Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Journal of Computational and Applied Mathematics 202 (2007) 122–132
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

The generalized Weierstrass system inducing surfaces of constant and nonconstant mean curvature in Euclidean three space

Paul Bracken

Department of Mathematics, University of Texas, Edinburg, TX, 78541-2999, USA

Received 3 March 2005; received in revised form 22 June 2005

Paper based on talk delivered at Joint Mathematics Meetings, Atlanta, GA, January 8, 2005

Abstract

The generalized Weierstrass (GW) system is introduced and its correspondence with the associated two-dimensional nonlinear sigma model is reviewed. The method of symmetry reduction is systematically applied to derive several classes of invariant solutions for the GW system. The solutions can be used to induce constant mean curvature surfaces in Euclidean three space. Some properties of the system for the case of nonconstant mean curvature are introduced as well.

1. Introduction

The dynamics of surfaces is an important part of many interesting phenomena in mathematics and especially in physics. Such subjects as waves, deformation of membranes, dynamics of vortex sheets, two-dimensional gravity and string theory to mention a few have areas of common interest that overlap with the theory of surfaces. Such dynamics can be modeled by nonlinear partial differential equations (PDEs) that describe the evolution of surfaces in time [12].

One of the classical problems of differential geometry is the study of the connection between the geometry of submanifolds and nonlinear PDEs. The Liouville and sine-Gordon equations, which describe minimal and pseudospherical surfaces are well-known examples. These two equations were also the first nonlinear PDEs to reveal a deep connection between differential geometry and soliton theory.

Let us consider a surface in three-dimensional Euclidean space R^3 and denote local coordinates for the surface by

$$X^i = x^i(u^1, u^2), \quad i = 1, 2, 3,$$

where X^i are the coordinates of the variable point of the surface and x^i are scalar functions. The basic characteristics of the surface are given by the first and second fundamental forms

$$I = g_{\alpha\beta}\,\mathrm{d}u^\alpha\,\mathrm{d}u^\beta, \quad II = h_{\alpha\beta}\,\mathrm{d}u^\alpha\,\mathrm{d}u^\beta,$$

E-mail address: bracken@panam.edu.

0377-0427/$ - see front matter
doi:10.1016/j.cam.2005.07.043

where $\alpha, \beta = 1, 2, 3$ and $g_{\alpha\beta}$ and $h_{\alpha\beta}$ are symmetric tensors such that

$$g_{\alpha\beta} = \frac{\partial X^i}{\partial u^\alpha}\frac{\partial X^i}{\partial u^\beta}, \quad h_{\alpha\beta} = \frac{\partial^2 X^i}{\partial u^\alpha \partial u^\beta} N^i.$$

Here N^i are the components of the normal vector

$$N^i = (\det g)^{-1/2} \varepsilon^{ikl} \frac{\partial X^k}{\partial u^1}\frac{\partial X^l}{\partial u^2}.$$

The matrix $g_{\alpha\beta}$ completely defines the intrinsic properties of the surface. The Gaussian curvature K of the surface is given by $K = R_{1212}(\det g)^{-1}$, where $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor.

Extrinsic properties of surfaces are described by the Gaussian curvature K and the mean curvature $H = g^{\alpha\beta} h_{\alpha\beta}$. Embedding the surface into R^3 is described by both $g_{\alpha\beta}$ and $h_{\alpha\beta}$ as governed by the Gauss–Codazzi equations

$$\frac{\partial^2 X^i}{\partial u^\alpha \partial u^\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial X^i}{\partial u^\gamma} - h_{\alpha\beta} N^i = 0, \quad \frac{\partial N^i}{\partial u^\alpha} + h_{\alpha\gamma} g^{\gamma\beta} \frac{\partial X^i}{\partial u^\beta} = 0.$$

Here, $\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols which can be determined from the first fundamental form. Among the global characteristics of surfaces, there is the Euler characteristic

$$\chi = \frac{1}{2\pi}\int_S K(\det g)^{1/2}\,\mathrm{d}u^1\,\mathrm{d}u^2,$$

where integration is performed over the entire surface. For compact, oriented surfaces, $\chi = 2(1-n)$, where n is the genus of the surface.

Weierstrass and Enneper started by introducing two holomorphic functions $\psi(z)$ and $\phi(z)$ and three complex-valued functions ω_1, ω_2 and ω_3 which satisfy the following system of equations

$$\partial\omega_1 = \mathrm{i}(\psi^2 + \varphi^2), \quad \partial\omega_2 = \psi^2 - \varphi^2, \quad \partial\omega_3 = -2\psi\varphi, \quad \bar{\partial}\psi = 0, \ \ \bar{\partial}\varphi = 0,$$

where the derivatives are abbreviated $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$, and the bar denotes complex conjugation. They show that if the system of three real-valued functions $X^i(z, \bar{z})$ are considered to be a coordinate system for a surface immersed in R^3 defined by

$$X^1 = \mathrm{Re}\,\omega_1 = \mathrm{Re}\int_C \mathrm{i}(\psi^2 + \varphi^2)\,\mathrm{d}z,$$

$$X^2 = \mathrm{Re}\,\omega_2 = \mathrm{Re}\int_C (\psi^2 - \varphi^2)\,\mathrm{d}z,$$

$$X^3 = \mathrm{Re}\,\omega_3 = -\mathrm{Re}\int_C 2\psi\varphi\,\mathrm{d}z,$$

where C is any contour in the domain of holomorphicity of ψ and φ. Then the functions $X^i(z, \bar{z})$ determine a minimal surface.

This idea has been generalized by Konopelchenko, who established the connection between certain classes of constant mean curvature surfaces and the trajectories of an infinite-dimensional Hamiltonian system [13,9]. He considered the nonlinear Dirac-type system of equations for two complex valued functions ψ_1 and ψ_2 defined by

$$\begin{aligned} &\partial\psi_1 = p\psi_2, \quad \bar{\partial}\psi_2 = -p\psi_1, \\ &\bar{\partial}\bar{\psi}_1 = p\bar{\psi}_2, \quad \partial\bar{\psi}_2 = -p\bar{\psi}_1, \\ &p = |\psi_1|^2 + |\psi_2|^2. \end{aligned} \tag{1.1}$$

This will be referred to as the generalized Weierstrass (GW) system [8,4]. Using system (1.1), it can be verified that the following conservation laws apply, namely,

$$\partial(\psi_1^2)+\bar{\partial}(\psi_2^2)=0,\quad \bar{\partial}(\bar{\psi}_1^2)+\partial(\bar{\psi}_2^2)=0,\quad \partial(\psi_1\bar{\psi}_2)+\bar{\partial}(\bar{\psi}_1\psi_2)=0.$$

Making use of these conserved quantities, there exist three real-valued quantities $X^i(z,\bar{z})$ such that

$$\begin{aligned} X^1+\mathrm{i}X^2&=2\mathrm{i}\int_\gamma(\bar{\psi}_1^2\,\mathrm{d}z'-\bar{\psi}_2^2\,\mathrm{d}\bar{z}'),\\ X^1-\mathrm{i}X^2&=2\mathrm{i}\int_\gamma(\psi_2^2\,\mathrm{d}z'-\psi_1^2\,\mathrm{d}\bar{z}'),\\ X^3&=-2\int_\gamma(\bar{\psi}_1\psi_2\,\mathrm{d}z'+\psi_1\bar{\psi}_2\,\mathrm{d}\bar{z}'). \end{aligned}\tag{1.2}$$

The functions $X^i(z,\bar{z})$ can be treated as the coordinates of a surface immersed into R^3. The Gaussian curvature, the constant mean curvature and the first fundamental form for the surface are given by

$$K=-\frac{\partial\bar{\partial}(\ln p)}{p^2},\quad H=1,\quad \Omega=4p^2\,\mathrm{d}z\,\mathrm{d}\bar{z},\tag{1.3}$$

in isothermic coordinates.

2. Second order system associated with the GW system

Introduce the new variables defined in terms of the ψ_i as follows [8]

$$\rho=\frac{\psi_1}{\bar{\psi}_2}.\tag{2.1}$$

Using system (1.1), it is found that

$$\partial\rho=\frac{\partial\psi_1}{\bar{\psi}_2}-\frac{\psi_1}{\bar{\psi}_2^2}\partial\bar{\psi}_2=(1+|\rho|^2)\psi_2^2.\tag{2.2}$$

Solving for ψ_2^2, Eq. (2.2) and then (2.1) generate the following transformation from the variable ρ to the set of variables ψ_i:

$$\psi_1=\varepsilon\rho\frac{(\bar{\partial}\bar{\rho})^{1/2}}{1+|\rho|^2},\quad \psi_2=\varepsilon\frac{(\partial\rho)^{1/2}}{1+|\rho|^2},\quad \varepsilon=\pm 1.\tag{2.3}$$

Proposition 1. *If ψ_1 and ψ_2 are solutions of the GW system* (1.1), *then the function ρ defined by* (2.1) *is a solution of the second order sigma model system*

$$\partial\bar{\partial}\rho-\frac{2\bar{\rho}}{1+|\rho|^2}\partial\rho\,\bar{\partial}\rho=0,\quad \partial\bar{\partial}\bar{\rho}-\frac{2\rho}{1+|\rho|^2}\partial\bar{\rho}\bar{\partial}\bar{\rho}=0.\tag{2.4}$$

Proposition 2. *If ρ is a solution to system* (2.4), *then the functions ψ_1 and ψ_2 defined in terms of ρ by*

$$\psi_1=\varepsilon\rho\frac{(\bar{\partial}\bar{\rho})^{1/2}}{1+|\rho|^2},\quad \psi_2=\varepsilon\frac{(\partial\rho)^{1/2}}{1+|\rho|^2}\tag{2.5}$$

satisfy GW system (1.1).

So equivalently, given a solution to the sigma model system (2.4), a surface can be determined by calculating the ψ_i using Proposition 2 and then using (1.2) to obtain the coordinates of the surface [8,4].

3. Group invariant solutions of the sigma model

Explicit solutions of GW system (1.1) based on transformation (2.5) will be found which uses a variety of invariant solutions of sigma model system (2.4) obtained by means of the symmetry reduction method to ODEs. In this approach, we need to find its symmetry group G, and then classify all subgroups G_i of G having generic orbits of codimension one in the space of independent variables. We then find the associated invariants of each of its subgroups G_i, and perform for each of these invariants the symmetry reduction of (2.4) to a system of ODEs which are then solved [4].

Let us examine the system of real PDEs which are equivalent to the two-dimensional Euclidean sigma model (2.4). Introduce into (2.4) a polar representation for ρ

$$\rho = R e^{i\phi}. \tag{3.1}$$

The real and imaginary parts which result from (2.4) are given by

$$\phi_{xx} + \phi_{yy} + \frac{2(1-R^2)}{R(1+R^2)}(R_x\phi_x + R_y\phi_y) = 0,$$

$$R_{xx} + R_{yy} - \frac{R(1-R^2)}{1+R^2}(\phi_2^2 + \phi_y^2) - \frac{2R}{1+R^2}(R_x^2 + R_y^2) = 0. \tag{3.2}$$

Note that if we put $R = 1$, then (3.2(ii)) is identically satisfied and the first one reduces to the Laplace equation for the phase ϕ. This implies that ϕ has to be a periodic, harmonic function with a period equal to 2π. If the period of ϕ is not 2π, then the solution may become multivalued.

The symmetry algebra $\mathscr{L}$ can be decomposed into a direct sum of two infinite-dimensional simple Lie subalgebras with direct sum of a one-dimensional algebra generated by $\Phi = \partial_\phi$

$$\mathscr{L} = \{\alpha^+\} \oplus \{\alpha^-\} \oplus \{\Phi\}. \tag{3.3}$$

There is only one finite-dimensional subalgebra spanned by the vector fields

$$P_1 = \partial_x, \quad P_2 = \partial_y,$$

$$D = x\partial_x + y\partial_y, \quad L_3 = y\partial_x - x\partial_y,$$

$$C_1 = (x^2 - y^2)\partial_x + 2xy\partial_y, \quad C_2 = 2xy\partial_x - (x^2 - y^2)\partial_y. \tag{3.4}$$

The operators P_1 and P_2 generate translations in the xy directions, D and L_3 correspond to dilation and rotation, C_1 and C_2 generate two different types of conformal transformations. The nonvanishing commutation relations for algebra (3.4) can be calculated showing it closes and that this algebra is isomorphic to the O(3, 1) algebra. In this case, among all nonconjugate subalgebras, the ones that have generic orbits of codimension one in the space of independent variables and three in the space of independent and dependent variables $\{x, y, R, \phi\}$ reduce the original system (3.2) to a system of ODEs by means of the symmetry reduction method. The one-dimensional subalgebras are given by

$$P_1 + b\Phi, \quad L_3 + b\Phi, \quad D + b\Phi, \quad D + aL_3 + b\Phi, \tag{3.5}$$

where a and b are real parameters. In order to find the relation associated with subalgebras (3.5), we compute for each of them the corresponding invariants by solving the PDE

$$XH(x, y, R, \phi) = 0, \tag{3.6}$$

where H is an auxiliary function of four variables (x, y, R, ϕ), and X is one of the generators listed in (3.5). The solution of (3.6) is found by integrating the associated characteristic system. Three invariants ξ, R and F are found which are given in Table 1. The orbits of the subgroups of G can be expressed in terms of two functions R and ϕ in the following form:

$$R = R(\xi), \quad \phi = \alpha(x, y) + F(\xi), \quad \xi = \xi(x, y), \tag{3.7}$$

Table 1
The symmetry reduction for (3.2)

Algebra and symmetry variable	One-dimensional orbits of subgroups	Coefficients of the reduction to ODEs (3.8)	Reduction to second order ODE
$P_1 + b\Phi$	$R = R(\xi)$	$g = g_0,\ h = 0,\ s = 0$	$\ddot{R} - \frac{2R}{1+R^2}\dot{R}^2 - A^2 g_0^2 \frac{(1-R^2)(1+R^2)^3}{R^3}$
y	$\phi = bx + F(\xi)$	$l = 1,\ m = 0$	$-\frac{R(1-R^2)}{1+R^2} = 0$
$L_3 + b\Phi$	$R = R(\xi)$	$\frac{\dot{g}}{g} = -\frac{1}{\xi},\ h = 0,\ s = 0$	$\ddot{R} - \frac{2R}{1+R^2}\dot{R}^2 + \frac{1}{\xi}\dot{R}$
$\sqrt{x^2+y^2}$	$\phi = b\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}$	$l = \frac{b^2}{\xi^2},\ m = 0$	$-\frac{A^2C^2}{\xi^2}\frac{(1-R^2)(1+R^2)^3}{R^3}$
	$+F(\xi)$		$-\frac{b^2}{\xi^2}\frac{R(1-R^2)}{1+R^2} = 0$
$D + b\Phi$	$R = R(\xi)$	$\frac{\dot{g}}{g} = -\frac{2\xi}{1+\xi^2},\ h = \frac{b}{\xi(1+\xi^2)}$	$\ddot{R} - \frac{2R}{1+R^2}\dot{R}^2 + \frac{2\xi}{1+\xi^2}\dot{R}$
$\frac{x}{y}$	$\phi = b\ln x + F(\xi)$	$s = -\frac{b}{\xi^2(1+\xi^2)},\ l = \frac{b^2}{\xi^2(1+\xi^2)}$	$-\frac{A(\xi)^2C^2}{(1+\xi^2)^2}\frac{(1-R^2)(1+R^2)^3}{R^3}$
		$m = -\frac{4b}{(1+\xi^2)^2}$	$-\frac{R(1-R^2)}{(1+\xi^2)^2(1+R^2)} = 0$
$D + aL_3 + b\Phi$,	$R = R(\xi)$	$g = g_0,\ h = -\frac{2b}{1+a^2}$	$\ddot{R} - \frac{2R}{1+R^2}\dot{R}^2$
$\ln\sqrt{x^2+y^2}$	$\phi = -\frac{b}{a}\tan^{-1}\frac{x}{y}$	$l = \frac{b^2}{1+a^2},\ s = 0,\ m = 0$	$-A^2 g_0^2 \frac{(1-R^2)(1+R^2)^3}{R^3}$
$+\frac{1}{a}\tan^{-1}\frac{x}{y}$ $a > 0$	$+F(\xi)$		$+\frac{R(1-R^2)}{1+R^2}b^2\left(\frac{3-a^2}{(1+a^2)^2}\right) = 0$

where α and ξ are given functions of x and y for each subalgebra. The function ξ is the symmetry variable of the invariance subgroup having generic orbits of codimension one. Substituting each specific form (3.7) into system (3.2) leads to the coupled system of ODEs in terms of the symmetry variable ξ

$$\ddot{R} - \frac{2R}{1+R^2}\dot{R}^2 - \frac{R(1-R^2)}{1+R^2}\dot{\phi}^2 - \frac{\dot{g}}{g}\dot{R} - 2h\frac{R(1-R^2)}{1+R^2}\dot{\phi} - l\frac{R(1-R^2)}{1+R^2} = 0,$$
$$\ddot{\phi} + 2\frac{1-R^2}{R(1+R^2)}\dot{R}\dot{\phi} - \frac{\dot{g}}{g}\dot{\phi} + 2h\frac{1-R^2}{R(1+R^2)}\dot{R} + s = 0, \tag{3.8}$$

where the functions g, h, l and s are given for each of the subalgebras in Table 1, and dot means differentiation with respect to ξ. The differential equations for R and ϕ in (3.8) can be decoupled if we perform the transformation

$$\dot{\phi} = V - h \tag{3.9}$$

on (3.8 (ii)). Then V must satisfy the equation

$$\dot{V} + 2\dot{R}\frac{1-R^2}{R(1+R^2)}V - \frac{\dot{g}}{g}V - m = 0, \tag{3.10}$$

where $m = \dot{h} - (\dot{g}/g)h - s$. Let us consider separately two cases:

(1) When $m = 0$, the V equation is a homogeneous ODE in V. This corresponds to the three subalgebras $P_1 + b\Phi$, $L_3 + b\Phi$, $D + aL_3 + b\Phi$ and the general integral has the form

$$V = Ag\left(\frac{1+R^2}{R^2}\right)^2, \quad \dot{\phi} = Ag\left(\frac{1+R^2}{R^2}\right)^2 - h. \tag{3.11}$$

Eliminating $\dot{\phi}$ from the R equation gives

$$\ddot{R} - \frac{2R}{1+R^2}\dot{R}^2 - \frac{\dot{g}}{g}\dot{R} - A^2 g^2 \frac{(1-R^2)(1+R^2)}{R^3} + (h^2 - l)\frac{R(1-R^2)}{1+R^2} = 0. \tag{3.12}$$

(2) When $m \neq 0$ corresponding to the subalgebra $D + b\Phi$, the general solution is obtained by the method of variation of parameters, and has the following form

$$V = A(\xi) g \left(\frac{1+R^2}{R^2}\right)^2, \quad A(\xi) = \int \frac{mR^2}{g(1+R^2)^2}\, \mathrm{d}\xi'. \tag{3.13}$$

These can be substituted to eliminate ϕ to obtain an equation

$$\ddot{R} - \frac{2R}{1+R^2}\dot{R}^2 - A^2 g^2 \frac{(1-R^2)(1+R^2)^3}{R^3} - \frac{\dot{g}}{g}\dot{R} + (h^2 - l)\frac{R(1-R^2)}{1+R^2} = 0,$$

which is of the same form as (3.12). This ODE has the Painlevé property, that is, no movable singularities other than simple poles, so we can transform to the new variable U as follows:

$$R(\xi) = (-U(\xi))^{1/2},$$

Choosing a new variable $\eta = \int g(\xi)\, \mathrm{d}\xi$, Eq. (3.12) is transformed into the following ODE in the η variable

$$\ddot{U} = \left(\frac{1}{2U} - \frac{1}{1-U}\right)\dot{U}^2 + \frac{2C^2}{U}(1+U)(1-U)^3. \tag{3.14}$$

If $C = 0$, the solution is given by

$$U = \tanh^2(K_1\eta + K_2), \quad K_1, K_2 \in \mathbb{R},$$

and $\phi = \phi_0$. If $C \neq 0$, the equation can be found in Ince's [11] classification where $\beta = -\alpha = 2C^2$ and $\gamma = \delta = 0$ in Ince's notation. This equation admits a first integral given by

$$\dot{U} = -4C^2U^4 + 4C_1U^3 - 2(B + 4C_1)U^2 + 4C_1U - 4C^2, \tag{3.15}$$

where differentiation is with respect to η. Eq. (3.15) can be written for $C \neq 0$ in equivalent form

$$\dot{U}^2 = -4C^2(U - U_1)(U - U_2)(U - U_3)(U - U_4), \tag{3.16}$$

where U_i for $i = 1, \ldots, 4$ denote the roots of the right-hand side of (3.15), and can be expressed in terms of B, C and C_1. The behavior of the solutions to (3.16) depends upon the relationships between the roots of the quartic polynomial on the right-hand side of (3.16). It can be solved in terms of elliptic Jacobi functions or in terms of elementary algebraic functions with one or two simple poles, and trigonometric and hyperbolic solutions, which have been extensively tabulated [4], and for reasons of space will not be given here. For example, when $C = C_1 = 0$, (3.15) can be integrated easily to give

$$U = D \exp\left(\pm\sqrt{-2B}\eta\right).$$

Given U we can obtain R and ϕ thus giving ρ by means of (3.1) which can be used in (2.3) to produce ψ_1 and ψ_2 thus generating a surface by means of (1.2).

4. Multisoliton solutions

Proposition 3. *Suppose that for each $i=1,\dots,N$ the complex valued functions ρ_i satisfy the sigma model system* (2.4) *and the conditions $|\rho_i|^2=1$. Then the product function*

$$\rho=\prod_{i=1}^{N}\rho_i \tag{4.1}$$

is also a solution of system (2.4) [4].

Let us now discuss the construction of an algebraic multi-soliton solution of system (1.1) using Proposition 3. First we look for a particular class of rational solutions of (2.4) which admit simple poles at the points $\bar{z}=\bar{a}_j$,

$$\rho_j=\frac{z-a_j}{\bar{z}-\bar{a}_j},\quad a_j\in\mathbb{C},\quad j=1,\dots,N.$$

By Proposition 3, a more general class of rational solution to (2.4) admitting simple poles is given by

$$\rho=\prod_{k=1}^{N}\frac{z-a_k}{\bar{z}-\bar{a}_k}.$$

In this case, the function ρ satisfies $\partial\bar{\partial}\rho\neq 0$ and $|\rho|^2=1$ as well. The first derivatives of ρ are given by

$$\partial\rho=\sum_{j=1}^{N}\frac{\rho}{z-a_j}=F(z)\rho,\quad \bar{\partial}\rho=-\sum_{j=1}^{N}\frac{\rho}{\bar{z}-\bar{a}_j}=-\bar{F}(\bar{z})\rho.$$

From these derivatives, an algebraic multi-soliton solution of system (1.1) is determined by

$$\psi_1=\frac{\varepsilon}{2}\left(\sum_{j=1}^{N}\frac{1}{\bar{z}-\bar{a}_j}\rho\right)^{1/2},\quad \psi_2=\frac{\varepsilon}{2}\left(\sum_{j=1}^{N}\frac{1}{z-a_j}\rho\right)^{1/2}.$$

Moreover, p and J are given as follows

$$p=\frac{1}{2}\left|\sum_{j=1}^{N}\frac{1}{z-a_j}\right|,\quad J=\frac{1}{4}\left(\sum_{k=1}^{N}\frac{1}{z-a_k}\right)^2.$$

When $N=1$ the functions ψ_1 and ψ_2 can be substituted into the relations which give the X^i, namely (1.2). The corresponding constant mean curvature surface is determined by

$$((X^1)^2+(X^2)^2)^2-\left(2+\frac{a^2}{4}e^{2X^3}\right)((X^1)^2+(X^2)^2)+\frac{a^2}{2}e^{2X^3}X^2+1-\frac{a^2}{4}e^{2X^3}=0.$$

5. A physical application

To give the previous considerations a physical perspective [2,3], consider the classical spin vector $\mathbf{S}=(S_1,S_2,S_3)$, where each S_j depends on the variable $t=x_0$ as well as two spatial degrees of freedom x_1,x_2. The S_j are real functions which satisfy

$$S_3^2+\kappa^2(S_1^2+S_2^2)=1, \tag{5.1}$$

and $\kappa^2=\pm1$ represents the curvature of spin phase space. It is associated with the sphere S^2 when $\kappa^2=1$, or the pseudosphere when $\kappa^2=-1$.

The Landau–Lifshitz equation describes the time evolution of the spin vector and is given by

$$\partial_t \mathbf{S} = \mathbf{S} \times \nabla^2 \mathbf{S}. \tag{5.2}$$

Let $\mathscr{S}$ be a matrix defined by

$$\mathscr{S} = \begin{pmatrix} S_3 & \kappa \bar{S}_+ \\ \kappa S_+ & -S_3 \end{pmatrix}, \tag{5.3}$$

where $S_+ = S_1 + \mathrm{i} S_2$ and the bar denotes complex conjugation. In terms of the matrix $\mathscr{S}$, the Landau–Lifshitz equation can be written as

$$\partial_t \mathscr{S} = \frac{1}{2\mathrm{i}} [\mathscr{S}, \nabla^2 \mathscr{S}]. \tag{5.4}$$

Introduce the variable r defined in terms of the S_j by

$$r = \frac{S_1 + \mathrm{i} S_2}{1 + S_3},$$

then the Cartesian components of the magnetization can be shown to be given by

$$S_1 = \frac{r + \bar{r}}{1 + |r|^2}, \quad S_2 = \frac{r - \bar{r}}{\mathrm{i}(1 + |r|^2)}, \quad S_3 = \frac{1 - |r|^2}{1 + |r|^2}. \tag{5.4}$$

It is clear that these components S_j satisfy constraint (5.1) and $\Delta \mathscr{S}$ and $\dot{\mathscr{S}}$ can be evaluated from (5.4). Substituting the derivatives into the second form of the Landau–Lifshitz equation, two independent equations in terms of the derivatives $\dot{r}$ and $\dot{\bar{r}}$ are obtained and are given explicitly by

$$\mathrm{i}\dot{r} = -\Delta r + 2\bar{r} \frac{(\nabla r)^2}{1 + |r|^2}, \quad -\mathrm{i}\dot{\bar{r}} = -\Delta \bar{r} + 2r \frac{(\nabla \bar{r})^2}{1 + |r|^2}. \tag{5.5}$$

Transforming the derivatives into complex form, we have

$$\frac{\mathrm{i}}{4} \dot{r} + \partial \bar{\partial} r = 2\bar{r} \frac{\partial r \bar{\partial} r}{1 + |r|^2}, \quad -\frac{\mathrm{i}}{4} \dot{\bar{r}} + \bar{\partial} \partial \bar{r} = 2r \frac{\bar{\partial} \bar{r}\, \partial \bar{r}}{1 + |r|^2}. \tag{5.6}$$

The related case $\kappa^2 = -1$ can be analyzed in a similar way. The stereographic projection of the pseudosphere onto the (S_1, S_2) plane is used

$$S_1 = \frac{\xi + \bar{\xi}}{1 - |\xi|^2}, \quad S_2 = \frac{\xi - \bar{\xi}}{\mathrm{i}(1 - |\xi|^2)}, \quad S_3 = \frac{1 + |\xi|^2}{1 - |\xi|^2},$$

and the equations of motion are

$$\mathrm{i}\dot{\xi} = -\Delta \xi - 2\bar{\xi} \frac{(\nabla \xi)^2}{1 - |\xi|^2}, \quad -\mathrm{i}\dot{\bar{\xi}} = -\Delta \bar{\xi} - 2\xi \frac{(\nabla \bar{\xi})^2}{1 - |\xi|^2}. \tag{5.7}$$

If t is held fixed, or r and ξ are t-independent, then systems (5.6) and (5.7) reduce to the usual nonlinear sigma model equations of form (2.4).

6. Non-constant mean curvature system

We would now like to turn to the case in which the mean curvature H is not strictly constant [12]. The set of functions which determine a surface with mean curvature function H are given by

$$\begin{aligned} &\partial \psi_1 = pH\psi_2, \quad \bar{\partial} \psi_2 = -pH\psi_1, \\ &\bar{\partial} \bar{\psi}_1 = pH\bar{\psi}_2, \quad \partial \bar{\psi}_2 = -pH\bar{\psi}_1, \\ &p = |\psi_1|^2 + |\psi_2|^2. \end{aligned} \tag{6.1}$$

The function $H = H(z, \bar{z})$ denotes the mean curvature of the surface.

Proposition 4. *If ψ_1 and ψ_2 are solutions of the system in* (6.1) *and $\rho=\psi_1/\bar{\psi}_2$, then ψ_1 and ψ_2 are related to ρ through the equations*

$$\psi_1=\varepsilon\rho\frac{(\bar{\partial}\bar{\rho})^{1/2}}{H^{1/2}(1+|\rho|^2)},\quad \psi_2=\frac{(\partial\rho)^{1/2}}{H^{1/2}(1+|\rho|^2)},\quad \varepsilon=\pm 1. \tag{6.2}$$

Moreover, ρ is a solution to the following second order system:

$$\partial\bar{\partial}\rho-\frac{2\bar{\rho}}{1+|\rho|^2}\partial\rho\bar{\partial}\rho=\bar{\partial}(\ln H)\partial\rho,\quad \partial\bar{\partial}\bar{\rho}-\frac{2\rho}{1+|\rho|^2}\bar{\partial}\bar{\rho}\partial\bar{\rho}=\partial(\ln H)\bar{\partial}\bar{\rho}. \tag{6.3}$$

The converse of this holds as well. Note the close similarity between system (6.3) and (2.4). In fact, putting H constant in (6.3), we obtain (2.4) exactly. Using system (6.1), the following conservation laws are found to hold

$$\partial(\psi_1\bar{\psi}_2)+\bar{\partial}(\bar{\psi}_1\psi_2)=0,\quad \partial(\psi_1^2)+\bar{\partial}(\psi_2^2)=0,\quad \bar{\partial}(\bar{\psi}_1^2)+\partial(\bar{\psi}_2^2)=0.$$

Surfaces can be induced by using solutions of either (6.1) or both Eqs. (6.3) and (6.2).

Proposition 5. *If $J=\bar{\psi}_1\partial\psi_2-\psi_2\partial\bar{\psi}_1$, then the quantity $\mathscr{J}$ defined by*

$$\mathscr{J}=J+\int_{\bar{z}_0}^{\bar{z}}p^2(z,\tau)\partial H(z,\tau)\,\mathrm{d}\tau \tag{6.4}$$

is conserved under differentiation with respect to $\bar{z}$,

$$\bar{\partial}\mathscr{J}=0.$$

Proposition 6. *With p defined in* (6.1) *and J defined in Proposition* 5, *then p satisfies a second order differential equation which involves p, J and the mean curvature function H. The equation is given by*

$$\partial\bar{\partial}\ln p=\frac{|J|}{p^2}-H^2p^2. \tag{6.5}$$

It has been shown [8] that when H is constant, there is a connection between the time independent Landau–Lifshitz equation which from (5.4) takes the form

$$[\mathscr{S},\partial\bar{\partial}\mathscr{S}]=0, \tag{6.6}$$

and the two-dimensional nonlinear sigma model. The matrix $\mathscr{S}$ will be referred to as the spin matrix. In terms of the sigma model variable ρ, the matrix $\mathscr{S}$ is given by

$$\mathscr{S}=\frac{1}{1+|\rho|^2}\begin{pmatrix}1-|\rho|^2 & 2\bar{\rho}\\ 2\rho & -1+|\rho|^2\end{pmatrix}. \tag{6.7}$$

Define f and $\bar{f}$ to be the ρ dependent factors on the left-hand sides of the sigma model equations in (2.4). System (2.4) can be written as $f=0$ and $\bar{f}=0$. In terms of f and $\bar{f}$, the matrix generated by (6.6) is of the form

$$[\mathscr{S},\partial\bar{\partial}\mathscr{S}]=\frac{4}{(1+|\rho|^2)^2}\begin{pmatrix}\bar{\rho}f-\rho\bar{f} & \bar{\rho}^2f-\bar{f}\\ -f+\rho^2\bar{f} & -\bar{\rho}f+\rho\bar{f}\end{pmatrix}.$$

These results can be summarized as follows.

Proposition 7. *If ρ is a solution of the nonlinear sigma model system* (2.4), *then the spin matrix $\mathscr{S}$ defined by* (6.7) *is a solution of the Landau–Lifshitz* (6.6).

Proposition 7 can be adapted to the case in which H is not constant. Define the matrices $\mathscr{R}$ and $\mathscr{H}$ as follows:

$$\mathscr{R} = \frac{4}{(1+|\rho|^2)^2}\begin{pmatrix} -\bar{\rho}\partial\rho & \rho\bar{\partial}\bar{\rho} \\ \partial\rho & -\rho^2\bar{\partial}\bar{\rho} \end{pmatrix}, \quad \mathscr{H} = \begin{pmatrix} \bar{\partial}\ln H & \bar{\rho}\bar{\partial}\ln H \\ \partial\ln H & \dfrac{1}{\rho}\partial\ln H \end{pmatrix}, \tag{6.8}$$

where the matrix $\mathscr{R}$ depends only on the ρ variable.

Proposition 8. *If ρ is a solution of sigma model equations* (6.3) *and the matrices $\mathscr{R}$ and $\mathscr{H}$ are defined in* (6.8), *then the spin matrix $\mathscr{S}$ of* (6.7) *is a solution of the nonhomogeneous Landau–Lifshitz equation*

$$[\mathscr{S}, \partial\bar{\partial}\mathscr{S}] + \mathscr{R}\mathscr{H} = 0, \tag{6.9}$$

modulo (6.3).

Finally, let us give an example of a solution to system (6.1). Such solutions are more difficult to determine than in the case of constant mean curvature. Consider the mean curvature function H specified by the rational function

$$H(z,\bar{z}) = \frac{1}{1+\lambda^2(z+\bar{z})^2}, \tag{6.10}$$

where λ is a real constant, and H is real valued. Substituting H in (6.10) into (6.3), it can be verified that

$$\rho = \lambda(z+\bar{z}) = \bar{\rho}$$

is a solution to (6.3). Moreover, $\partial\rho = \lambda = \bar{\partial}\bar{\rho}$, and by (6.2) the corresponding solution to (6.1) is given by

$$\psi_1 = \varepsilon\frac{\lambda^{3/2}(z+\bar{z})}{(1+\lambda^2(z+\bar{z})^2)^{1/2}}, \quad \psi_2 = \varepsilon\frac{\lambda^{1/2}}{(1+\lambda^2(z+\bar{z})^2)^{1/2}}.$$

Using these functions, the coordinates of a surface can be calculated using (1.2).

7. Conclusions

This work has been found to be very useful in producing many different types of solutions to the nonlinear sigma model and thereby to the GW system (1.1) which can be used to determine surfaces in R^3. The integrability of the model has also been shown [6]. Some of this work has even been extended to a consideration of surfaces in four-dimensional Euclidean space [7]. Many of these types of solutions can be interpreted as multisolitons, and the work has many physical applications, for example, to the area of two-dimensional gravity, quantum field theory, statistical physics and fluid dynamics [5,10]. We have recently shown that many of our solutions satisfy equations in string theory, that is, they are common solutions of the Nambu–Goto–Polyakov action and GW system (1.1) [5,14,1]. There is also an application to the construction of periodic constant mean curvature surfaces. From the group point of view, we have found a way to construct doubly periodic surfaces in terms of Jacobi functions.

References

[1] P. Bracken, A surface model for classical strings in Minkowski space, Phys. Lett. B 541 (2002) 166–170.
[2] P. Bracken, Spin model equations, connections with integrable systems and applications to magnetic vortices, Int. J. Phys. B 17 (2003) 4525–4537.
[3] P. Bracken, Reductions of Chern–Simons theory to integrable systems which have geometric applications, Int. J. Phys. B 18 (2004) 1261–1275.
[4] P. Bracken, A.M. Grundland, Symmetry properties and explicit solutions of the generalized Weierstrass system, J. Math. Phys. 42 (2001) 1250–1282.
[5] P. Bracken, A.M. Grundland, Geometrical aspects of the complex Eikonal equations associated to the generalized Weierstrass representation, Czech. J. Phys. 51 (4) (2001) 293–300.
[6] P. Bracken, A.M. Grundland, On complete integrability of the generalized Weierstrass system, J. Nonlinear Math. Phys. 9 (2) (2002) 229–247.

[7] P. Bracken, A.M. Grundland, Solutions of the generalized Weierstrass representation in four-dimensional Euclidean space, J. Nonlinear Math. Phys. 9 (3) (2002) 357–381;
B.G. Konopelchenko, Weierstrass representations for surfaces in 4D spaces and their integrable deformations via DS hierarchy, Ann. Global Anal. Geometry 18 (1999) 61–74;
B.G. Konopelchenko, G. Landolfi, Induced surfaces and their integrable dynamics II, Stud. Appl. Math. 104 (1999) 129–169

[8] P. Bracken, A.M. Grundland, L. Martina, The Weierstrass–Enneper system for constant mean curvature surfaces and the completely integrable sigma model, J. Math. Phys. 40 (1999) 3379–3403.

[9] R. Carroll, B.G. Konopelchenko, Generalized Weierstrass–Enneper inducing conformal immersions and gravity, Int. J. Mod. Phys. A 11 (1996) 1183–1216.

[10] B.G. Gross, C.N. Pope, S. Weinberg, Two-Dimensional Quantum Gravity and Random Surfaces, World Scientific, Singapore, 1992.

[11] E.L. Ince, Ordinary Differential Equations, Dover, New York, 1956.

[12] B.G. Konopelchenko, Induced surfaces and their integrable dynamics, Stud. Appl. Math. 96 (1996) 9–51.

[13] B.G. Konopelchenko, I.A. Taimanov, Constant mean curvature surfaces via an integrable dynamical system, J. Phys. A 29 (1996) 1261–1265.

[14] D. Nelson, T. Piran, S. Weinberg, Statistical Mechanics of Membranes and Surfaces, World Scientific, Singapore, 1992.