{x \in A} \inf_{y \in B} \delta(x, y)\right], \left[\sup_{x \in B} \inf_{y \in A} \delta(x, y)\right]\right\}$$

The result expressed by the transitive property of the degree of implication may be stated using modal notation in the form

$$q \Rightarrow \Pi_\alpha r \text{ and } r \Rightarrow \Pi_\beta p \quad \text{imply that} \quad q \Rightarrow \Pi_{\alpha \circledast \beta} p$$

as the simplest form of the generalized modus ponens rule of Zadeh.

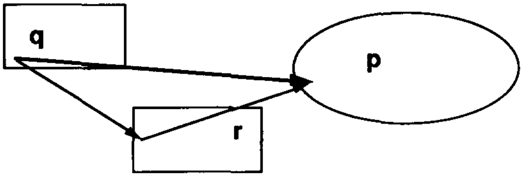

Figure 1. The generalized modus ponens.

The relationship between this rule and the classical modus ponens is easier to perceive if it is remembered that classical conditional propositions of the form "If q, then p" simply state that the set of q-worlds is a subset of the set of p-worlds. Such relationships of inclusion can also be described in metric terms by saying that every q-world has a p-world (i.e., itself) that is as similar as possible to it.

Logic structures, however, only allow us to say either that q implies p or that q implies its negation $\neg p$, or that neither of those statements is true. By contrast, similarity relations allow measurement of the amount by which a set must be "stretched" (as illustrated in Figure 1) to enclose another set. Using such metrics, we can describe the generalized modus ponens as a relation between the stretching required to reach p from any point of the set r, the stretching required to reach r from any point of the set q, and the stretching required to reach p from any point of the set q.

In the section Generalized Inference, we will derive alternative expressions for the generalized modus ponens that allow us to propagate both measures characterizing degree of implication and degree of consistency; a dual concept that plays, with respect to the notion of possibility, the function that is fulfilled by the degree of implication function with respect to necessity. In those derivations, by introducing sharper bounds for certain conditional concepts, we will also be able to improve the quality of the bounds provided by generalized modus ponens rules while being closer in spirit to its usual fuzzy-logic formulation.

DEGREE OF CONSISTENCY A notion that is dual to that of degree of implication is given by a function that measures the pointwise proximity between pairs of possible worlds from an "optimistic" point of view characterizing the degree to which statements that are true in some worlds *may* apply in others. By contrast, the degree of implication measures the extent to which statements that are true in p-worlds *must* hold in q-worlds.

DEFINITION 2 *The* degree of consistency *of p and q is the value*

$$\mathbf{C}(p \mid q) = \sup_{w' \vdash q} \sup_{w \vdash p} S(w, w')$$

An immediate consequence of this definition that $\mathbf{C}(\cdot \mid \cdot)$ is a symmetric function that is increasingly monotonic in both arguments (with respect to the $\Rightarrow$). It is also easy to see that the values of the degree of consistency function are never smaller than the corresponding values of the degree of consistence function,

$$\mathbf{I}(p \mid q) \leq \mathbf{C}(p \mid q)$$

as the amount of stretching required to reach p from some "convenient" q-world is smaller (i.e., higher values of S) than that required to reach p from any q-world. In general, however, the degree of consistency function is not transitive, preventing the statement of a "compatibility" counterpart of the generalized modus ponens rule. Its relationship with the degree of implication function expressed by the expression

$$\mathbf{C}(p \mid q) = \sup_{w' \vdash q} \mathbf{I}(p \mid w') = \sup_{w \vdash p} \mathbf{I}(q \mid w)$$

will permit us, nonetheless, to derive a useful bound-propagation expression.

POSSIBILITY AND NECESSITY DISTRIBUTIONS

This section presents interpretations of the major constructs of fuzzy logic—possibility and necessity distributions—in terms of similarity-based structures. Possibility and necessity distributions are functions that measure the proximity of either all or some of the worlds in the evidential set to worlds in other sets that are employed as reference landmarks.

The role played by possibility and necessity distributions is similar to that performed by lower and upper bounds of probability distributions (or by the belief and plausibility functions of the Dempster-Shafer calculus of evidence) with respect to probability distributions. The essential difference between these bounds and those provided by possibility/necessity pairs lies in the fundamentally dissimilar character of what is being bound—metric structures relating pairs of worlds in one case and measures of set size in the other. Furthermore, in the model of possibilistic structures that is presented in this paper, necessity (possibility) distributions are any lower (upper) bounds of a certain metric function rather than its "best" or "sharpest" bounds. The operations of fuzzy logic allow computation of bounds for some of these measures as a function of bounds of other measures.

Inverse of a Triangular Norm

When working in ordinary metric spaces, it is often convenient to express the conventional statement of the triangular inequality,

$$\delta(w, w') \leq \delta(w, w'') + \delta(w'', w')$$

Table 1. Triangular Norms, Conorms, and Pseudoinverses

Name	T-Norm $a \circledast b$	Conorm $a \oplus b$	Pseudoinverse $a \oslash b$
Lukasiewicz	$\max(a+b-1,0))$	$\min(a+b,1)$	$\min(1+a-b,1)$
Product	ab	$a+b-ab$	a/b if $b>a$ 1 otherwise
Zadeh	$\min(a,b)$	$\max(a,b)$	a if $b>a$ 1 otherwise

in the equivalent form

$$\delta(w, w') \geq |\delta(w, w'') - \delta(w', w'')|,$$

which utilizes a form of inverse (i.e., the subtraction operator $-$) of the function used to express the original inequality (i.e., the addition operator $+$). This notion of inverse can be directly generalized (Schweizer and Sklar [37]) to provide us with the tools required to define possibility and necessity functions and to derive useful forms of the generalized modus ponens involving either type of these constructs.

DEFINITION 3 *If* $\circledast$ *is a triangular norm, its* pseudoinverse $\oslash$ *is the function defined over pairs of numbers in the unit interval of the real line by the expression*

$$a \oslash b = \sup\{c : b \circledast c \leq a\}$$

From this definition it is clear that $a \oslash b$ is nondecreasing in a and nonincreasing in b. Furthermore, $a \oslash 0 = 1$ and $a \oslash 1 = a$ for any a in $[0, 1]$. Other important properties of the pseudoinverse function are given in the works of Schweizer and Sklar [37], Trillas and Valverde [43], and Valverde [45].

Examples of the pseudoinverses of important triangular norms are given in Table 1 together with the corresponding conorms.

Unconditioned Necessity Distributions

We introduce first a family of functions that bound from below the value of the similarity between any evidential world in $\mathscr{E}$ and some world where another proposition p is true. These *unconditioned necessity* distributions are lower bounds for values of the degree of implication $\mathbf{I}(p \mid \mathscr{E})$, which measures the extent to which statements that are true in a reference set (i.e., the subset of p-worlds) must hold in the evidential set.

As observed before, whenever $\mathbf{I}(p \mid \mathscr{E}) = 1$, it is true, under minimal assumptions, that the evidential subset $\mathscr{E}$ is a subset of the set of all p-worlds, or that p necessarily holds in $\mathscr{E}$. If, on the other hand, $\mathbf{I}(p \mid \mathscr{E}) = \alpha < 1$, then p must be stretched a certain amount—with smaller α corresponding to greater stretching—in order for one of its neighborhoods to encompass $\mathscr{E}$.

DEFINITION 4 *If $\mathscr{E}$ is an evidential set, then a function* Nec($\cdot$) *defined over propositions in the language $\mathscr{L}$ is called an* unconditioned necessity distribution *for $\mathscr{E}$ if*

$$\operatorname{Nec}(p) \leq \mathbf{I}(p \mid \mathscr{E})$$

Unconditioned Possibility Distributions

The dual counterpart of the unconditioned necessity distribution is provided by upper bounds of the degree of consistency $\mathbf{C}(p \mid \mathscr{E})$. Whenever $\mathbf{C}(p \mid \mathscr{E}) = 1$, it is easy to see that, under minimal assumptions, there exists a p-world w that is in the evidential set $\mathscr{E}$ or, equivalently, that p (for all we know) is possibly true. If, on the other hand, $\mathbf{C}(p \mid \mathscr{E}) = \alpha < 1$, then there exists a neighborhood (of "size" α) of some p-world that intersects the evidential set.

DEFINITION 5 *If $\mathscr{E}$ is an evidential set, then a function* Poss($\cdot$) *defined over propositions in the language $\mathscr{L}$ is called an* unconditioned possibility distribution *for $\mathscr{E}$ if*

$$\operatorname{Poss}(p) \geq \mathbf{C}(p \mid \mathscr{E})$$

Since the value Poss(p) of any possibility function Poss($\cdot$) is an upper bound of the value $\mathbf{C}(p \mid \mathscr{E})$ of the degree of consistence, the corresponding value Nec(p) of any necessity function Nec($\cdot$) is a lower bound of $\mathbf{I}(p \mid q)$, it follows that values of a possibility function can never be smaller than the corresponding values of any necessity function, that is, that

$$\operatorname{Nec}(p) \leq \operatorname{Poss}(p)$$

Properties of Possibility and Necessity Distributions

In this subsection we will develop similarity-based interpretations for some basic formulas of possibilistic calculus. These expressions may be thought of as mechanisms that allow the extension of a partially known possibility distribution. For example, the property that

$$\max[\operatorname{Poss}(p), \operatorname{Poss}(q)] \geq \mathbf{C}(p \vee q \mid \mathscr{E})$$

which is proved below, is the similarity interpretation of the standard rule that allows computation of the value of the possibility of a disjunction in fuzzy logic, that is,

$$\operatorname{Poss}(p \vee q) = \max[\operatorname{Poss}(p), \operatorname{Poss}(q)]$$

THEOREM 2 *If p and q are propositions, and if the quantities Poss(p), Poss(q), Nec(p), and Nec(q) are such that*

$$\mathrm{Nec}(p) \le \mathbf{I}(p \mid \mathscr{E}), \qquad \mathrm{Nec}(q) \le \mathbf{I}(q \mid \mathscr{E})$$

$$\mathrm{Poss}(p) \ge \mathbf{C}(p \mid \mathscr{E}), \qquad \mathrm{Poss}(q) \ge \mathbf{C}(q \mid \mathscr{E})$$

then the following statements (similarity-based interpretations of the basic laws of fuzzy logic) are valid:

$$\max[\mathrm{Nec}(p), \mathrm{Nec}(q)] \le \mathbf{I}(p \vee q \mid \mathscr{E})$$

$$\max[\mathrm{Poss}(p), \mathrm{Poss}(q)] \ge \mathbf{C}(p \vee q \mid \mathscr{E})$$

$$\min[\mathrm{Poss}(p), \mathrm{Poss}(q)] \ge \mathbf{C}(p \wedge q \mid \mathscr{E})$$

Proof Note first that since $\mathbf{C}(\cdot \mid \cdot)$ is nondecreasing (with respect to the $\Rightarrow$ order) in its arguments, it is true that

$$\mathrm{Poss}(p) \ge \mathbf{C}(p \mid \mathscr{E}) \ge \mathbf{C}(p \wedge q \mid \mathscr{E})$$

$$\mathrm{Poss}(q) \ge \mathbf{C}(q \mid \mathscr{E}) \ge \mathbf{C}(p \wedge q \mid \mathscr{E})$$

whenever $p \wedge q$ is satisfiable, from which it is easy to see that

$$\min[\mathrm{Poss}(p), \mathrm{Poss}(q)] \ge \mathbf{C}(p \wedge q \mid \mathscr{E})$$

The corresponding result is obvious when $p \wedge q$ is nonsatisfiable.

A similar argument shows, for necessity functions, that

$$\max[\mathrm{Nec}(p), \mathrm{Nec}(q)] \le \mathbf{I}(p \vee q \mid \mathscr{E})$$

To prove the disjunctive law for possibilities, notice that if f is any function mapping elements of a general domain D into real numbers, then

$$\sup\{f(d) : d \in A \cup B\} = \max[\sup\{f(d) : d \in A\}, \sup\{f(d) : d \in B\}]$$

From this equality, it is easy to see that if Poss(p) and Poss(q) are upper bounds of $\mathbf{I}(p \mid \mathscr{E})$ and $\mathbf{I}(q \mid \mathscr{E})$, respectively, then

$$\max[\mathrm{Poss}(p), \mathrm{Poss}(q)] \ge \mathbf{C}(p \vee q \mid \mathscr{E})$$

which completes the proof of the theorem. ∎

Note, however, that another law commonly given as an axiom for necessity functions does not hold valid in our interpretation. As illustrated in Figure 2, the distance from a point to the intersection of two sets may be strictly larger than the distance to either set (i.e., the similarity will be strictly smaller). In general, therefore,

$$\min[\mathrm{Nec}(p), \mathrm{Nec}(q)] \not\le \mathbf{I}(p \wedge q \mid \mathscr{E})$$

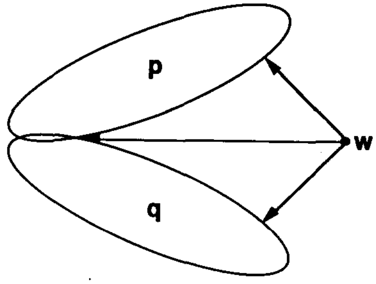

Figure 2. Failure of conjunctive necessity.

making invalid, under this interpretation, the conjunctive law for necessities (Dubois and Prade [11])

$$\mathrm{Nec}(p \wedge q) = \min[\mathrm{Nec}(p), \mathrm{Nec}(q)]$$

We may also note in this regard that the similarity-based model that is discussed here does not make use of the notion of negation either as a mechanism to generate dual concepts or in its own right as an important logical concept. It is my intent to study, in the immediate future, alternative models in which notions of negation and maximal dissimilarity play more substantive roles.

Conditional Possibilities and Necessities

The concepts of conditional possibility and necessity are closely related to the previously introduced unconditioned structures. These structures may be thought of as a characterization of the proximity of a world w to some or all of the worlds where a proposition p is true, ***given that w is similar in the degree 1 to the evidential set*** $\mathscr{E}$ (i.e., $w \vdash \mathscr{E}$). With this fact in mind, we could have used the somewhat baroque formulation

$$\mathbf{C}(p \mid \mathscr{E}) = \sup_{w \vdash \mathscr{E}} [\mathbf{I}(p \mid w) \oslash \mathbf{I}(\mathscr{E} \mid w)]$$

to define unconditioned possibility distributions—a rather unnecessary effort if we consider that $\mathbf{I}(\mathscr{E} \mid w) = 1$ whenever $w \vdash \mathscr{E}$, showing its obvious equivalence to the simpler form used in the previous section. In spite of such observation, the above identity is important in understanding the purpose of the definitions that follow. Those definitions interpret conditional possibilities and necessities as a measure of the proximity of worlds on the evidential set $\mathscr{E}$ to (some or all) worlds satisfying a (conditioned) proposition p relative to their proximity to (some or all of) the worlds that satisfy another (conditioning) proposition q.

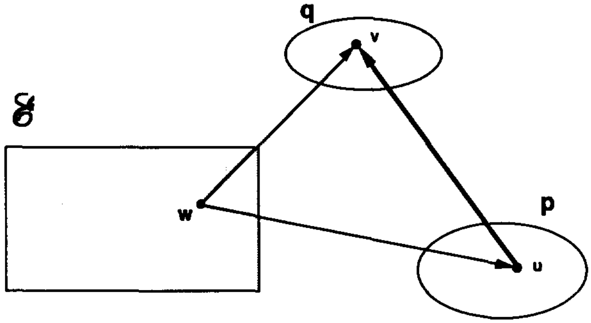

Figure 3. Similarities as viewed from the evidential set.

The mechanism used to specify that relationship, which is closely related in spirit to results of Valverde [45] on the structure of indistinguishability relations, is based on the pseudoinverse function introduced earlier. The basic idea used by these definitions is also illustrated in Figure 3, where, from the perspective of the evidential world w, the similarity between the p-world u and the q-world v is estimated by means of an inequality that generalizes the "absolute value" form of the triangular inequality,

$$\delta(u, v) \geq |\delta(u, w) - \delta(v, w)|$$

to its similarity-based form

$$S(u, v) \leq \min[S(u, w) \oslash S(v, w), S(v, w) \oslash S(u, w)]$$

The required interplay between similarities to conditioning and conditioned sets is captured by the following definitions.

Definition 6 *Let $\mathcal{E}$ be an evidential set. A function $Nec(\cdot \mid \cdot)$ mapping pairs of propositions in the language $\mathcal{L}$ into* [0, 1] *is called a* conditional necessity distribution *for $\mathcal{E}$ if*

$$\mathrm{Nec}(q \mid p) \leq \inf_{w \vdash \mathcal{E}} [\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid w)]$$

for any propositions p and q in $\mathcal{L}$.

Definition 7 *Let $\mathcal{E}$ be an evidential set. A function $Poss(\cdot \mid \cdot)$ mapping pairs of propositions in the language $\mathcal{L}$ into* [0, 1] *is called a* conditional possibility distribution *for $\mathcal{E}$ if*

$$\mathrm{Poss}(q \mid p) \geq \sup_{w \vdash \mathcal{E}} [\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid w)]$$

for any propositions p and q in $\mathcal{L}$.

It is easy to see from these definitions that the values of a conditional necessity distribution are never larger than the corresponding values of a conditional possibility distribution, that is,

$$\mathrm{Nec}(q \mid p) \leq \mathrm{Poss}(q \mid p)$$

Furthermore, since $\mathbf{I}(\cdot \mid \cdot)$ is $\circledast$-transitive, it is

$$\mathbf{I}(q \mid w) \geq \mathbf{I}(q \mid p) \circledast \mathbf{I}(p \mid w)$$

From this inequality and the definition of psuedoinverse of a triangular norm, it is easy to see that $\mathbf{I}(q \mid p)$ is a conditional necessity function, showing also that the bounds provided by the evidential-set perspective are better than those that can be obtained by direct use of the degree of implication as the definition of conditional necessity.[8]

Note also that if $\mathrm{Nec}(p) = 1$, indicating that $\mathbf{I}(p \mid \mathscr{E}) = 1$, and if $\mathrm{Nec}(q \mid p) = 1$, then the above definition of conditional necessity shows that $\mathbf{I}(q \mid \mathscr{E}) = 1$, indicating that $\mathrm{Nec}(q)$ may be taken to be equal to 1, thus generalizing the well-known axiom (consequential closure) of certain modal systems (e.g., the system **T**, as discussed in Hughes and Creswell [21])

$$\text{If } \mathbf{N}p \text{ and } \mathbf{N}(p \to q), \text{ then } \mathbf{N}q.$$

The definitions above can also be further interpreted as a way to compare the similarities between evidential worlds and those in the conditioning and conditioned sets by noting that whenever

$$\mathbf{I}(q \mid w) \geq \mathbf{I}(p \mid w)$$

for every evidential world $w \vdash \mathscr{E}$, then $\mathrm{Nec}(q \mid p)$ may be chosen to be equal to 1. Similarly, if there exists some world $w \vdash \mathscr{E}$ where this inequality holds, then it is $\mathrm{Poss}(q \mid p) = 1$. In either case, however, the maximum value for the conditional distribution (i.e., 1) is reached when the proximity of one evidential world w, in the case of possibilities, or of every one of them, in the case of necessities, to a world w_q in the conditioned set exceeds the proximity of w to the conditioning set p. In either case, once again returning to an apparent notational overkill, we may state this fact by means of the identity function τ in the unit interval:

$$\tau : [0, 1] \mapsto [0, 1] : \alpha \mapsto \alpha$$

in the form

$$\mathbf{I}(q \mid w) \geq \tau(\mathbf{I}(p \mid w))$$

for some $w \vdash \mathscr{E}$ in the case of possibilities, with the same inequality holding

[8]A dual result for possibilities involving $\mathbf{C}(q \mid p)$ does not hold in general. It is easy to see, however, that $\mathbf{C}(q \mid \mathscr{E}) \oslash \mathbf{I}(p \mid \mathscr{E})$ is a possibility function for q given p.

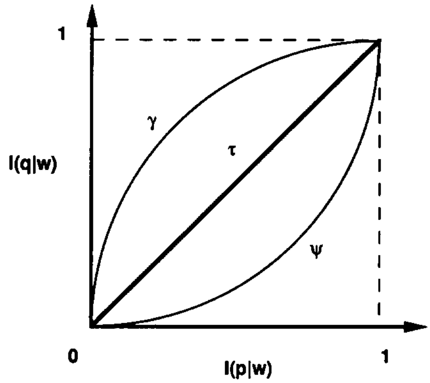

Figure 4. Examples of possible similarity relationships between conditioning and conditioned sets.

for every $w \vdash \mathscr{E}$ in the case of necessities. We can, however, conceive of other functions

$$\gamma : [0,1] \mapsto [0,1] : \alpha \mapsto \gamma(\alpha)$$

with $\gamma(\alpha) \geq \alpha$ to specify a stronger form of implication, as illustrated in Figure 4, that is,

$$\mathbf{I}(q \mid w) \geq \gamma(\mathbf{I}(p \mid w))$$

Similarly, we can also conceive of functions ψ with $\psi(\alpha) \leq \alpha$ that can be used to model weaker forms of implication.

Possibilistic calculi based on the propagation of truth mappings of this type, first proposed by Baldwin [2], are utilized in the RUM (Bonissone and Decker [4], Bonissone et al. [5]) and MILORD (Godo et al. [18]) expert systems. The particular case when $\gamma = \tau$, stating that every α-cut of the conditioning proposition p is fully enclosed (in the conventional sense) in the α-cut of the conditioned proposition q, has been called *truth mapping* in fuzzy logic literature.

The primary purpose of conditional distributions, however, is to provide a quantitative measure of the degree to which one proposition may be said to imply another with a view to extending inferential procedures by means of structures that superimpose the topological notion of continuity upon a logical framework concerned with propositional validity.

GENERALIZED INFERENCE

The major inferential tool of fuzzy logic is the *compositional rule of inference* of Zadeh [53], which generalizes the corresponding classical rule of inference by its ability to infer valid statements even when a perfect match between facts and rule antecedent does not exist, that is, from

$$\begin{array}{l} p \\ p \to q \\ \hline \quad\;\; q \end{array}$$

to its "approximate" version

$$\begin{array}{l} p' \\ p \to q \\ \hline \quad\;\; q' \end{array}$$

where p' and q' are similar to p and q, respectively. In this sense, the generalized modus ponens operates as an "interpolation" (or, more precisely, as an "extrapolation") procedure in possible-world space.

Unlike the interpolation procedures of numerical analysis, however, which yield estimates of function value, this extrapolation procedure approximates truth in the sense that it produces a proposition that is more general than the consequent of the inferential rule but resembles it to some degree (which is a function of the degree to which p' resembles p). The "extrapolated conclusion," however, is a correctly derived proposition, that is, the result of a sound logical procedure rather than of an approximate heuristic technique.

Generalized Modus Ponens

The theorems that are proved below are based on the use of a family $\mathscr{P}$ of propositions that partitions the universe of discourse $\mathscr{U}$ in the sense that every possible world will satisfy at least one proposition in $\mathscr{P}$.

DEFINITION 8 *If $\mathscr{P}$ is a subset of satisfiable propositions in $\mathscr{L}$ such that if w is a possible world in the universe $\mathscr{U}$, then there exists a proposition p in $\mathscr{P}$ such that $w \vdash p$, then the family $\mathscr{P}$ is called a* partition *of $\mathscr{U}$.*

These results make use of information such as the values of the unconditioned necessity or possibility distributions for antecedent propositions p in the family $\mathscr{P}$ together with the values $\mathrm{Nec}(q \mid p)$ or, respectively, $\mathrm{Poss}(q \mid p)$ to "extend" the unconditioned distributions to the "consequent" proposition q. In this sense, these findings interpret, in the same spirit used in Theorem 2 for other basic laws, the generalized modus ponens laws of fuzzy logic:

$$\mathrm{Nec}(q) = \sup_{\mathscr{P}} \left[\mathrm{Nec}(q \mid p) \circledast \mathrm{Nec}(p)\right]$$

$$\mathrm{Poss}(q) = \sup_{\mathscr{P}} \left[\mathrm{Poss}(q \mid p) \circledast \mathrm{Poss}(p)\right]$$

THEOREM 3 (GENERALIZED MODUS PONENS FOR NECESSITY FUNCTIONS) *Let $\mathscr{P}$ be a partition of $\mathscr{U}$ and let q be a proposition. If $\mathrm{Nec}(p)$ and*

Nec(q | p) are real values defined for every proposition p in the partition $\mathcal{P}$ such that

$$\mathrm{Nec}(p) \leq \mathbf{I}(p \mid \mathcal{E})$$

$$\mathrm{Nec}(q \mid p) \leq \inf_{w \vdash \mathcal{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid w)\right]$$

then the following inequality is valid:

$$\sup_{\mathcal{P}} \left[\mathrm{Nec}(q \mid p) \circledast \mathrm{Nec}(p)\right] \leq \mathbf{I}(q \mid \mathcal{E})$$

Proof Note first that since $\oslash$ is nonincreasing in its second argument and since

$$\mathbf{I}(p \mid \mathcal{E}) \leq \mathbf{I}(p \mid w)$$

for every evidential world w,

$$\mathrm{Nec}(q \mid p) \leq \inf_{w \vdash \mathcal{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid w)\right] \leq \inf_{w \vdash \mathcal{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid \mathcal{E})\right]$$

It follows then from the monotonicity and continuity of $\circledast$ with respect to its arguments that

$$\begin{aligned}
\mathrm{Nec}(p) \circledast \mathrm{Nec}(q \mid p) &\leq \mathbf{I}(p \mid \mathcal{E}) \circledast \inf_{w \vdash \mathcal{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid \mathcal{E})\right] \\
&= \inf_{w \vdash \mathcal{E}} \left\{\mathbf{I}(p \mid \mathcal{E}) \circledast \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid \mathcal{E})\right]\right\} \\
&\leq \inf_{w \vdash \mathcal{E}} \mathbf{I}(q \mid w) \\
&= \mathbf{I}(q \mid \mathcal{E})
\end{aligned}$$

since

$$\mathbf{I}(p \mid \mathcal{E}) \circledast \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid \mathcal{E})\right] \leq \mathbf{I}(q \mid w)$$

because of the definition of $\oslash$ and the continuity of $\circledast$.

Since the above inequality is valid for any proposition p in $\mathcal{P}$, Theorem 3 follows. ■

A dual result also holds for possibility functions.

THEOREM 4 (GENERALIZED MODUS PONENS FOR POSSIBILITY FUNCTIONS *Let $\mathcal{P}$ be a partition of $\mathcal{U}$ and let q be a proposition. If Poss(p) and Poss(q | p) are real values, defined for every proposition p in $\mathcal{P}$, such that*

$$\mathrm{Poss}(p) \geq \mathbf{C}(p \mid \mathcal{E})$$

$$\mathrm{Poss}(q \mid p) \geq \sup_{w \vdash \mathcal{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid w)\right]$$

then the following inequality is valid:

$$\sup_{\mathscr{P}} \left[\mathrm{Poss}(q \mid p) \circledast \mathrm{Poss}(p)\right] \geq \mathbf{C}(q \mid \mathscr{E})$$

Proof Note first that if w is an evidential world, then

$$\mathbf{C}(p \mid \mathscr{E}) \geq \mathbf{I}(p \mid w)$$

It follows then from the nonincreasing nature of $\oslash$ with respect to its second argument that

$$\begin{aligned}\mathrm{Poss}(q \mid p) &\geq \sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid w)\right] \\ &\geq \sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{C}(p \mid \mathscr{E})\right]\end{aligned}$$

and therefore that

$$\mathrm{Poss}(q \mid p) \circledast \mathrm{Poss}(p) \geq \sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{C}(p \mid \mathscr{E})\right] \circledast \mathbf{C}(p \mid \mathscr{E})$$

Taking now, in the above expression, the supremum with respect to all propositions p in $\mathscr{P}$, it is

$$\sup_{\mathscr{P}} \left[\mathrm{Poss}(q \mid p) \circledast \mathrm{Poss}(p)\right] \geq \sup_{\mathscr{P}} \left\{ \sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{C}(p \mid \mathscr{E})\right] \circledast \mathbf{C}(p \mid \mathscr{E}) \right\} \quad (1)$$

Note, however, that since $\mathscr{P}$ is a partition, there always exists a proposition $\hat{p}$ in $\mathscr{P}$ such that $\mathbf{C}(\hat{p} \mid \mathscr{E}) = 1$ (i.e., $\hat{p}$ "intersects" $\mathscr{E}$), and therefore

$$\begin{aligned}&\sup_{\mathscr{P}} \left\{ \sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{C}(p \mid \mathscr{E})\right] \circledast \mathbf{C}(p \mid \mathscr{E}) \right\} \\ &\quad \geq \sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{C}(\hat{p} \mid \mathscr{E})\right] \circledast \mathbf{C}(\hat{p} \mid \mathscr{E}) \\ &\quad = \sup_{w \vdash \mathscr{E}} \mathbf{I}(q \mid w) \\ &\quad = \mathbf{C}(q \mid \mathscr{E}) \qquad (2)\end{aligned}$$

Theorem 4 follows at once by combination of the inequalities (1) and (2). ■

Finally, notice also that, although Theorems 3 and 4 have been characterized as duals, it is not necessary that $\mathscr{P}$ be a partition for the generalized modus ponens for necessities to hold, although the proof of its possibilistic counterpart relies on such an assumption. It should be clear, however, that richer propositional collections $\mathcal{P}$ would lead to better lower bounds for values of the degree of implication $\mathbf{I}(q \mid \mathcal{E})$.

Variables

The $\circledast$-transitivity property of $\mathbf{I}$ is the essential fact expressing the relationships between the degrees of implication of the propositions that were proved in the previous section. The statements of these relations in most works devoted to fuzzy logic are made, however, using special subsets of the universe of discourse that are described through the important notion of *variable*. Introduction of this concept, which is also central to other approximate reasoning methodologies, permits us to make a clearer distinction between similarities defined, in some absolute sense, from the several viewpoints and related proximity measures that compare objects (in our case, possible worlds) from the marginal viewpoint of one or more variables.

In what follows, we will assume that only certain propositions, specifying the value of a system variable belonging to a finite set

$$\mathcal{V} = \{X, Y, Z, \ldots\}$$

will be used to characterize possible worlds.

The propositions of interest are those formed by logical combination of statements of the type

"The value of the variable V is v."

where V is in the variable set $\mathcal{V}$ and v is a specific value in the domain $\mathcal{D}(V)$ of the variable V.

We will also assume that, in any possible world, the value of any variable is a member of the corresponding domain of definition of the variable. In the context of our discussion, we will not need to make special assumptions about the scalar or numeric nature of the state variables, using the notion in the same primitive and general sense in which it is customarily used in predicate calculus.

We will be specially interested in subsets, called *variable sets*, of the universe $\mathcal{U}$ consisting of worlds where the value of some variable V is equal to a specified value v. We will denote by $[X = x]$ (similarly $[Y = y]$, etc.) the set of all possible worlds where the proposition "The value of the variable X is x" is true. Clearly, the variable-sets in the collection

$$\{[X = x] : x \text{ is in } \mathcal{D}(X)\}$$

partition the universe into disjoint subsets. These collections have been used to characterize the concept of *rough sets* (Pavlak [30]), of importance in many information system analysis problems, including some that arise in the context of approximate reasoning. A similar notion has been used also to describe algorithms for the combination of probabilities and of belief functions (Shafer et al. [39]).

To simplify the notation we will write

$$w \vdash x, \quad w \vdash y, \quad \ldots$$

as shorthand for $w \vdash [X = x]$, $w \vdash [Y = y], \ldots,$ respectively.

POSSIBILISTIC STRUCTURES AND LAWS The usual statements of the laws of fuzzy logic are made, as mentioned before, through the use of variables rather than by means of general propositional expressions. It is customary, for example, to speak of the possibility of the variable X taking the value x to describe the value that a possibility function for an evidential set $\mathscr{E}$ attains for the proposition $[X = x]$.

In our model, we will therefore say that a function

$$\mathrm{Poss}(\cdot) : \mathscr{D}(X) \mapsto [0, 1]$$

is a possibility function for the evidential set $\mathscr{E}$ and the variable X whenever

$$\mathrm{Poss}(x) \geq \mathbf{C}([X = x] \mid \mathscr{E})$$

for all values x in the domain $\mathscr{D}(X)$. Similarly, we will say that $\mathrm{Nec}(\cdot)$ is a necessity function for X whenever

$$\mathrm{Nec}(x) \leq \mathbf{I}([X = x] \mid \mathscr{E})$$

for all values x in $\mathscr{D}(X)$.

If possibility distributions are defined in this way as point functions in the variable domain $\mathscr{D}(X)$, then it is possible to use the disjunctive laws of fuzzy logic proved in the section Properties of Possibility and Necessity Functions to extend their definition over the power set of $\mathscr{D}(X)$, that is,

$$\mathrm{Nec}(A \cup B) = \max\left[\mathrm{Nec}(A), \mathrm{Nec}(B)\right]$$

$$\mathrm{Poss}(A \cup B) = \max\left[\mathrm{Poss}(A), \mathrm{Poss}(B)\right]$$

where A and B are subsets of the domain $\mathscr{D}(X)$. These equations are usually given as the basic disjunctive laws of possibility distributions.

Note that, using such extensions, both possibility and necessity functions are nondecreasing functions (with respect to the order induced by set inclusion). The value of $\mathrm{Nec}(A)$ measures the extent to which the evidence supports the statement that the variable value necessarily lies in the subset A of its domain of definition, with a dual interpretation being applicable for possibility distributions.

MARGINAL AND JOINT POSSIBILITIES The original similarity relation introduced earlier may be considered to be a measure of proximity between possible worlds from the joint viewpoint of all system variables. The notion of variable, however, permits the definition of similarities from the restricted viewpoint of some variables or subsets of variables.

These restricted perspectives play a role with respect to the original similarity S that is analogous to that of marginal probability distributions with respect to joint probability distributions. To derive useful expressions that describe similarities between two values x and x' of the same variable X, it should be noted first that the degree of implication $\mathbf{I}(\cdot \mid \cdot)$ is transitive. This fact permits the application of a theorem of Valverde [45] to define a function S_X by means of the expression

$$S_X : \mathscr{D}(X) \times \mathscr{D}(X) \mapsto [0,1] : (x, x') \mapsto \min[\mathbf{I}(x \mid x'), \mathbf{I}(x' \mid x)]$$

Defined in this way as a "symmetrization" of the *preorder* induced by the degree of implication $\mathbf{I}(\cdot \mid \cdot)$, the marginal similarity S_X has the properties of a similarity function. Furthermore, the "projection" operation entailed by the use of $\mathbf{I}(x \mid x')$, based on the projection of every x'-world into the set of x-worlds, may be considered to be the basic mechanism to transform the original similarity function into one that discerns differences only in the values of the variable X.

It must be noted, however, that unless additional assumptions are made about the nature of the original similarity S, the function S_X fails to satisfy the intuitive requirement

$$S(w, w') \le S_X(w, w')$$

whenever $w \vdash x$ and $w' \vdash x'$, that is, the similarity between two objects from a restricted viewpoint is always higher than their similarity from more general viewpoints that encompass additional criteria of comparison.

Although considerable research remains to be done to identify alternative definitions of marginal similarities that are not hampered by this problem, a basic result of Valverde [45] presented later in this paper, appears to provide the essential tool that must be employed to produce the required coarser measures. Additional reasonable assumptions that might be demanded from S to facilitate the construction of marginal similarities with desirable characteristics are also an object of current investigation.

CONDITIONAL DISTRIBUTIONS AND GENERALIZED INFERENCE The basic conditional structures of fuzzy logic are usually defined as elastic constraints that restrict the values of one variable given those of another. By simple extension of our previous convention to conditional structures, we will write $\mathrm{Nec}(y \mid x)$ and $\mathrm{Poss}(y \mid x)$ as shorthand for

$$\mathrm{Nec}([Y = y] \mid [X = x]) \qquad \text{and} \qquad \mathrm{Poss}([Y = y] \mid [X = x])$$

respectively.

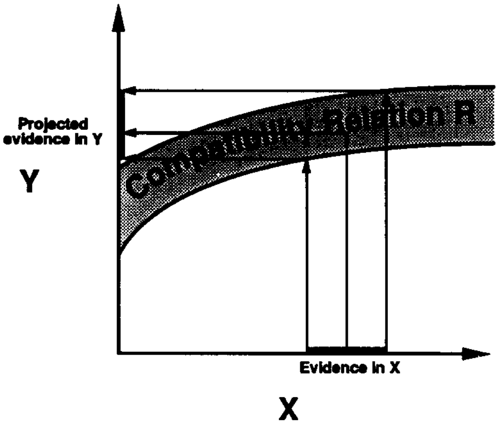

Figure 5. Inference as a compatibility relation.

If a classical (i.e., Boolean) inferential rule of the type

$$\text{If } X = x, \text{ then } Y \text{ is in } R(x).$$

is thought of as the definition of a relation R defined over pairs (x, y) in the Cartesian product $X \times Y$, then such a relation may be used to define a multivalued mapping that maps possible values of X into possible values of Y as illustrated in Figure 5.

Such a *compatibility relation* perspective is an essential element of the original formulations of both the Dempster–Shafer calculus of evidence (Dempster [8]), where distributions in some space (i.e., the domain of some variable X) are mapped into distributions of another variable (i.e., the domain of another variable Y) by direct transfer of "mass" from individual values to their mapped projections, and of the compositional rule of inference (Zadeh [51]).

Note that whenever $\operatorname{Poss}(y \mid x) = 1$, if the bound is actually attained, that is, if

$$\sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(y \mid w) \oslash \mathbf{I}(x \mid w)\right] = 1$$

then it is possible for an evidential world w in $[X = x]$ [i.e., $\mathbf{I}(x \mid w) = 1$] to be such that $w \vdash y$. Pairs (x, y) such that $\operatorname{Poss}(y \mid x) = 1$ may be considered to approximate the *core*[9] of a generalized inferential relation that allows us to determine bounds for the similarity between evidential worlds and those in the variable set $[Y = y]$ on the basis of knowledge of similar bounds applicable to the variable set $[X = x]$. This relation, which is the fuzzy extension of the classical compatibility mapping R illustrated in Figure 5, may be thought of as a descriptor of the behavior, for x-worlds, of the values of the variable Y

[9]The core of a fuzzy set μ: $\mathscr{U} \mapsto [0, 1]$ is the set of all points w such that $\mu(w) = 1$, that is, the points that "fully" belong to μ.

"near" R. The compatibility relation is itself approximated by (or embedded in) the core of the conditional possibility distribution, that is, worlds w such that $w \vdash x$ and $w \vdash y$, and such that $\mathrm{Poss}(y \mid x) = 1$.

Since the collection of the sets $[X = x]$ partitions the universe $\mathscr{U}$ into disjoint sets, then the generalized modus ponens laws can be readily stated in terms of variable values as

$$\mathrm{Nec}(y) = \sup_x \left[\mathrm{Nec}(y \mid x) \circledast \mathrm{Nec}(x)\right]$$

$$\mathrm{Poss}(y) = \sup_x \left[\mathrm{Poss}(y \mid x) \circledast \mathrm{Poss}(x)\right]$$

which clearly shows the basic nature of inferential mapping as the composition of relational combination (i.e., $\circledast$-"intersection") and projection (i.e., maximization).

FUZZY IMPLICATION RULES We will now examine proposed interpretations for conditional rules, usually stated in the form

If X is A, then Y is B.

within the context of possibilistic logic. Whereas in two-valued logic any such rule simply states that whenever a condition A is true, another condition B also holds, various interpretations have been proposed for rules expressing other notions of conditional truth.

In the case of probabilities, for example, degrees of conditionality have been modeled either by means of conditional probability values $\mathrm{Prob}(A \mid B)$, which measure the likelihood of B given the assumed truth of A, or by the alternative interpretation $\mathrm{Prob}(\neg A \vee B)$, used by Nilsson [29] in his probabilistic logic, which essentially quantifies the probability that a rule is a valid component of a knowledge base. Either one of these interpretations is valid in particular contexts being, respectively, the probabilistic extensions of the so-called *de re*, that is,

$$p \to \Pi q$$

and *de dicto*, that is,

$$\Pi(p \to q)$$

interpretations of conditionals in modal logic.

In fuzzy logic, two major interpretations have been advanced to translate conditional rules,[10] with A and B corresponding to the fuzzy sets

$$\mu_A : X \mapsto [0, 1] \qquad \text{and} \qquad \mu_B : Y \mapsto [0, 1]$$

[10]A rather encompassing account of potential fuzzy reasoning mechanisms may be found in a paper by Mizumoto et al. [27].

The first interpretation was originally proposed by Zadeh [52], as a formal translation of the statement

If μ_A is a possibility for X, then μ_B is a possibility distribution for Y.

This conditional statement, which may be regarded as a constraint on the values of one variable given those of another, states the existence of a conditional possibility function Poss($\cdot \mid \cdot$) such that

$$\mu_B(y) \geq \sup_x \left[\operatorname{Poss}(y \mid x) \circledast \mu_A(x)\right] \geq \operatorname{Poss}(y \mid x) \circledast \mu_A(x)$$

Recalling now the definition and properties of the pseudoinverse, we may restate this particular interpretation as

$$\operatorname{Poss}(y \mid x) = \mu_B(y) \oslash \mu_A(x) \geq \mathbf{I}(y \mid w) \oslash \mathbf{I}(x \mid w)$$

for every world $w \vdash \mathscr{E}$.

In Zadeh's original formulation, made within the context of a calculus based on the minimum function as the T-norm, conditionals were, however, formally translated by means of the pseudoinverse of the Lukasiewicz T-norm. Certain formal problems associated with such a combination were pointed out by Trillas and Valverde [42], who developed translations consistent with the T-norm used as the basis for the possibilistic calculus.

Using the characterization of conditionals introduced earlier, this relation may also be thought of as a measure of the degree to which a possibility for Y exceeds a fraction (measured by the conditional possibility distribution) of a given possibility distribution for X. In particular, whenever $\operatorname{Poss}(y \mid x) = 1$, then $\mu_B(y) \geq \mu_A(x)$, indicating the *possible* existence, since $\operatorname{Poss}(y \mid x)$ is only an upper bound of $\mathbf{I}(y \mid w) \oslash \mathbf{I}(x \mid w)$, of an evidential world such that $w \vdash x$ and $w \vdash y$, with x in A and y in B.

As illustrated in Figure 6, where it has been assumed that the underlying metric (i.e., dissimilarity) is proportional to the Euclidean distance in the plane, the core of the corresponding conditional possibility distribution is an (upper) approximant of a classical compatibility relation (indicated by the shaded area in the figure) that fans outward from the Cartesian product of the cores of A and B. If this interpretation is taken whenever several such rules are available, then each one of these rules will lead to a separate possibility distribution. Combination of these upper bounds by minimization results in a sharper possibility estimate that represents the "integrated" effect of the rule set.

The second interpretation of conditional relations, leading to a wide variety of practical applications (Sugeno [41]), was utilized by Mamdani and Assilian [26] to develop fuzzy controllers. The basic idea underlying this explanation follows an approach originally outlined by Zadeh [47, 48, 49, 50, 51]. In this

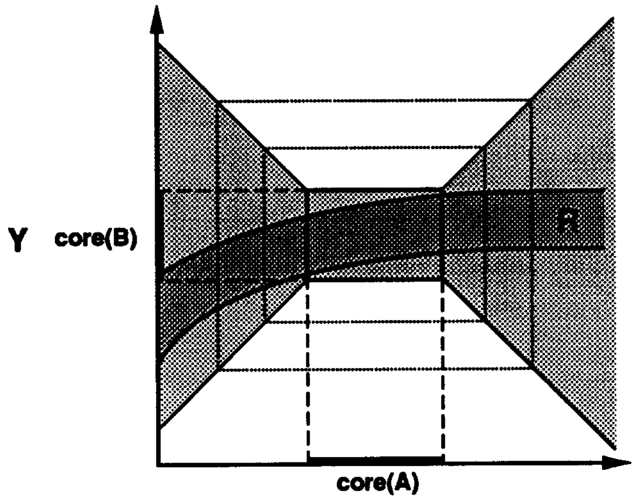

Figure 6. Rules as possibilistic approximants of a compatibility relation.

case, a number of conditional statements of the form

$$\text{If } X \text{ is } A_k, \text{ then } Y \text{ is } B_k, \qquad k = 1, 2, \ldots, n$$

are given as a combined "disjunctive" description of the relation between X and Y, rather than as a set of independently valid rules. The purpose of this rule set is the approximation of the compatibility relation by a "fuzzy curve" generated by disjunction of all the rules in the set, as shown in Figure 7.

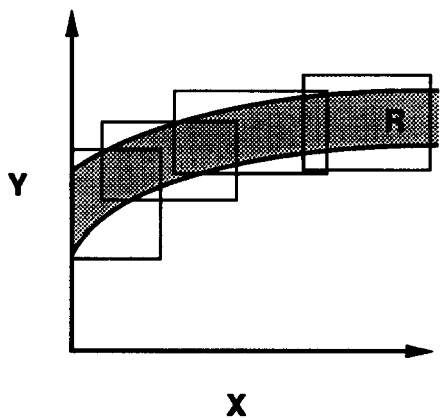

Figure 7. Rule sets as disjunct approximants of a compatibility relation.

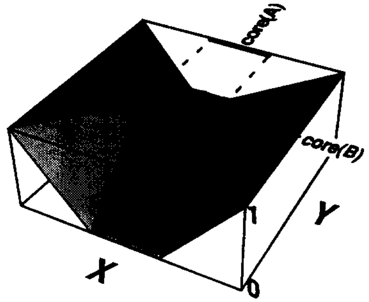

Figure 8. A possibilistic conditional rule (ZTV).

Recalling the characterization of conditioning as an extension of a classical compatibility relation, we may say that the core of the compatibility relation is approximated by above by the union

$$\bigcup_{k=1}^{n} \left[\operatorname{core}(\mu_{A_k}) \times \operatorname{core}(\mu_{B_k})\right]$$

of the Cartesian products of the cores of the fuzzy sets for A_k and B_k. In this case the multiple rules are meant to approximate some region of possible (X, Y) values, and the results of application of individual component rules must be combined using maximization to produce a conditional possibility function. We may say, therefore, that under the Zadeh–Mamdani–Assilian (ZMA) interpretation, the function

$$\operatorname{Poss}(y \mid x) = \sup_{k} \left\{\min\left[\mu_A(x), \mu_B(y)\right]\right\}$$

is a conditional possibility for Y given X.

It is important to note that the two interpretations of fuzzy rules that we have just examined are based on different approaches to the approximation (by above) of the value

$$\sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(y \mid w) \oslash \mathbf{I}(x \mid w)\right]$$

being, in the case of the Zadeh–Trillas–Valverde (ZTV) method, the result of the *conjunction* of multiple fuzzy relations such as that illustrated in Figure 8, while in the case of the ZMA logic the construction requires *disjunction* of relations such as that illustrated in Figure 9.

The difference between the two approaches when combining several rules is illustrated also in Figures 10 and 11, showing the contour plots for the α-cuts of the fuzzy relations that are obtained in a simple example involving four rules. In these figures, the rectangles with a dark outline correspond to the Cartesian products of the cores of the antecedents A_k and B_k. Darker shades of gray correspond to higher degrees of membership.

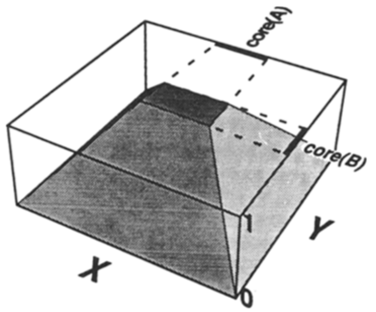

Figure 9. A component of a disjunctive rule set (ZMA).

The reader should be cautioned, however, about the potential for invalid comparisons that may result from hasty examination of these figures. Each formalism should be regarded as a procedure for the approximation of a compatibility relation that is based on a different approach for the description of relationships between variables. In the case of the ZMA interpretation, the intent is to generalize the interpolation procedures that are normally employed in functional approximation. As such, this approach may be said to be inspired by the methodology of classical system analysis. The ZTV approach, by contrast, is a generalization of classical logical formulations and may be regarded, from a relational viewpoint, as a procedure to describe a function as the locus of points that satisfies a set of constraints rather than as a subset of "fuzzy points" of a Cartesian product.

Figures 10 and 11, while showing that the same rule sets would lead to radically different results, should not be considered, therefore, to discredit interpolative approaches, as such techniques, proceeding from a different

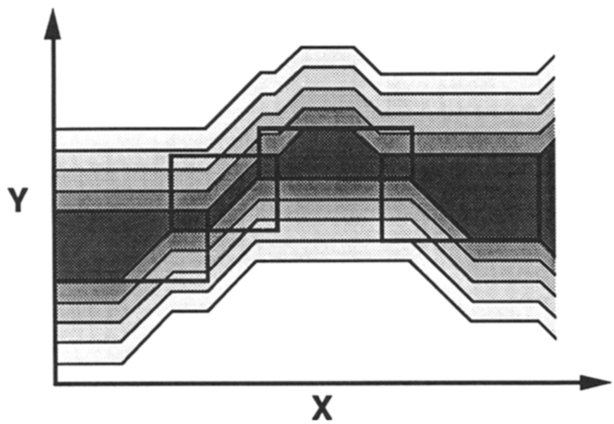

Figure 10. Contour plots for a rule set (ZTV).

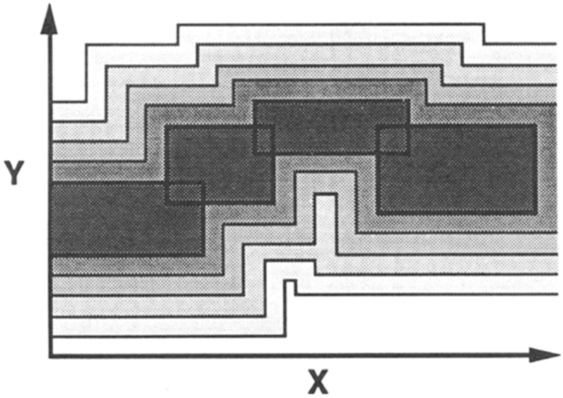

Figure 11. Contour plots for a rule set (ZMA).

perspective, should normally be based on rule sets that are different from those used when rules are thought of as independent constraints.

THE NATURE OF SIMILARITY RELATIONS

In this closing section, we will examine issues that arise naturally from our previous examination of the role of similarities as the semantic basis for possibility theory.

Our discussion focuses on two topics. We look first at the requirements that our theory imposes upon the nature of the scales used to measure proximity or resemblance between possible worlds. Finally, our examination of the interplay between similarities and possibilities turns to issues related to the generation of similarity relations from such sources as domain knowledge that describe significant relations between system variables.

On Similarity Scales

Our previous interpretation of possibilistic concepts and structures was based on the use of measures of proximity that quantify interobject resemblance using real numbers between 0 and 1. Our assumptions about the use of the [0, 1] interval as a similarity scale have been made primarily, however, as a matter of convenience to simplify the description of our model while being consistent with the customary definitions of possibility and necessity distributions as functions taking values in that interval.

Close examination of the actual requirements imposed upon our similarity scales reveals, however, that our measurement domain may be quite general so as to include symbolic structures such as

$$\{\textit{identical}, \textit{very similar}, \ldots, \textit{completely dissimilar}\}$$

Our model is based on the use of a partially ordered set having a maximal and a minimal element representing identity and complete dissimilarity, respectively. Furthermore, we have assumed the existence of a binary operation (the triangular norm $\circledast$) mapping pairs of possible worlds into real numbers, with certain desirable order-preserving and transitive properties. The concept of triangular norm, however, does not rely substantially on the use of real numbers as its range and may be readily extended to more general partially ordered sets with maximal and minimal elements.

We have also assumed a continuity property for the triangular norm operation. This property, however, simply requires that a notion of proximity also exist among similarity values so as to provide a form of (order-consistent) topology in that space. While, in general, more precise scales will result in more detailed representations of interworld similarity, it is important to stress that the similarity-based model presented here does not rely on "density" assumptions such as the existence of an intermediate value c between any different values a and b in the similarity-measurement scale.

From a practical viewpoint, the major requirement is to quantify proximity in such a way as to be able to determine that two quantities are similar to some degree (i.e., approximate matching). The degree of precision that such a matching entails is problem-dependent and will typically be the result of conflicting impositions between the desire, on the one hand, to keep granularity relatively low to reduce complexity, and the need, on the other, to describe system behavior at an acceptable level of accuracy. The work of Bonissone and Decker [4] is a significant example of the type of systematic study that must be carried out to define similarity scales that are both useful and tractable.

The Origin of Similarity Functions

The model of fuzzy logic presented in this paper is centered on the metric notion of similarity as a primitive concept that is useful in explaining the nature of possibilistic constructs and the meaning of possibilistic reasoning. In this formulation, similarities are defined as real functions defined over pairs of possible worlds.

From this perspective, similarities describe relations of resemblance between objects of high complexity, which, typically, result from consideration of a large number of system variables. Reliance on such complex structures has been the direct consequence of a research program that stressed conceptual clarification as its primary objective. In practice, however, it will be generally difficult to define complex measures that quantify similarity between complex objects on the basis of a large number of criteria.

Similarities provide the framework that is required to understand approximate relations of corelevance, usually stated as generalized conditional rules. The practical generation of similarity functions typically proceeds, however, in the opposite direction, from separate statements about limited aspects of system behavior to general metric structures. Once such resemblance measures are defined, they may be used to express and acquire new laws of system behavior determined, for example, from historical experience with similar systems. Furthermore, such similarity notions may be used as the basis for analogical reasoning systems that try to determine the system's state on the basis of similarity to known cases (Kolodner [23]).

Perhaps the simplest mechanism that may be devised to generate complex metrics from simpler ones is that which starts with measures of resemblance that quantify proximity from a limited viewpoint. These metrics are usually derived, using a variety of techniques, in unsupervised pattern classification (or clustering) problems (Hartigan [20]). In many important applications, hierarchical taxonomies—a feature of many representation approaches in artificial intelligence—may be used, often in connection with a variety of weighing schemes, quantifying branching importance, to generate metrics that often satisfy the more stringent requirements of an ultrametric (Jardine and Sibson [22]).

Classification hierarchies such as those may be though of as sets of general rules, having a particularly useful structure, that specify interest proximity from relevant, but restricted, viewpoints, eventually providing measures of similarity between variable values (i.e., the "leaves" of the taxonomic tree). More generally, however, we may expect that sets of possibilistic rules (i.e., a general knowledge base) defining a general semantic network of corelevance relations may be available as the source for the determination of interobject proximity. These possibilistic semantic networks resemble conventional semantic networks in most regards, being more general in that, in addition to specifying knowledge about system behavior in some subsets of state-space,[11] they also specify characteristics of behavior in neighborhoods of those subsets.

We may think, therefore, that the antecedents of implicational rules define general regions in state-space where existence of relevant knowledge may increase insight through application of inferential rules. Using Zadeh's terminology, these antecedents define "granules" that identify important regions of state-space and indicate the level of accuracy (or *granularity*) that is required to perform effective system analysis. In this case, the possibilistic granules correspond to fuzzy sets that are used to specify both what is true in the core of the granule and, with decreasing specificity, what is true in a nested set (i.e., the α-cuts) of its neighborhoods. The ability to specify behavior using such a topological structure results in inferential gains that are the direct consequence of our ability to reason by similarity—an ability that is made possible by the approximate matching property of the generalized modus ponens. From an-

[11]The expression "state-space" is loosely used here to indicate the space defined by all system variables.

other perspective yet, the fuzzy granules identified by possibilistic rules may also be thought of as generalizations of the arbitrary variable sets used in a variety of artificial intelligence efforts aimed at understanding system behavior using qualitative descriptions of reality (Forbus [16]).

A number of heuristics may be easily formulated to integrate "marginal" measures of resemblance into joint similarity relations. More generally, however, we may state the problem of similarity construction as that of defining metric structures on the basis of knowledge of the aspects of system behavior that are important to its understanding—the previously mentioned granules, which define what must be distinguished. Since generally those granules are fuzzy sets, the relevance to similarity construction of the following representation theorem, due to Valverde, may be immediately seen.

THEOREM 5 (VALVERDE)) *A binary function S mapping pairs of objects of a universe of discourse $\mathscr{U}$ into* [0, 1] *is a similarity relation if and only if there exists a family $\mathscr{H}$ of fuzzy subsets of $\mathscr{U}$ such that*

$$S(w, w') = \inf_{\mathscr{H}} \{\min[h(w) \oslash h(w'), h(w') \oslash h(w)]\}$$

for all w and w′ in $\mathscr{U}$, where the infimum is taken over all fuzzy subsets h in the family $\mathscr{H}$.

Besides its obvious relevance to the generation of similarity relations from knowledge of important sets in the domain of discourse, Valverde's theorem—resulting originally from studies in pattern recognition—is also of potential significance to the solution of knowledge acquisition problems because of the important relations that exist between learning procedures and structure-discovery techniques such as cluster analysis.

CONCLUSION

This paper has presented a similarity-based model that provides a clear interpretation of the major structures and methods of possibilistic logic using metric concepts that are formally different from the set-measure constructs of probability theory. Regardless of the potential existence, so far unestablished, of probability-based interpretations for possibilistic structures, this metric model makes clear that there are no compelling reasons to confuse two rather different aspects of uncertainty into a single notion simply because one's favorite theoretical framework, in spite of its otherwise many remarkable virtues, fails to fully capture reality.

Succinctly stated, being in a situation that resembles a state of affairs S does not make S likely or vice versa. Furthermore, our reference state may not even be possible in the current circumstances, which would make it completely unlikely, but we may still find it useful as a comparison landmark. This use of "impossible" examples as a way to illustrate system behavior is very prevalent in human culture, being exemplified by such utterances as "he had the strength of a horse and the swiftness of a swallow," even if it is obvious to all that no such beast exists other than for such metaphorical purposes.

The insight provided by this model makes it rather obvious that very little can be gained by continuing to assert a potential—although never revealed—encompassing probabilistic interpretation for possibilistic structures that, presumably, would render them unnecessary as serious objects of scientific discourse. In addition, and quite beyond whatever understanding theory may provide, the current success of possibilistic logic as the basis for major systems of important human value (Sugeno [41]), often unmatched by other approaches, should be enough to convince those having more pragmatic perspectives as to its utility.

The task for approximate reasoning researchers is to proceed now beyond unnecessary controversy into the study of the issues that arise from models such as the one presented in this paper. Among such questions, further studies of the relations between the notions of possibility, similarity, and negation and of those between probability and possibility are of major importance.

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