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On the simultaneous edge-coloring conjecture

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Abstract

At the 16th British Combinatorial Conference (1997), Cameron introduced a new concept called 2-simultaneous edge-coloring and conjectured that every bipartite graphic sequence, with all degrees at least 2, has a 2-simultaneous edge-colorable realization. In fact, this conjecture is a reformulation of a conjecture of Keedwell (Graph Theory, Combinatorics, Algorithms and Applications, Proceedings of Third China–USA International Conference, Beijing, June 1–5, 1993, World Scientific Publ. Co., Singapore, 1994, pp. 111–124) on the existence of critical partial latin squares (CPLS) of a given type. In this paper, using some classical results about nowhere-zero 4-flows and oriented cycle double covers, we prove that this conjecture is true for all bipartite graphic sequences with all degrees at least 4.

1. Introduction

In this paper we consider finite simple graphs. For notations not defined here we refer to [1]. Our results are based on some classical results about cycle double covers and nowhere-zero integer flows. In this section we discuss some necessary preliminaries about these concepts.

A **cycle double cover** (CDC) $\mathscr{C}$ of a graph G is a collection of cycles in G such that every edge of G belongs to exactly two cycles of $\mathscr{C}$. It can be easily seen that a necessary condition for a graph to have a CDC is that the graph be bridgeless. Seymour [8] in 1979 conjectured that this condition is also sufficient.

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Conjecture (*CDC conjecture, Seymour* [8]). Every bridgeless graph has a cycle double cover.

The idea of this conjecture comes from the fact that the set of faces of a planar graph (including the infinite face) is a set of cycles that cover every edge exactly twice. The CDC conjecture has some stronger forms one of them is the following which can be found in [13].

Conjecture (*Oriented CDC conjecture*). Every bridgeless graph has a cycle double cover in which every cycle can be oriented in such a way that every edge of the graph is covered by exactly two directed cycle in two different directions.

There are many relations between the concept of cycle double cover and other concepts in graph theory. Perhaps the most related concept is the concept of an integer flow. A **nowhere-zero integer k-flow** (or simply a **k-flow**) of a graph is an assignment to each edge a direction and a weight from $\{1,2,\ldots,k\}$, such that for each vertex v the sum of the weights on the edges directed into v equals the sum of the weights on the edges directed away from v. The concept of an integer flow was introduced by Tutte [11] as a refinement and generalization of the face coloring problem of planar graphs. He proposed the following conjecture as a generalization of the five color theorem for planar graphs.

Conjecture (*5-flow conjecture, Tutte* [12]). Every bridgeless graph admits a 5-flow.

The best result in direction of Tutte's conjecture is obtained by Seymour [9]. He proved that every bridgeless graph has a 6-flow. Also the following theorem, due to Jaeger, shows that stronger assumptions about connectivity imply the existence of stronger integer flows.

Theorem A (Jaeger [4]). *Every 4-edge connected graph admits a 4-flow.*

There are many results about the relation between integer flows and cycle double covers in [13]. The following theorem which follows from the results in [13] plays a key role in this paper.

Theorem B. *If a graph admits a 4-flow then it has an orientable cycle double cover.*

In the next section we state a conjecture related to latin squares and study the relation between this conjecture and the cycle double cover conjecture.

2. The SE conjecture

Recently Cameron [2] stated a conjecture called simultaneous edge-coloring (SE) conjecture which is in fact a reformulation of a conjecture by Keedwell [5] on the existence of critical partial latin squares (CPLS) of a given type. Before stating the conjecture, we need to define the concept of a 2-simultaneous edge-coloring.

Definition. Let G be a graph. A 2-*simultaneous edge-coloring* of G is a pair of proper edge-colorings of G such that

- for each vertex, the sets of colors appearing on the edges incident to that vertex are the same in both colorings;
- no edge receives the same color in both colorings.

If G has a 2-simultaneous edge-coloring, then G is called a 2-simultaneous edge-colorable graph.

In fact, 2-simultaneous edge-colorable graphs for edge-coloring of graphs play a role similar to the role of trades in block designs [10]. Therefore, this concept has applications in the study of the defining sets of graph colorings and uniquely colorable graphs. For a survey on these concepts see [7].

Let G be a bipartite graph with bipartition (X, Y). The bipartite degree sequence of G is the sequence $(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m)$, where $(x_1, x_2, \ldots, x_n)$ is the degree sequence of the vertices in X and $(y_1, y_2, \ldots, y_m)$ is the degree sequence of the vertices in Y. A sequence $S = (x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m)$ is called a bipartite graphic sequence if there exists a bipartite graph G whose bipartite degree sequence is S. It is well known that a necessary and sufficient condition for a sequence $S = (x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m)$, where x_i's and y_j's are two nonegative and nondecreasing sequences to be bipartite graphic, is given by the *Gale–Ryser Theorem*

$$\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_i,$$

$$\sum_{i=1}^{k} x_i \leqslant \sum_{j=1}^{m} \min\{k, y_j\} \quad \text{for } 1 \leqslant k \leqslant n.$$

Conjecture. *SE Conjecture, Cameron* [2]. Every bipartite graphic sequence, with all degrees at least 2, has a 2-simultaneous edge-colorable realization.

Cameron [2] noted that it is not true that any bipartite graph G with $\delta(G) > 1$ is 2-simultaneous edge-colorable. His example is a graph which consists of two 4-cycles with an extra edge joining them together. It is easy to verify that this graph is not 2-simultaneous edge-colorable.

In [6] it is proved that every bipartite graph which has a cut edge does not have a 2-simultaneous edge-coloring. And it is conjectured that the converse is true.

Conjecture (*Strong SE Conjecture, Mahdian et al.* [6]). Every bridgeless bipartite graph has a 2-simultaneous edge-coloring.

It is proved there that the conjecture above is equivalent to the oriented CDC conjecture. In particular, the following theorem follows from their results.

Theorem C (Mahdian et al. [6]). *Every bipartite graph has an oriented cycle double cover if and only if it is* 2*-simultaneous edge-colorable.*

In the next section we use this relation between oriented CDC conjecture and SE conjecture to obtain some results about SE conjecture.

3. The results

In this section we deal with some problems on connectivity and for this we need the following terminology and notations. Let G be a graph and S be a nonempty proper subset of $V(G)$. The notation $[S,\bar{S}]$, called an edge cut (or simply a cut), specifies the set of edges having one endpoint in S and the other in $\bar{S}$. The value of a cut, denoted by $|[S,\bar{S}]|$, is the number of edges in $[S,\bar{S}]$. In [3], Edmonds proved that every graphic degree sequence, with all degrees at least $k\geqslant 2$, has a k–edge connected realization. Here we prove a similar theorem about bipartite graphs.

Theorem 1. *Every bipartite graphic sequence D, with all degrees at least $2k$ $(k\geqslant 1)$, has a $2k$-edge connected realization.*

Proof. For each bipartite realization G of D define $\mathscr{S}_G=\{[S,\bar{S}]\colon |[S,\bar{S}]|\leqslant 2k-1\}$. Let $d=\min\{|\mathscr{S}_G|\colon G$ is a bipartite realization of $D\}$. If $d=0$, then there is nothing to prove. Suppose $d\geqslant 1$ and H is a bipartite realization of D such that $|\mathscr{S}_H|=d$ and $\sum_{[S,\bar{S}]\in\mathscr{S}_H}|[S,\bar{S}]|$ is a maximum. We show a contradiction.

Suppose that $[X,\bar{X}]\in\mathscr{S}_H$. Let $A\subset X$ and $B\subset\bar{X}$ be subsets, each with a minimum size, such that $[A,\bar{A}]\in\mathscr{S}_H$ and $[B,\bar{B}]\in\mathscr{S}_H$. Obviously, A and B are disjoint sets.

There are at most $2k-1$ vertices in A adjacent to the vertices of $\bar{A}$, and since the degree of each vertex is at least $2k$, every vertex in A has a neighbor $a_1\in A$ which is not adjacent to any vertex of $\bar{A}$. Therefore, we can find two adjacent vertices a_1 and a_2 in A (b_1 and b_2 in B, respectively) which are not adjacent to the vertices of $\bar{A}$ ($\bar{B}$, respectively). Without loss of generality, suppose that a_1 and b_1 are in the same part of H.

By removing the edges a_1a_2 and b_1b_2 and inserting edges a_1b_2 and a_2b_1, we construct a new bipartite graph H' with the same degree sequence, in which $|[A,\bar{A}]|$ and $|[B,\bar{B}]|$ are increased. Now extremal conditions on H imply that there exists a cut $[C,\bar{C}]$ in

H' such that $|[C,\bar{C}]|\leqslant 2k-1$ and $|[C,\bar{C}]|$ is decreased. Clearly, in every cut $[C,\bar{C}]$ whose value is decreased in H', the vertices a_1 and b_2 together are in either C or in $\bar{C}$, and a_2 and b_1 are in $\bar{C}$ or C. Assume that a_1 and b_2 are in C. Consider nonempty and mutually disjoint sets $A\cap C$, $A\cap\bar{C}$, $B\cap C$ and $B\cap\bar{C}$. Since $|[C,\bar{C}]|\leqslant 2k-1$, there exists at most $k-1$ edges from $A\cap C$ to $A\cap\bar{C}$ or from $B\cap C$ to $B\cap\bar{C}$. By symmetry, assume that the former is the case. By counting a_1a_2, there exists at most k edges from $A\cap C$ to $A\cap\bar{C}$. On the other hand, $|[A,\bar{A}]|\leqslant 2k-1$ and without loss of generality we can assume that the number of edges from $A\cap C$ to $\bar{A}$ is at most $k-1$. Thus, we have $|[A\cap C,\overline{A\cap C}]|\leqslant k-1+k=2k-1$. Obviously, $A\cap C$ is a proper subset of A and this contradicts the choice of A. Thus $\mathscr{S}_H$ is empty and this completes the proof. □

By the theorem above for every bipartite graphic sequence, with all degrees at least 4, we can find a 4-edge connected realization, say G. Theorem A and Theorem B imply that G has an oriented cycle double cover and by Theorem C, G is 2-simultaneous edge-colorable. Thus we have proved the following theorem.

Theorem 2. *Every bipartite graphic sequence, with all degrees at least* 4, *has a* 2-*simultaneous edge-colorable realization.*

In fact, theorem above shows that the SE conjecture is true except possibly when some of the degrees are 2 or 3. The following result which was stated in Cameron's homepage, shows that the conjecture is true if all degrees are 2 or 3.

Theorem 3. *Every sequence* $D=(x_1,x_2,\ldots,x_m;\,y_1,y_2,\ldots,y_n)$, *where each component is equal to* 2 *or* 3, $\min\{m,n\}\geqslant 3$, *and* $\sum x_i=\sum y_i$, *has a* 2-*simultaneous edge-colorable bipartite realization.*

Proof. We proceed by induction. The result for $\min\{m,n\}\leqslant 5$ can be easily verified by checking all cases. Thus assume that $\min\{m,n\}\geqslant 6$. There are at least three equal elements in each part. Therefore, one of the patterns $(2,2;2,2),(3,3;2,2,2)$, $(2,2,2;3,3)$, or $(3,3,3;3,3,3)$ can be found in D. By deleting this pattern from D, we obtain a degree sequence D' which again satisfies the conditions of the statement. Thus by induction hypothesis, D' has a 2-simultaneous edge-colorable bipartite realization. And since each of these patterns are also 2-simultaneous edge-colorable, so the union of these two realizations is a 2-simultaneous edge-colorable graph with degree sequence D as desired. □

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