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Global solutions of singular parabolic equations arising from electrostatic MEMS

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Abstract

We study dynamic solutions of the singular parabolic problem

$$\begin{cases} u_t - \Delta u = \dfrac{\lambda_* |x|^\alpha}{(1-u)^2}, & (x,t) \in B \times (0, \infty), \\ u(x,0) = u_0(x) \geqslant 0, & x \in B, \\ u(x,t) = 0, & x \in \partial B, \end{cases} \tag{P}$$

where $\alpha \geqslant 0$ and $\lambda_* > 0$ are two parameters, and B is the unit ball $\{x \in \mathbb{R}^N\colon |x| \leqslant 1\}$ with $N \geqslant 2$. Our interest is focussed on (P) with $\lambda_* := \frac{(2+\alpha)(3N+\alpha-4)}{9}$, for which (P) admits a singular stationary solution in the form $S(x) = 1 - |x|^{\frac{2+\alpha}{3}}$. This equation models dynamic deflection of a simple electrostatic Micro-Electro-Mechanical-System (MEMS) device. Under the assumption $u_0 \lneqq S(x)$, we address the existence, uniqueness, regularity, stability or instability, and asymptotic behavior of weak solutions for (P). Given $\alpha^{**} := \frac{4-6N+3\sqrt{6}(N-2)}{4}$, in particular we show that if either $N \geqslant 8$ and $\alpha > \alpha^{**}$ or $2 \leqslant N \leqslant 7$, then the minimal compact stationary solution u_{λ_*} of (P) is stable and while $S(x)$ is unstable. However, for $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$, (P) has no compact minimal solution, and $S(x)$ is an attractor from below not from above. Furthermore, the refined asymptotic behavior of global solutions for (P) is also discussed for the case where $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$, which is given by a certain matching of different asymptotic developments in the large outer region closer to the boundary and the thin inner region near the singularity.

Keywords: MEMS; Existence and uniqueness; Asymptotic behavior; Minimal solutions; Stability

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1. Introduction

The singular parabolic problem

$$\begin{cases} u_t - \Delta u = \dfrac{\lambda f(x)}{(1-u)^2}, & (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geqslant 0, & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

was recently proposed in [11,14], where $\lambda > 0$ is a parameter, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $f(x)$ is a nonnegative function satisfying

$$f \in C^\alpha(\bar{\Omega}) \quad \text{for some } \alpha \in (0,1],\ 0 \leqslant f \leqslant 1, \quad \text{and}$$
$$f > 0 \quad \text{on a subset of } \Omega \text{ of positive measure.} \tag{1.2}$$

When $N = 1$ or 2, this equation models a simple electrostatic Micro-Electro-Mechanical-System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 below a rigid plate located at $+1$. The dynamic solution $u(x,t)$ of (1.1) characterizes the dynamic deflection of the elastic membrane. When a voltage—represented here by λ—is applied to the surface of the membrane, the membrane deflects towards the ceiling plate and a snap-through may occur when it exceeds a certain critical value λ^* (pull-in voltage). This creates a so-called "pull-in instability" which greatly affects the design of many devices. In an effort to achieve better MEMS designs, the material properties of the membrane can be technologically fabricated with a spatially varying dielectric permittivity profile $f(x)$ (see [11,14] and references therein for more detailed discussions on MEMS devices).

Concerning the associated stationary problem

$$\begin{cases} -\Delta v = \dfrac{\lambda f(x)}{(1-v)^2}, & x \in \Omega, \\ 0 < v < 1, & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases} \tag{$(S)_\lambda$}$$

we established in [8] the existence and some monotonicity properties of pull-in voltage λ^* which is defined as

$$\lambda^*(\Omega, f) = \sup\bigl\{\lambda > 0 \bigm| (S)_\lambda \text{ possesses at least one solution}\bigr\}. \tag{1.3}$$

In other words, λ^* is called pull-in voltage if there exist uncollapsed states for $0 < \lambda < \lambda^*$ while there are none for $\lambda > \lambda^*$. Fine properties of steady states—such as regularity, stability, uniqueness, multiplicity, energy estimates, and comparison results—were also shown in [4,8] to depend on the dimension of the ambient space and on the permittivity profile. For

$$\alpha^{**} := \frac{4 - 6N + 3\sqrt{6}(N-2)}{4}, \qquad \lambda_* := \frac{(2+\alpha)(3N+\alpha-4)}{9}, \tag{1.4}$$

if $f(x) \equiv |x|^\alpha$ with $\alpha \geqslant 0$ and $\Omega = B = \{x \in \mathbb{R}^N\colon |x| \leqslant 1\}$ is a unit ball, in particular we then obtained in [4,8] the following refined properties of $(S)_\lambda$.

1. If either $2 \leqslant N \leqslant 7$ or $N \geqslant 8$ and $\alpha > \alpha^{**}$, then $(S)_\lambda$ with $\lambda = \lambda_*$ admits at least two solutions: one is the singular radial solution $S(x) = 1 - |x|^{\frac{2+\alpha}{3}}$, and the other is the regular minimal (radial) solution u_λ.
2. If $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$, then the singular function $S(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ is the unique solution of $(S)_\lambda$ at $\lambda = \lambda_*$. Moreover, we have $\lambda^* = \lambda_*$ and hence $S(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ is also called the extremal solution of $(S)_\lambda$.

Note that the minimal solution u_λ of $(S)_\lambda$ is defined in the following sense: a solution $0 < u_\lambda(x) < 1$ is said to be a minimal (positive) solution of $(S)_\lambda$, if for any solution $0 < u(x) < 1$ of $(S)_\lambda$ we have $u_\lambda(x) \leqslant u(x)$ in Ω. The limit of u_λ as $\lambda \uparrow \lambda^*$ is called the extremal solution u^* of $(S)_\lambda$. As observed in Fig. 1, if $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$, then $(S)_\lambda$ has a unique radial solution which is the minimal solution u_λ for any $\lambda \leqslant \lambda^*$. While if either $2 \leqslant N \leqslant 7$ or $N \geqslant 8$ and $\alpha > \alpha^{**}$, then $(S)_\lambda$ with $\lambda = \lambda_*$ admits infinitely multiple radial solutions, including a singular radial solution $S(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ and a minimal radial solution u_λ.

For the parabolic problem (1.1), we recall that a point $x_0 \in \bar{\Omega}$ is said to be a *touchdown point* for a solution $u(x,t)$ of (1.1), if for some $T \in (0, +\infty]$, we have $\lim_{t_n \to T} u(x_0, t_n) = 1$. T is then said to be a—finite or infinite—touchdown time. The global convergence or touchdown behavior of (1.1) with zero initial data was recently studied in [9,10], where we focused on the classic solutions of (1.1). Particularly, we proved in [9] that the unique classic solution of (1.1) globally converges to its unique minimal positive steady-state when $\lambda \leqslant \lambda^*$, with a possibility of touchdown at infinite time when $\lambda = \lambda^*$ and $N \geqslant 8$. The latter essentially occurs only if $\lambda = \lambda^*$ and the associated extremal steady-state is singular. It is now natural to ask whether such dynamic properties can be extended to (1.1) with non-zero initial data, even with singular initial data.

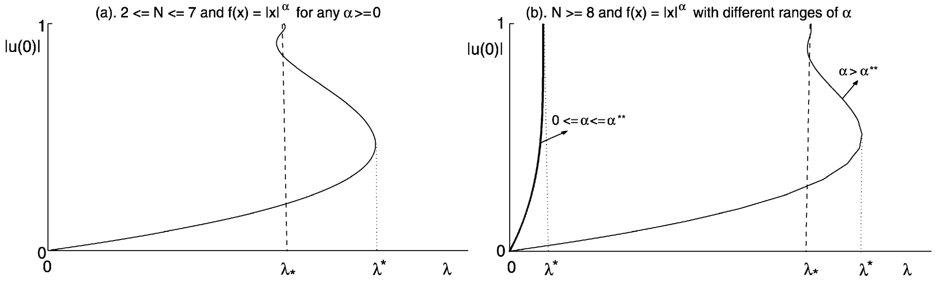

Fig. 1. Left figure: Plots of $u(0)$ versus λ for the power-law permittivity profile $f(x)=|x|^\alpha$ ($\alpha \geqslant 0$) defined in the unit ball $B \subset \mathbb{R}^N$ with $2 \leqslant N \leqslant 7$. In this case, $u(0)$ oscillates around the value λ_* and u^* is regular. Right figure: Plots of $u(0)$ versus λ for the power-law permittivity profile $f(x)=|x|^\alpha$ ($\alpha \geqslant 0$) defined in the unit ball $B \subset \mathbb{R}^N$ with $N \geqslant 8$. The characters of bifurcation diagrams depend on different ranges of α: when $0 \leqslant \alpha \leqslant \alpha^{**}$, there exists a unique solution for $(S)_\lambda$ with $\lambda \in (0, \lambda^*)$ and u^* is singular; when $\alpha > \alpha^{**}$, $u(0)$ oscillates around the value λ_* and u^* is regular.

Motivated by the above analytic results and observations, our interest of this paper is to study the singular parabolic problem

$$\begin{cases} u_t - \Delta u = \dfrac{\lambda_* |x|^\alpha}{(1-u)^2}, & (x,t) \in B \times (0,\infty), \\ u(x,0) = u_0(x) \geqslant 0, & x \in B, \\ u(x,t) = 0, & x \in \partial B, \end{cases} \tag{P}$$

where $\alpha \geqslant 0$, $\lambda_* := \frac{(2+\alpha)(3N+\alpha-4)}{9}$, and B is the unit ball $\{x \in \mathbb{R}^N\colon |x| \leqslant 1\}$ with $N \geqslant 2$. In this case, (P) admits a singular stationary solution in the form $S(x) = 1 - |x|^{\frac{2+\alpha}{3}}$. We take initial data as a nonnegative and measurable function u_0 which is bounded above by the singular stationary solution, i.e.,

$$0 \leqslant u_0(x) \leqslant S(x) := 1 - |x|^{\frac{2+\alpha}{3}}, \tag{1.5}$$

and $u_0 \not\equiv S$ is taken in the obvious measure sense. The assumption (1.5) implies that the initial data $u_0(x)$ is allowed to be singular at the origin. Since in principle the solutions $u(x,t)$ of (P) are singular, we consider weak solutions of (P) satisfying

$$0 \leqslant u(x,t) \leqslant S(x) := 1 - |x|^{\frac{2+\alpha}{3}} \quad \text{in } Q = B \times (0,\infty). \tag{1.6}$$

Therefore, we now define

Definition 1.1. A solution $u = u(x,t)$ is called a weak solution of (P) with $u_0(x) \leqslant S(x)$, if $u \in C((0,\infty) : H_0^1(B))$ satisfies

1. u_t, Δu and $\frac{|x|^\alpha}{(1-u)^2} \in L^1(B \times [\tau, T])$ for every $0 < \tau < T < \infty$;
2. Eq. (P) is satisfied almost everywhere in Q;
3. $u(\cdot,t) \to u_0$ in $L^2(B)$ as $t \to 0$.

The main purpose of this paper is to address the existence, uniqueness, regularity, stability or instability, and asymptotic behavior of weak solutions for (P). Throughout this paper and unless mentioned otherwise, λ_* and α^{**} are defined as in (1.4), the initial data u_0 of (P) is assumed to satisfy (1.5), and weak solutions of (P) are considered to satisfy (1.6).

This paper is organized as follows: stimulated by I. Peral and J.L. Vazquez' work [15], in Section 2 we discuss the existence, uniqueness and regularity of weak solutions for (P). More exactly, the following existence and regularity of weak solutions for (P) are first studied by iterative methods in Section 2.

Theorem 1.1. *Under the assumption* (1.5), (P) *has a minimal solution* v *and a maximal solution* w *satisfying the bound* (1.6). *Moreover, each solution* u *of* (P) *satisfies*

$$\|u_t\|_{L^2(B\times(\tau,\infty))} < C, \tag{1.7}$$

$$\sup_{t>\tau} \|\nabla u\|_{L^2(B)} < C. \tag{1.8}$$

Finally, the solutions of (P) *are bounded and are* C^∞*-smooth for any* $t \geqslant \tau > 0$.

Note that Theorem 1.1 gives the existence of minimal solution and maximal solution of (P). In general, one cannot expect that the uniqueness of weak solutions for (P) holds. However, we are able to prove in Section 2.1 that the uniqueness of weak solutions for (P) does hold if either the dimension $N \geqslant 8$ or the initial data u_0 is not too singular. The main results in this direction can be stated by the following theorem.

Theorem 1.2. *Under the assumption* (1.5), (P) *has a unique solution in* $(0,\infty)$ *provided that either*

1. $N \geqslant 8$ *and* $0 \leqslant \alpha \leqslant \alpha^{**}$, *or*
2. *the initial data* u_0 *also satisfies*

$$0 \leqslant u_0(x) \leqslant 1 - |x|^\beta \quad \text{with } |x| \leqslant R \tag{1.9}$$

for some $0 \leqslant \beta < \frac{2+\alpha}{3}$ *and* $0 < R = R(\beta) < 1$.

We now deduce from Theorem 1.1 that there exists a small $\varepsilon > 0$ such that any solution $u(x,\varepsilon)$ is regular. This further implies that $u(x,\varepsilon)$ satisfies the condition (1.9). Therefore, Theorem 1.2 essentially shows that, in general, there exists a small $\varepsilon \geqslant 0$ such that the uniqueness of global solutions for (P) holds for $t \in [\varepsilon,\infty)$.

Section 2.2 is devoted to the global stability or instability of dynamic solutions for (P), where we can obtain the following global convergence or instantaneous touchdown behavior.

Theorem 1.3. *Suppose* u *is a global solution of* (P) *satisfying* (1.6), *then we have the followings.*

1. *If the initial data* $u_0(x)$ *satisfies* $u_0(x) \lneqq S(x)$ *on* B, *then for the case*

$$\text{either} \quad N \geqslant 8 \text{ and } \alpha > \alpha^{**} \quad \text{or} \quad 2 \leqslant N \leqslant 7,$$

we have

$$\lim_{t\to\infty} u(x,t) = u_{\lambda_*}(x) \quad \text{in } L^2(B),$$

where $u_{\lambda_*}(x)$ *is the minimal stationary solution of* (P).

2. *If the initial data $u_0(x)$ satisfies $u_0(x) \lneqq S(x)$ on B, then for the case $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$, we have*

$$\lim_{t\to\infty} u(x,t) = S(x) \quad \text{in } L^2(B).$$

3. *While if the initial data $u_0(x)$ satisfies $u_0(x) \gneqq S(x)$ on B, then u must instantaneously touchdown at the time $t = 0$.*

Theorem 1.3 shows that if either $N \geqslant 8$ and $\alpha > \alpha^{**}$ or $2 \leqslant N \leqslant 7$, then the minimal compact stationary solution u_{λ_*} of (P) is stable and while $S(x)$ is unstable. However, for $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$, (P) has no compact minimal solution, and $S(x)$ is an attractor from below not from above. The proof of Theorem 1.3 needs to use an ordering property of weak-$H_0^1(B)$ stationary solutions for (P), and the details are given in Appendix A.

Sections 3–5 are focussed on the asymptotic behavior of global solutions for (P) in the case where $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$, such that the unique solution u of (P) globally converges to the singular steady-state $S(x)$. Similar to [3,6], we shall show that such an asymptotic behavior is not governed by a self-similar nature, but by a certain matching of different asymptotic developments in the large outer region closer to the boundary and the thin inner region near the singularity, see Sections 3 and 4 for more details. We finally reach the following refined asymptotic profile in Section 5.

Theorem 1.4. *Suppose $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$ such that (P) has a unique global solution u, and let λ_1 be the first eigenvalue of operator $\mathcal{A} := -\Delta - \frac{2\lambda_*}{|x|^2}$ defined in B.*

1. *If $0 \leqslant \alpha < \alpha^{**}$, then we have $\lambda_1 > 0$ and*

$$\left\| \ln \frac{1}{1-u(x,t)} \right\|_{L^\infty(B)} = \frac{\lambda_1(2+\alpha)}{2+\alpha-3\gamma_+} t + O(1) \quad \text{as } t \to \infty, \tag{1.10}$$

where γ_+ satisfies

$$\gamma_+ = \frac{1}{2}\left[2 - N + \sqrt{-8\alpha^2 - (24N-16)\alpha + \left(9N^2 - 84N + 100\right)}\,\right] < 0.$$

2. *If $\alpha = \alpha^{**}$, then we have $\lambda_1 < 0$ and*

$$\left\| \ln \frac{1}{1-u(x,t)} \right\|_{L^\infty(B)} \geqslant \frac{2(2+\alpha)|\lambda_1|}{3N+2\alpha-2} t + C_1 \quad \text{as } t \to \infty. \tag{1.11}$$

Remark 1.1. For the case where $N \geqslant 8$ and $\alpha = \alpha^{**}$, unfortunately we are unable to obtain the upper bound estimate of $\| \ln \frac{1}{1-u(x,t)} \|_{L^\infty(B)}$ as $t \to \infty$. However, in Section 5.2 we can derive the following formal expansion

$$\left\| \ln \frac{1}{1-u(x,t)} \right\|_{L^\infty(B)} \sim \frac{8(2+\alpha)\nu_1^2}{(3N+2\alpha-2)(N-2)^2} t + O(\ln t) \quad \text{as } t \to \infty,$$

where ν_1 is the first zero of the zeroth-order Bessel function: $J_0(\frac{2\nu_1}{N-2}) = 0$, and therefore the second term C_1 in (1.11) is not optimal.

2. Basic properties of dynamic solutions

In this section, we study some basic properties of weak solutions for (P). As stated in the introduction, we take initial data as a nonnegative and measurable function u_0, which is bounded above by the singular stationary solution, i.e.,

$$0 \leqslant u_0(x) \leqslant S(x) := 1 - |x|^{\frac{2+\alpha}{3}}, \tag{2.1}$$

where $u_0 \not\equiv S$ is taken in the obvious measure sense, and we consider weak solutions for (P) satisfying

$$0 \leqslant u(x,t) \leqslant S(x) := 1 - |x|^{\frac{2+\alpha}{3}} \quad \text{in } Q = B \times (0, \infty). \tag{2.2}$$

Our purpose of this section is to address the existence, uniqueness, regularity, and stability of weak solutions for (P). We start with the following existence and regularity.

Theorem 2.1. *Under the assumption* (2.1), (P) *has a minimal solution* v *and a maximal solution* w *satisfying the bound* (2.2). *Moreover, each solution* u *of* (P) *satisfies*

$$\|u_t\|_{L^2(B\times(\tau,\infty))} < C, \tag{2.3}$$

$$\sup_{t>\tau} \|\nabla u\|_{L^2(B)} < C, \tag{2.4}$$

where $\tau > 0$ *is arbitrary. Finally, the solutions of* (P) *are bounded away from* 1 *and are* C^∞*-smooth for any* $t \geqslant \tau > 0$.

Proof. We first prove the existence of minimal and maximal solutions for (P) by iteration: Let $v_0 = 0$ and $w_0 = S(x)$, which are stationary subsolution and supersolution of (P), respectively, and consider the problems

$$v_{kt} = \Delta v_k + \frac{\lambda_*|x|^\alpha}{(1 - v_{k-1})^2}, \qquad v_k(x,0) = u_0, \qquad v_k|_{\partial B} = 0, \tag{P_1}$$

$$w_{kt} = \Delta w_k + \frac{\lambda_*|x|^\alpha}{(1 - w_{k-1})^2}, \qquad w_k(x,0) = u_0, \qquad w_k|_{\partial B} = 0, \tag{P_2}$$

where $k = 1, 2, \ldots.$ By a standard comparison of weak solutions for the heat equation $u_t - \Delta u = g$, we have

$$0 = v_0 \leqslant v_1 \leqslant v_2 \leqslant \cdots \leqslant w_2 \leqslant w_1 \leqslant w_0 = S.$$

It is then clear that the respective limits v and w are the minimal and maximal solutions of (P) satisfying $0 \leqslant v \leqslant w \leqslant S$.

Let u be any solution of (P). Since $0 \leqslant u_0 \leqslant S$, the iteration of the above construction gives $v_n \leqslant u \leqslant w_n$, and hence we have $v \leqslant u \leqslant w \leqslant S$. Multiplying (1.1) by u and integrating by parts, we have the estimate

$$\frac{1}{2}\int_B u^2(x,t)\,dx + \int_0^t \int_B |\nabla u|^2\,dx\,dt = \frac{1}{2}\int_B u_0^2(x)\,dx + \lambda_* \int_0^t \int_B \frac{u|x|^\alpha}{(1-u)^2}\,dx\,dt.$$

Since $N \geqslant 2$ and $0 \leqslant u \leqslant S(x)$, we get that

$$\int_B \frac{|x|^\alpha}{(1-u)^2}\,dx \leqslant \int_B \frac{|x|^\alpha}{(1-S(x))^2}\,dx = \int_B |x|^{\frac{\alpha-4}{3}}\,dx < \infty.$$

These give that $\nabla u \in L^2(B \times (0,t))$. Moreover, multiplying (1.1) by u_t and taking integration in any interval $[\tau, T]$, we obtain

$$\int_\tau^T \int_B u_t^2\,dx\,dt + \frac{1}{2}\int_B \int_\tau^T \frac{d}{dt}|\nabla u|^2\,dt\,dx = \lambda_* \int_B \int_\tau^T \frac{d}{dt}\frac{|x|^\alpha}{1-u}\,dt\,dx.$$

Therefore, we have

$$\int_B \left|\nabla u(x,T)\right|^2 dx \leqslant 2\lambda_* \int_B \left[\frac{|x|^\alpha}{1-S(x)} - \frac{|x|^\alpha}{1-u(x,\tau)}\right] dx + \int_B \left|\nabla u(x,\tau)\right|^2 dx < \infty,$$

which leads to (2.4). As a consequence, it then turns out that $\int_\tau^T \int_B u_t^2\,dx\,dt \leqslant C$ for any $T > 0$, which yields (2.3).

We next discuss the boundedness and smoothness of weak solutions for (P). This is not immediate unless the initial data u_0 is bounded away from 1, but it follows from a delicate argument. Without loss of generality, in the following we may as well consider only the maximal solution w of (P). Given any $T > 0$, we introduce a new variable

$$\phi(x,t) = S(x) - w_1(x,t) \quad \text{in } B \times (0,T),$$

where w_1 is defined as in (P_2). Then ϕ satisfies

$$\phi_t - \Delta\phi = 0 \quad \text{in } B \times (0,T), \qquad \phi(x,0) = S(x) - u_0(x) \geqslant 0 \quad \text{in } B, \qquad \phi = 0 \quad \text{on } \partial B. \tag{2.5}$$

It is then clear that ϕ is a smooth function in $B \times (0,T)$. Furthermore, the strong maximum principle implies that $\phi(x,t) = S(x) - w_1(x,t) > 0$ in $B \times (0,T)$, and hence $w_n(x,t) \leqslant \cdots \leqslant w_2(x,t) \leqslant w_1(x,t) < S(x) \leqslant 1$ for any $t \geqslant \tau > 0$. This gives that the maximal solution $w = \lim_{n\to\infty} w_n$ of (P) is bounded away from 1. Finally, once we prove that the maximal solution w of (P) is bounded away from 1, the C^∞-smoothness of w immediately follows from the standard bootstrap argument of heat equation $u_t - \Delta u = g$. This completes the proof of Theorem 2.1. $\square$

We note that the boundedness of Theorem 2.1 does not depend on N and α. Further, for the case

$$\text{either} \quad N \geqslant 8 \text{ and } \alpha > \alpha^{**} \quad \text{or} \quad 2 \leqslant N \leqslant 7, \tag{2.6}$$

the solutions of (P) are uniformly bounded for all large times, which was essentially proved in [9], if the initial data u_0 is no larger than its minimal steady-state u_{λ_*}. But the uniformly boundedness is not true for the case where $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$, since the solutions are attracted by the singular steady-state $S(x)$ as $t \to \infty$, see Theorem 2.6 below for more details.

2.1. Uniqueness of dynamic solutions

The purpose of this subsection is to discuss the uniqueness of global solutions for (P). By applying the Hardy inequality, we first prove the following uniqueness in higher dimensional case.

Theorem 2.2. *If $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$, then the solutions of (P) must be unique.*

Proof. Suppose that u is any solution of (P). Let $\phi(x,t) = S(x) - u(x,t)$, then $\phi(x,t) \geqslant 0$ on Q and satisfies

$$\phi_t - \Delta\phi = \lambda_* |x|^\alpha \left[\frac{1}{(r^{\frac{2+\alpha}{3}})^2} - \frac{1}{(r^{\frac{2+\alpha}{3}} + \phi)^2} \right]. \tag{2.7}$$

It now suffices to prove the uniqueness of solutions ϕ for (2.7). Actually, suppose that there is another solution v for (P) and set $\psi = S - v$ which also satisfies (2.7), then multiplying the difference of the two equations by $\phi - \psi$ and integrating on B, we obtain

$$\begin{aligned}
\frac{1}{2}\int_B |\phi - \psi|^2\,dx + \int_0^t \int_B \left|\nabla(\phi - \psi)\right|^2 dx\,dt &\leqslant \int_0^t \int_B \frac{2\lambda_*}{|x|^2} |\phi - \psi|^2\,dx\,dt \\
&\leqslant \frac{8\lambda_*}{(N-2)^2} \int_0^t \int_B \left|\nabla(\phi - \psi)\right|^2 dx\,dt,
\end{aligned} \tag{2.8}$$

where the last inequality is derived by applying the Hardy inequality ($N \geqslant 2$):

$$\frac{(N-2)^2}{4} \int_B \frac{\phi^2}{|x|^2}\,dx \leqslant \int_B |\nabla\phi|^2\,dx \quad \text{for any } \phi \in H_0^1(B). \tag{2.9}$$

Note that if $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$, then we have $\frac{8\lambda_*}{(N-2)^2} \leqslant 1$, and therefore, the uniqueness follows from (2.8). $\square$

For the case (2.6), in general one cannot expect the uniqueness of global solutions for (P). Essentially, motivated by [7], we conjecture that the non-uniqueness of global solutions for (P)

holds under the assumptions (2.6) and $u_0(x) \equiv S(x) = 1 - |x|^{\frac{2+\alpha}{3}}$. However, the following theorem shows that the uniqueness does hold if the initial data u_0 has weaker singularity than $S(x)$ near the origin.

Theorem 2.3. *In addition to* (2.1)*, if the initial data u_0 also satisfies*

$$0 \leqslant u_0(x) \leqslant 1 - |x|^{\beta} \quad \text{with } |x| \leqslant R \tag{2.10}$$

for some $0 \leqslant \beta < \frac{2+\alpha}{3}$ *and* $0 < R = R(\beta) < 1$*, then the solutions of* (P) *must be unique.*

Remark 2.1. Given any solution $u(x,t)$ of (P), one can deduce from Theorem 2.1 that there exists a small $\varepsilon > 0$ such that any solution $u(x,\varepsilon)$ is regular. This further implies that $u(x,\varepsilon)$ satisfies (2.10). Therefore, Theorem 2.3 essentially shows that, in general, there exists a small $\varepsilon \geqslant 0$ such that the uniqueness of global solutions for (P) holds for $t \in [\varepsilon, \infty)$, and while the uniqueness maybe fail in the time interval $[0, \varepsilon)$.

The proof of Theorem 2.3 is based on the following lemma.

Lemma 2.4. *Suppose that u is a solution of* (P) *defined on the interval* $[0,T]$ *such that* $\frac{|x|^\alpha}{(1-u)^3} \in X_p = L^\infty([0,T]: L^p(B))$ *for some* $p > \frac{N}{2}$*. Then the solutions of* (P) *must be unique in this class.*

Proof. Suppose $u_1, u_2 \in X_p$ are two solutions of (P). The difference $U = u_1 - u_2$ then satisfies

$$U_t - \Delta U = \alpha U \quad \text{in } B \times (0,T) \tag{2.11}$$

with zero initial data and zero boundary condition, where

$$0 \leqslant \alpha(x,t) = \lambda_* |x|^\alpha \frac{\frac{1}{(1-u_1)^2} - \frac{1}{(1-u_2)^2}}{u_1 - u_2} \leqslant 2\lambda_* |x|^\alpha \max\left\{\frac{1}{(1-u_1)^3}, \frac{1}{(1-u_2)^3}\right\},$$

which implies that $\alpha(x,t) \in X_p$. We now fix $T_1 \in (0,T]$ and consider the solution ϕ of the problem

$$\begin{cases} \phi_t + \Delta\phi + \alpha\phi = 0, & x \in B,\ 0 < t < T_1, \\ \phi(x,T_1) = \theta(x) \in C_0(B), \\ \phi(x,t) = 0, & x \in \partial B. \end{cases} \tag{2.12}$$

The standard linear theory (cf. Theorem 8.1 of [13]) gives that the solution of (2.12) is unique and bounded. Now multiplying (2.11) by ϕ, and integrating it on $B \times [0,T_1]$, together with (2.12), yield that

$$\int_B U(x,T_1)\theta(x)\,dx = 0$$

for arbitrary T_1 and $\theta(x)$, which implies that $U \equiv 0$, and we are done. □

In order to prove Theorem 2.3, we need to obtain X_p-estimate of $\frac{|x|^\alpha}{(1-u)^3}$ for any solution u of (P) in view of Lemma 2.4. However, the given restriction $u \leqslant S(x)$ is not enough for such an estimate, since $\frac{|x|^\alpha}{(1-S)^3} = |x|^{-2} \in L^p(B)$ holds only for $p < \frac{N}{2}$. By a more delicate analysis, we next proceed the proof as follows. For any solution $u = u(x,t)$ of (P), introduce a new transformation

$$w = -\log(1-u), \tag{2.13}$$

then $w \geqslant 0$ satisfies

$$\begin{cases} w_t - \Delta w = \lambda_* |x|^\alpha e^{3w} - |\nabla w|^2, & (x,t) \in B \times (0,\infty), \\ w(x,0) = w_0(x) \lneqq W_0 \equiv -\dfrac{2+\alpha}{3} \log|x|, & x \in B, \\ w(x,t) = 0, & x \in \partial B. \end{cases} \tag{2.14}$$

Note that $W_0 \equiv -\frac{2+\alpha}{3} \log|x|$ is an unbounded steady-state of (2.14). In order to complete the proof of Theorem 2.3, it now suffices to prove the following X_p-estimate on w.

Lemma 2.5. *Suppose that w is any solution of* (2.14) *defined on the interval* $[0,T]$, *and set* $W_0(x) \equiv -\frac{2+\alpha}{3} \log|x|$. *If the initial data w_0 satisfies*

$$0 \leqslant w_0(x) \leqslant \begin{cases} \beta W_0(x) & \text{for } |x| \leqslant R, \\ W_0(x) & \text{for } R < |x| \leqslant 1, \end{cases} \tag{2.15}$$

for some $0 \leqslant \beta < 1$ and $0 < R = R(\beta) < 1$, then we have $|x|^\alpha e^{3w} \in X_p = L^\infty([0,T] : L^p(B))$ for some $p > \frac{N}{2}$.

Proof. We first construct the unique maximal solution $K = K(x,t)$ of (2.14) by iteration: Let $K_0 = W_0$, which is a stationary supersolution of (2.14), and consider the problem

$$(K_n)_t - \Delta K_n = \lambda_* |x|^\alpha e^{3K_{n-1}} - |\nabla K_{n-1}|^2, \qquad K_n(x,0) = w_0, \qquad K_n|_{\partial B} = 0, \tag{2.16}$$

where $n = 1, 2, \ldots$. Similar to Theorem 2.1, one can deduce that

$$0 < \cdots \leqslant K_n \leqslant \cdots \leqslant K_2 \leqslant K_1 \leqslant K_0 = W_0(x) \equiv -\frac{2+\alpha}{3} \log|x|.$$

It is then clear that the limit K of K_n is the unique maximal solutions of (2.14) satisfying $0 < K \leqslant W_0$. Under the assumption (2.15), in the following we need only to prove that $|x|^\alpha e^{3K} \in X_p = L^\infty([0,T] : L^p(B))$ for some $p > \frac{N}{2}$.

Since $W_0(x) \equiv -\frac{2+\alpha}{3} \log|x|$, we have

$$\lambda_* |x|^\alpha e^{3W_0} - |\nabla W_0|^2 = \frac{(2+\alpha)(N-2)}{3|x|^2}. \tag{2.17}$$

We define U as the solution of the problem

$$U_t - \Delta U = \lambda_* |x|^\alpha e^{3W_0} - |\nabla W_0|^2 \quad \text{in } B \times (0,T) \tag{2.18}$$

with zero boundary and initial data. Note that (2.17) now implies $U \equiv 0$ for $N = 2$. We also define $v = v(x,t)$ as the solution of

$$v_t - \Delta v = 0 \quad \text{in } B \times (0,T), \qquad v(x,0) = W_0(x), \qquad v|_{\partial B} = 0. \tag{2.19}$$

For any $0 < \tau < T$ the solution $v(x,\tau)$ is C^∞-smooth in the closed ball and goes to zero as $\tau \to \infty$. Since W_0 is a stationary solution of (2.14), we have

$$W_0(x) = U(x,t) + v(x,t),$$

and $W_0 = v(x,t)$ for $N = 2$.

We next estimate the maximal solution K of (P) by repeating the iteration defined by (2.16). Under the assumption (2.15), without loss of generality we may assume

$$w_0(x) = \beta W_0 + \text{const}$$

for some $0 \leqslant \beta < 1$. The first iteration K_1 satisfies

$$(K_1)_t - \Delta K_1 = \lambda_* |x|^\alpha e^{3W_0} - |\nabla W_0|^2 \quad \text{in } B \times (0,T)$$

with zero boundary data and with initial function w_0. We then have

$$K_1 = U + \beta v + \text{const}, \quad 0 \leqslant \beta < 1. \tag{2.20}$$

We now separately consider the following two cases.

Case 1. $N = 2$. In this case, $U \equiv 0$ and $W_0(x) = v(x,t)$. Since $\{K_n\}$ is a decreasing sequence, we obtain from (2.20) that

$$|x|^\alpha e^{3K} \leqslant |x|^\alpha e^{3K_1} \leqslant C|x|^\alpha e^{3\beta v} = C|x|^\alpha e^{3\beta W_0} \leqslant C|x|^{-2\beta} \quad \text{in } B \times [0,T],$$

and hence $|x|^\alpha e^{3K} \in X_p = L^\infty([0,T] : L^p(B))$ for some $p < \frac{N}{2\beta}$ with $0 \leqslant \beta < 1$, where p can be taken larger than $\frac{N}{2}$.

Case 2. $N > 2$. In this case, we obtain from (2.20) that

$$K_1 = U + \beta v + \text{const} = W_0 - (1-\beta)v + \text{const}, \quad 0 \leqslant \beta < 1,$$

and hence

$$\lambda_* |x|^\alpha e^{3K_1} \leqslant C \frac{e^{-3(1-\beta)v}}{|x|^2} = \frac{c(x,t)}{|x|^2},$$

where $c(x,t) \to 0$ as $(x,t) \to (0,0)$. Therefore, for any small $0 < R = R(\beta) < 1$ there exists a positive constant $C_* = C_*(R,\beta,T) < 1$ such that

$$\lambda_*|x|^\alpha e^{3K_1} - |\nabla K_1|^2 \leqslant \lambda_*|x|^\alpha e^{3K_1} \leqslant \frac{C_*(2+\alpha)(N-2)}{3|x|^2}$$
$$= C_*\big[\lambda_*|x|^\alpha e^{3W_0} - |\nabla W_0|^2\big] \quad \text{in } B \times [0,T],$$

where (2.17) is used in the equality. Taking $C_* \in (\beta, 1)$, we then get that

$$K_2 \leqslant C_* U + \beta v + \text{const} = C_* W_0 - (C_* - \beta) v + \text{const} \leqslant C_* W_0 + \text{const},$$

and hence

$$|x|^\alpha e^{3K} \leqslant |x|^\alpha e^{3K_2} \leqslant C|x|^\alpha e^{3C_* W_0} \leqslant C|x|^{-2C_*} \quad \text{in } B \times [0,T].$$

This implies again that $|x|^\alpha e^{3K} \in X_p = L^\infty([0,T]: L^p(B))$ for some $p < \frac{N}{2C_*}$ with $\beta < C_* < 1$, where p can be taken larger than $\frac{N}{2}$. This completes the proof of Lemma 2.5. □

2.2. *Global convergence or instantaneous touchdown*

In this subsection, we study the stability or instability of the solutions as $t \to \infty$. We shall prove that for $u_0 \lneqq S(x)$, the solution u of (P) globally converges to the minimal steady-state of (P); and while for $u_0 \gneqq S(x)$, the solution u of (P) instantaneously touches down at the time $t = 0$.

We first discuss the case where the initial data satisfies $u_0 \leqslant S(x)$.

Theorem 2.6. *Let u be a global solution of (P) satisfying* (2.1) *and* (2.2).

1. *If*

$$\textit{either} \quad N \geqslant 8 \textit{ and } \alpha > \alpha^{**} \quad \textit{or} \quad 2 \leqslant N \leqslant 7,$$

then we have

$$\lim_{t\to\infty} u(x,t) = u_{\lambda_*}(x) \quad \textit{in } L^2(B),$$

where $u_{\lambda_*}(x)$ is the minimal stationary solution of (P).

2. *If*

$$N \geqslant 8 \quad \textit{and} \quad 0 \leqslant \alpha \leqslant \alpha^{**},$$

then we have

$$\lim_{t\to\infty} u(x,t) = S(x) \quad \textit{in } L^2(B).$$

Proof. Since u satisfies (2.1) and (2.2), Theorem 2.1 then implies the regularity $u_t \in L^2(B \times (t_0,\infty))$ for any $t_0 > 0$. So the Hölder inequality gives that

$$\int_B \left|u(x,t)-u(x,t+s)\right|^2 dx = \int_B \left| \int_t^{t+s} \frac{\partial u}{\partial t}(x,\tau)\,d\tau \right|^2 dx \leqslant \int_B \left(s \int_t^{t+s} \left| \frac{\partial u}{\partial t}(x,\tau)\right|^2 d\tau \right) dx. \tag{2.21}$$

Note from Theorem 2.1 that u satisfies the estimates (2.3) and (2.4). Applying (2.4) gives that there exist a function $v(x)$ and a sequence $\{t_n\}$ satisfying

$$u(x,t_n) \rightharpoonup v(x) \quad \text{weakly in } H_0^1(B),$$
$$u(x,t_n) \to v(x) \quad \text{in } L^2(B) \text{ and a.e.}$$

In particular, for fixed $s>0$ it reduces from (2.3) and (2.21) that

$$\lim_{n\to\infty} \left\|u(x,t_n+s)-u(x,t_n)\right\|_2 = 0.$$

Moreover, for $u_n(x,s)=u(x,t_n+s)$ we have

$$u_n(x,s) \to v(x) \quad \text{in } L^\infty_{\mathrm{loc}}\big((\tau,\infty):L^2(B)\big) \text{ as } n\to\infty.$$

Passing to the limit in the weak form of the equation satisfied by u_n, we now conclude that $v \leqslant S$ is a weak solution of the associated stationary problem. We next separately consider the following two cases.

Case 1. $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$. For this case, it is known from [8] that (P) has a unique stationary solution $S(x)$, and hence $v(x) \equiv S(x)$ in B. This gives Theorem 2.6(2).

Case 2. We now consider the second case where

$$\text{either} \quad N \geqslant 8 \text{ and } \alpha > \alpha^{**} \quad \text{or} \quad 2 \leqslant N \leqslant 7.$$

For this case, we recall from [4,8] that (P) has a regular minimal stationary solution u_{λ_*}.

Since Theorem 2.1 implies that any solution u of (P) is bounded away from 1 for all fixed $t>0$, after slightly moving the origin of the time, we may assume that u_0 satisfies

$$0 \leqslant u_0(x) \leqslant 1-|x|^\beta \quad \text{for some } 0<\beta<\frac{2+\alpha}{3}. \tag{2.22}$$

The solutions of (P) must be unique under such an assumption. Let w be a solution of (P) with initial data $w_0 = \gamma u_{\lambda_*} + (1-\gamma)S \geqslant u_0$ for some $0<\gamma<1$. We then have $u_{\lambda_*}(x) \leqslant w_0(x) < S(x)$ in Ω, and the inequality

$$\Delta w_0 + \frac{\lambda_*|x|^\alpha}{(1-w_0)^2} \leqslant \Delta w_0 + \gamma\frac{\lambda_*|x|^\alpha}{(1-u_{\lambda_*})^2} + (1-\gamma)\frac{\lambda_*|x|^\alpha}{(1-S)^2} = 0 = (w_0)_t$$

implies that w_0 is a supersolution of (P) with initial data w_0. Since u is a solution of (P) with $u_0 \leqslant w_0$, the comparison principle now yields that

$$u(x,t) \leqslant w(x,t) \leqslant w_0(x) < S(x) \quad \text{in } B \times (0,\infty),$$

which implies that $v(x) < S(x)$ in B. We now conclude from Proposition A.1 that it must have $v(x) \equiv u_{\lambda_*}(x)$ in B.

The uniqueness of the limit follows from a classical argument in dynamical systems. Indeed, if there were two different sequences $\{t_i\}$ and $\{t_i'\}$ satisfying $u(x,t_i) \to u_{\lambda_*}(x)$ as $t_i \to \infty$, and $u(x,t_i') \to S(x)$ as $t_i' \to \infty$, respectively, then for any constant c between $\|u_{\lambda_*}\|_2$ and $\|S\|_2$, one can find out another sequence $\{t_i''\}$ satisfying $\|u(x,t_i'')\|_2 = c$ as $t_i'' \to \infty$. Since the orbit is compact in $L^2(B)$, we have a limit $w(x) = \lim_{t_i'' \to \infty} u(x,t_i'')$, which would be the third stationary solution of (P) satisfying $u_{\lambda_*} \leqslant w \leqslant S$ in B. By choosing a proper constant c, it now reduces to a contradiction with Proposition A.1 again. Therefore, the proof of Theorem 2.6(1) is complete. □

For any $N \geqslant 2$ and $\alpha \geqslant 0$, we next consider the dynamic problem (P) with initial data larger than $S(x)$, and we obtain touchdown in the strongest possible sense: it happens for any kind of weak solutions and it is complete almost everywhere in the sense of [1]—instantaneous touchdown takes place at the time $t = 0$.

Theorem 2.7. *There is no weak solution u of (P), with $u_0(x) \gneqq S(x)$ in B, defined in a domain $Q_T := B \times (0,T)$ with $T > 0$ such that*

$$u(x,t) \gneqq S(x) \quad \text{in } Q_T. \tag{2.23}$$

Proof. Let u be any solution of (P) with initial data $u_0 \gneqq S$ on B. Setting $\psi = u - S$, then ψ satisfies

$$\psi_t - \Delta\psi = \lambda_* |x|^\alpha \left[\frac{1}{(|x|^{\frac{2+\alpha}{3}} - \psi)^2} - \frac{1}{(|x|^{\frac{2+\alpha}{3}})^2} \right] \quad \text{in } Q_T \tag{2.24}$$

with initial data $\psi_0 = u_0 - S$, and with zero boundary data. Now choose a function v_0 such that $0 \leqslant S - v_0 \leqslant u_0 - S$, and let v be the maximal solution of (P) constructed in the proof of Theorem 2.1. Define $\phi = S - v$, then ϕ satisfies

$$\phi_t - \Delta\phi = \lambda_* |x|^\alpha \left[\frac{1}{(|x|^{\frac{2+\alpha}{3}})^2} - \frac{1}{(|x|^{\frac{2+\alpha}{3}} + \phi)^2} \right] \quad \text{in } Q_T \tag{2.25}$$

with initial data $\phi_0 = S - v_0 \leqslant \psi_0$, and with zero boundary data. Recalling from the iterative construction of v in the proof of Theorem 2.1, we observe that ϕ is the limit of the sequence $\{\phi_n\}$. The first iteration ϕ_1 satisfies $(\phi_1)_t - \Delta\phi_1 = 0$ in Q_T. Making use of the inequality

$$\frac{1}{(|x|^{\frac{2+\alpha}{3}} - s)^2} - \frac{1}{(|x|^{\frac{2+\alpha}{3}})^2} \geqslant \frac{2s}{(|x|^{\frac{2+\alpha}{3}})^3} \geqslant \frac{1}{(|x|^{\frac{2+\alpha}{3}})^2} - \frac{1}{(|x|^{\frac{2+\alpha}{3}} + s)^2} \quad \text{for } s \geqslant 0,$$

the standard comparison theorem of heat equations gives that $\psi \geqslant \phi_1$ in Q_T. In a similar way, we have $\psi \geqslant \phi_n$ in Q_T, and it therefore reduces to the limit $\psi \geqslant \phi$ in Q_T. This leads to

$$u(x,t) \geqslant 2S(x) - v(x,t) \quad \text{in } Q_T. \tag{2.26}$$

Since for any $t \geqslant \tau > 0$, Theorem 2.1 implies that $\|v(x,t)\|_\infty \leqslant C(\tau) < 1$, and hence we have

$$u(x,t) \geqslant 2S(x) - C(\tau) \quad \text{if } t \geqslant \tau. \tag{2.27}$$

We now define

$$a_0 = a_0(\tau) := \left(\frac{1-C(\tau)}{2}\right)^{\frac{3}{2+\alpha}}, \qquad \beta := \min\left\{1 - a_0, \frac{a_0}{2}\right\}.$$

Defining the domain

$$D := \left\{x \in B:\ a_0 - \beta \leqslant |x| \leqslant a_0 + \beta\right\},$$

then for $t \geqslant \tau$, (2.27) gives the following estimate

$$\begin{aligned}
\int_B \frac{\lambda_*|x|^\alpha\,dx}{(1-u)^2} &\geqslant \int_B \frac{\lambda_*|x|^\alpha\,dx}{(C(\tau)-1+2|x|^{\frac{2+\alpha}{3}})^2} \geqslant \int_D \frac{\lambda_*|x|^\alpha\,dx}{(C(\tau)-1+2|x|^{\frac{2+\alpha}{3}})^2} \\
&= \lambda_* N w_N \int_{a_0-\beta}^{a_0+\beta} \frac{r^{N+\alpha-1}\,dr}{(C(\tau)-1+2r^{\frac{2+\alpha}{3}})^2} \\
&= \frac{3\lambda_* N w_N}{2+\alpha} \int_{a_0-\beta}^{a_0+\beta} \frac{r^{\frac{3N-2+2\alpha}{3}}\,dr^{\frac{2+\alpha}{3}}}{(C(\tau)-1+2r^{\frac{2+\alpha}{3}})^2} \\
&\geqslant \frac{3\lambda_* N w_N}{4(2+\alpha)}(a_0-\beta)^{\frac{3N-2+2\alpha}{3}} \int_{(a_0-\beta)^{\frac{2+\alpha}{3}}}^{(a_0+\beta)^{\frac{2+\alpha}{3}}} \frac{ds}{(s-\frac{1-C(\tau)}{2})^2} = +\infty,
\end{aligned}$$

where w_N refers to the volume of the unit ball B in $\mathbb{R}^N$. This implies that $\frac{\lambda_*|x|^\alpha}{(1-u)^2} \notin L^1(B\times(\tau,T))$ for any $0<\tau<T<\infty$. Therefore, the known result for heat equation, cf. Lemma 1.4 in [1], now yields that the solution u must have a complete touchdown at time τ. Since τ is arbitrary, we conclude that u must have a complete touchdown at the time $t=0$, which is also referred to as "instantaneous touchdown." □

3. Asymptotic behavior in the inner region

Borrowing the ideas from [3,6], the rest of this paper is devoted to asymptotic behavior of global solutions for (P), provided $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**} := \frac{4-6N+3\sqrt{6}(N-2)}{4}$. In this case, recall that (P) has a unique solution u which globally converges to the singular steady-state $S(x)$. In this section, we discuss the asymptotic behavior of u in the inner region. We shall prove that in the inner region, the solution u is given asymptotically by a quasi-steady problem so that it is close to a radially symmetric stationary solution. We therefore begin with studying a family of stationary solutions for (P).

Consider the symmetric stationary equation

$$\mathbb{S}(U) := \Delta U + \frac{\lambda_* r^\alpha}{(1-U)^2} = 0, \quad U = U(r),\ r > 0. \tag{3.1}$$

Let $U_0(r)$ be the radially symmetric solution of (3.1) with the conditions

$$U_0(0) = 0, \qquad U_0'(0) = 0. \tag{3.2}$$

It is clear that $U_0(r) < 0$ and $U_0'(r) < 0$ for any $r > 0$. For the case $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**} = \frac{4-6N+3\sqrt{6}(N-2)}{4}$, we have

$$U_0(r) < S(r) = 1 - r^{\frac{2+\alpha}{3}} \quad \text{for } r \geqslant 0. \tag{3.3}$$

Moreover, if $N \geqslant 8$ and $0 \leqslant \alpha < \alpha^{**}$, then for $r \to \infty$,

$$U_0(r) = S(r) - b_0 r^{\gamma_+}\big(1 + o(1)\big) \quad \text{with } b_0 = b_0(N, \alpha) > 0, \tag{3.4}$$

where $\gamma_+ = \gamma_+(N, \alpha) < 0$ is a constant explicitly computed in (4.10). For the critical case where $N \geqslant 8$ and $\alpha = \alpha^{**}$, the asymptotic expansion of $U_0(r)$ is different:

$$U_0(r) = S(r) - b_0 r^{\frac{2-N}{2}} \ln r\big(1 + o(1)\big) \quad \text{as } r \to \infty,\ b_0 = b_0(N) > 0. \tag{3.5}$$

Applying the scaling property of Eq. (3.1), one can deduce that for any given $0 \leqslant \mu < 1$, the solution $U_\mu(r)$ of (3.1) with $U_\mu(0) = \mu$ and $U_\mu'(0) = 0$ has the form

$$U_\mu(r) = \mu + (1-\mu) U_0\big((1-\mu)^{-\frac{3}{2+\alpha}} r\big) \tag{3.6}$$

and satisfies (3.3). Given sufficiently large $\delta > 0$, then for the case $N \geqslant 8$ and $0 \leqslant \alpha < \alpha^{**}$,

$$U_\mu(r) = S(r) - b_0 (1-\mu)^{1 - \frac{3\gamma_+}{2+\alpha}} r^{\gamma_+}\big(1 + o(1)\big) \quad \text{for } r \geqslant \delta \text{ as } \mu \nearrow 1, \tag{3.7}$$

while for the case $N \geqslant 8$ and $\alpha = \alpha^{**}$,

$$U_\mu(r) = S(r) - b_0 (1-\mu)^{1 + \frac{3(N-2)}{2(2+\alpha)}} r^{\frac{2-N}{2}} \ln\big((1-\mu)^{-\frac{3}{2+\alpha}} r\big)\big(1 + o(1)\big) \quad \text{for } r \geqslant \delta \text{ as } \mu \nearrow 1. \tag{3.8}$$

In both cases, it yields for $\mu \nearrow 1$

$$U_\mu(r) = S(r) - 0 \quad \text{uniformly on } [\delta, \infty). \tag{3.9}$$

Observe that $U_\mu(r)$ is strictly increasing in μ for all $r \geqslant 0$.

3.1. Inner analysis

In this subsection, we shall show that in the inner region, the asymptotic behavior of the unique solution u for (P) is given by a slow motion of the orbit $\{u(\cdot,t),\ t>0\}$ near the family of stationary states $\{U_\mu(r),\ \mu>0\}$. In view of the uniqueness of u, an evident symmetrization and comparison argument imply that one may now assume $u(r,t)\geqslant 0$ to be symmetric and decreasing in r for all $t\geqslant 0$. Therefore, in the following we define

$$\beta(t)\equiv\sup_r u(r,t)=u(0,t)\to 1 \quad \text{as } t\to\infty. \tag{3.10}$$

Moreover, intersection comparison with the family of stationary solutions implies that we may also assume strict monotonicity

$$\beta'(t)>0 \quad \text{for all } t\gg 1, \tag{3.11}$$

see a similar comparison in Chapter 4 of [17] or in the coming Lemma 3.2.

We first establish the following slowly varying stationary structure of the solution in the inner region, a result which is quite general for such a kind of asymptotic behavior.

Lemma 3.1. *Suppose $N\geqslant 8$ and $0\leqslant\alpha\leqslant\alpha^{**}$, and let u be the unique solution of (P). Then as $t\to\infty$,*

$$u(r,t)=U_{\beta(t)}(r)\big(1+o(1)\big) \quad \textit{with } \beta(t)=u(0,t) \tag{3.12}$$

uniformly on compact subsets $\{\xi=(1-\beta(t))^{-\frac{3}{2+\alpha}}r\leqslant C\}$ with any bounded $C>0$.

Proof. We introduce the rescaled function θ satisfying

$$u(r,t)=\beta(t)+\big(1-\beta(t)\big)\theta(\xi,t), \quad \xi=\big(1-\beta(t)\big)^{-\frac{3}{2+\alpha}}r. \tag{3.13}$$

It then follows from (3.10) that

$$\theta(0,t)\equiv 0 \quad \text{and} \quad \theta\leqslant 0. \tag{3.14}$$

Substituting (3.13) into (P), and introducing a new time variable

$$\tau=\int_0^t \big(1-\beta(s)\big)^{-\frac{6}{2+\alpha}}\,ds\to\infty \quad \text{as } t\to\infty, \tag{3.15}$$

we obtain that the function $\theta(\xi,\tau)\leqslant 0$ satisfies the following parabolic equation

$$\theta_\tau=\mathbb{S}(\theta)+g(\tau)\mathcal{C}\theta, \tag{3.16}$$

where $\mathbb{S}$ is the stationary operator (3.1), the function $g(\tau)$ and the linear first-order operator $\mathcal{C}$ satisfy

$$\mathcal{C}\theta = \frac{1}{2}\theta_\xi \xi + \frac{2+\alpha}{6}(1-\theta),$$

$$g(\tau) = -\frac{6}{2+\alpha}\big(1-\beta(t)\big)^{\frac{4-\alpha}{2+\alpha}}\beta'(t) = \big[\big(1-\beta(t)\big)^{\frac{6}{2+\alpha}}\big]' < 0. \tag{3.17}$$

Note that (3.16) looks more like a time-dependent perturbation of problem (P).

One can deduce from (3.15) and (3.17) that

$$\int^{\infty} g(\tau)\,d\tau = -\frac{6}{2+\alpha}\int^{\infty}\frac{d(\beta(t))}{1-\beta(t)} = -\infty,$$

which implies that the perturbation $g(\tau)$ is not integrable in time. However, on any compact subset of ξ, we have

$$\beta'(t) = u_t(0,t) \leqslant \lim_{r\to 0}\frac{\lambda_* r^\alpha}{(1-u)^2} \leqslant \lim_{r\to 0}\frac{\lambda_* r^\alpha}{(1-\beta(t))^2}$$
$$= \lim_{r\to 0}\frac{\lambda_*[\xi(1-\beta(t))^{\frac{3}{2+\alpha}}]^\alpha}{(1-\beta(t))^2} \leqslant C\big(1-\beta(t)\big)^{\frac{\alpha-4}{2+\alpha}}.$$

Together with (3.17), this estimate shows that $g(\tau)$ is uniformly bounded on compact subsets of ξ. Therefore, the standard C^∞-interior regularity of uniformly parabolic equations gives the uniform boundedness of $\frac{1}{1-\theta}$, θ_ξ, $\theta_{\xi\xi}$, θ_τ and $\theta_{\tau\xi}$ on any compact subset of ξ.

We now claim that $g(\tau)\to 0$ as $\tau\to\infty$ on any compact subset of ξ. We argue it by a contradiction. Since g is uniformly bounded, we then may assume that there exists a sequence $\tau_k\to\infty$ such that $g(\tau_k)\to -\gamma_0 < 0$. Then the interior regularity gives that $\theta(\cdot,\tau_k+s)\to h(\cdot,s)$ uniformly on any compact subset of ξ, where h solves the autonomous equation

$$h_s = \mathbb{S}(h) - \gamma_0\mathcal{C}h, \quad s \geqslant 0,$$

and moreover,

$$h(0,s)\equiv 0, \qquad h(\xi,s)\leqslant S(\xi).$$

By the strong maximum principle, this means that $h(\xi,s)$ is the stationary solution, $h(s)\equiv V_0$ which solves the stationary equation

$$\mathbb{S}(V_0) - \gamma_0\mathcal{C}V_0 = 0, \qquad V_0(0) = 0, \qquad V_0\leqslant S.$$

For any $\gamma_0>0$, the function V_0 comes from the following relation

$$u_*(x,t) = 1 - \big[\gamma_0(T-t)\big]^{\frac{2+\alpha}{6}}\big(1-V_0(\eta)\big), \quad \eta = \frac{x}{\sqrt{\gamma_0(T-t)}},$$

where $u_*(x,t)$ is the touchdown self-similar solution of (P) with finite touchdown time T. One then concludes that $V_0(\eta)$ must intersect $S(\eta)$, because otherwise no touchdown is possible at finite time T in this problem. But this is a contradiction under the assumption $u\leqslant S$. Therefore, $g(\tau)$ must vanish at infinity on any compact subset of ξ.

Now multiplying (3.16) with $\tau = \tau_k + s$ by the test function $\chi(\xi, s)$, integrating over $\mathbb{R}^N \times \mathbb{R}_+$ and passing to the limit as $k \to \infty$, it yields from the regularity that $\theta(\cdot, \tau_k + s) \to F(\cdot, s)$ uniformly on compact subsets of ξ, where the function $F(\cdot, s)$ satisfies the limit equation of (3.16), i.e.,

$$F_s = \mathbb{S}(F) \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+.$$

It then follows from (3.14) that F satisfies the same conditions as θ. Moreover, the standard regularity theory of uniformly parabolic equations with analytic coefficients gives that F is a C^∞-function and analytic in ξ. We finally claim that

$$F(\xi, s) \equiv U_0(\xi). \tag{3.18}$$

Indeed, if (3.18) is not true, then via the Sturmian argument, (3.14) for F implies that $F(\xi, s)$ intersects the stationary solution $U_0(\xi)$ infinitely many times for all $s \geqslant 0$. Since the Sturmian argument also implies that the number of intersections between $\theta(\xi, \tau)$ and $U_0(\xi)$ cannot increase with the time, and since it was finite at initial time $\tau = 0$ due to the analyticity of both solutions, it now reduces to a contradiction. This gives (3.18).

In view of (3.6), (3.12) now follows from (3.13) and (3.18), which completes the proof of Lemma 3.1. □

The following result shows that the stabilization in (3.12) is from above.

Lemma 3.2. *Suppose $N \geqslant 8$ and $0 \leqslant \alpha \leqslant \alpha^{**}$, and let u be the unique solution of (P). Then for $\theta(\xi, t)$ defined in* (3.13)*, we have as $t \to \infty$*

$$\theta(\xi, t) \geqslant U_0(\xi) \tag{3.19}$$

uniformly on compact subsets $\{\xi = (1 - \beta(t))^{-\frac{3}{2+\alpha}} r \leqslant C\}$ with any bounded $C > 0$.

Proof. We argue via the intersection comparison with a family of stationary solutions $\{U_\mu\}$, see the method in Chapter 7 of [17]. Since the strong maximum principle yields that $u'_r > S'_r$ with $t > 0$ at $r = 1$, (3.9) gives that any stationary solution U_μ intersects $u(r, 1)$ exactly once as μ sufficiently approaches to 1. On the other hand, the Sturmian argument implies that the number of intersections $J_\mu(t)$ between the solution $u(r, t)$ and $U_\mu(r)$ cannot increase in time. Hence, $J_\mu(t) \leqslant 1$ for all $t > 0$, which means that $u(r, t) \geqslant U_\mu(r)$ if $\mu = \beta(t)$ as $t \to \infty$, i.e., we have for $\xi = (1 - \beta(t))^{-\frac{3}{2+\alpha}} r$

$$u(r, t) = \beta(t) + \big(1 - \beta(t)\big)\theta(\xi, t) \geqslant U_{\beta(t)} \equiv \beta(t) + \big(1 - \beta(t)\big)U_0(\xi) \quad \text{as } t \to \infty, \tag{3.20}$$

uniformly on compact subsets $\{\xi = (1 - \beta(t))^{-\frac{3}{2+\alpha}} r \leqslant C\}$ with any bounded $C > 0$. This gives the validity of (3.19). □

4. Asymptotic behavior in the outer region

In this section, we study the asymptotic behavior of a unique solution $u(x,t)$ for (P) in the outer region, away from the origin. For that, we set $w(x,t) = S(x) - u(x,t)$, then $w(x,t) \to 0$ as $t \to \infty$ for any $|x| \in [0,1]$, and it satisfies

$$w_t - \Delta w = \lambda_* |x|^\alpha \left[\frac{1}{(r^{\frac{2+\alpha}{3}})^2} - \frac{1}{(r^{\frac{2+\alpha}{3}} + w)^2} \right] \quad \text{in } B \times (0,\infty),$$
$$w_0 := w(x,0) = S(|x|) - u_0(x) \geqslant 0 \quad \text{in } B, \qquad w = 0 \quad \text{on } \partial B \times (0,\infty). \tag{4.1}$$

We may assume that $\frac{|x|^\alpha}{(1-w_0)^2} \in L^2(B)$ so that a standard regularity theory can be used to deduce $w(x,t) \in C^\infty((B \setminus \{0\}) \times (0,\infty))$ and $w(x,t) \geqslant 0$ in B for all $t > 0$. Consider (4.1) in the form

$$w_t = -\mathcal{A}w - F(w) \quad \text{with } \mathcal{A}w = -\Delta w - \frac{2\lambda_*}{|x|^2} w, \tag{4.2}$$

where F is the nonlinear operator

$$F(w) = \lambda_* |x|^\alpha \left[\frac{1}{(r^{\frac{2+\alpha}{3}} + w)^2} - \frac{1}{(r^{\frac{2+\alpha}{3}})^2} + \frac{2w}{(r^{\frac{2+\alpha}{3}})^3} \right] \geqslant 0 \quad \text{for } w \geqslant 0. \tag{4.3}$$

4.1. Analysis of linearized operator $\mathcal{A}$

This subsection is focussed on the analysis of the linearized operator $\mathcal{A}$, and our main results can be stated in the following lemma.

Lemma 4.1. *Suppose $N \geqslant 8$, then the operator $\mathcal{A}$ defined in* (4.2) *admits the following properties.*

1. *If $0 \leqslant \alpha < \alpha^{**}$, then the operator $\mathcal{A}$ has a unique self-adjoint Friedrichs extension, which is positive definite with a purely discrete spectrum. Moreover, the first eigenvalue λ_1 of $\mathcal{A}$ satisfies*:
$$\lambda_1 \geqslant m = \mu_1 \cdot \left(1 - \frac{8(2+\alpha)(3N+\alpha-4)}{9(N-2)^2} \right) > 0, \tag{4.4}$$
where $\mu_1 > 0$ is the first eigenvalue of $-\Delta$ in B, and the orthonormal set of eigenfunctions $\{\psi_k\}$ for $\mathcal{A}$ is complete.
2. *If $\alpha = \alpha^{**}$, then the operator $\mathcal{A}$ has a unique self-adjoint Friedrichs extension with a purely discrete spectrum of simple eigenvalues $\sigma(B) = \{\cdots < \lambda_2 < \lambda_1 < 0\}$. Moreover, the orthonormal set of eigenfunctions $\{\psi_k\}$ for $\mathcal{A}$ is complete.*

Proof. 1. First, we consider $\mathcal{A}$ in the domain $D(\mathcal{A}) = H^2(B) \cap H_0^1(B)$. Using the following well-known Hardy inequality with $N \geqslant 5$ (see, for example, the Appendix of [3])

$$\int_B \frac{w^2}{|x|^4}\,dx \leqslant k^2 \int_B |\Delta w|^2\,dx,$$

we have $\mathcal{A}w \in L^2(B)$ for all $w \in D(\mathcal{A})$, and $\mathcal{A}$ is symmetric. Multiplying (4.2) by w in $L^2(B)$, and using the regularity of the classical solution w, we get that

$$0 \leqslant \frac{1}{2}\frac{d}{dt}\|w\|_2^2 \leqslant -\|\nabla w\|_2^2 + 2\lambda_* \int_B \frac{w^2}{|x|^2}\,dx \leqslant -\gamma\|\nabla w\|_2^2, \tag{4.5}$$

where

$$\gamma = 1 - \frac{8(2+\alpha)(3N+\alpha-4)}{9(N-2)^2} > 0$$
$$\iff \quad N \geqslant 8 \text{ and } 0 \leqslant \alpha < \alpha^{**} := \frac{4-6N+3\sqrt{6}(N-2)}{4}. \tag{4.6}$$

Note that the Hardy inequality (2.9) is applied in the last inequality of (4.5). Since $\|\nabla w\|_2^2 \geqslant \mu_1\|w\|_2^2$ for any $w \in H_0^1(B)$, where $\mu_1 > 0$ is the first eigenvalue of the problem

$$-\Delta\psi = \mu\psi, \quad \psi \in H_0^1(B),$$

it now follows from (4.5) that

$$(\mathcal{A}w, w) \geqslant \gamma\|\nabla w\|_2^2 \geqslant m\|w\|_2^2, \quad m = \gamma\mu_1 > 0, \tag{4.7}$$

provided that $N \geqslant 8$ and $0 \leqslant \alpha < \alpha^{**}$.

The above analysis shows that the operator $\mathcal{A}$ is lower semibounded and positive definite. Therefore, there exists a unique Friedrichs extension of $\mathcal{A}$ (still denoted by $\mathcal{A}$), which is obtained from the quadratic form associated with $\mathcal{A}$ and satisfies the same lower bound (4.7), see e.g. p. 228 in [2]. It then follows from (4.7) that $H_0^1(B) \subseteq D(\mathcal{A})$. As long as we consider radially symmetric functions (due to the symmetrization argument of parabolic equations), we can take $N \geqslant 8$ and $0 \leqslant \alpha < \alpha^{**}$.

Solving the homogeneous problem

$$\mathcal{A}\psi = 0 \quad \text{in } B,\ \psi = \psi(r), \tag{4.8}$$

we obtain the following two linearly independent solutions

$$\psi_+ = r^{\gamma_+} \quad \text{and} \quad \psi_- = r^{\gamma_-}, \tag{4.9}$$

where $\gamma_- < \gamma_+ < 0$ are two roots of the quadratic equation

$$\gamma^2 + (N-2)\gamma + \frac{2(2+\alpha)(3N+\alpha-4)}{9} = 0,$$

so that for $N \geqslant 8$, we have

$$\gamma_\pm = \frac{1}{2}[2-N\pm\sqrt{\Delta}] < 0 \quad \text{with } \Delta = -8\alpha^2 - (24N-16)\alpha + \big(9N^2-84N+100\big). \tag{4.10}$$

Note that

$$\Delta > 0 \quad \Longleftrightarrow \quad N \geqslant 8 \text{ and } 0 \leqslant \alpha < \alpha^{**},$$

and direct calculations show that

$$\psi_+ \in L^2(B) \quad \text{and} \quad \psi_+ \in H^1(B). \tag{4.11}$$

Defining

$$\alpha_+ := \frac{4 - 6N + \sqrt{54N^2 - 216N + 208}}{4},$$

we next separately discuss the following two cases.

Case 1. $0 \leqslant \alpha \leqslant \alpha_+$. In this case, one can check that $\psi_- \notin L^2(B)$ and $\mathcal{A}$ is essentially self-adjoint on $C_0^\infty(0,1)$, see [3]. This corresponds to the "limit-point" case of a singular endpoint $r = 0$, cf. [16]. Observing from the fact

$$w = S(r) - u \leqslant S(r) = 1 - r^{\frac{2+\alpha}{3}} \ll \psi_+(r) \quad \text{as } r \to 0,$$

we impose an extra boundary condition at the singular endpoint $r = 0$: the singularities of eigenfunctions ψ_k for the operator $\mathcal{A}$ are of the type $O(r^{\gamma_+})$ as $r \to 0$, for example,

$$\psi_1(r) = ar^{\gamma_+}\big(1 + o(1)\big) \quad \text{as } r \to 0 \tag{4.12}$$

with a constant $a_1 = a_1(N, \alpha) > 0$. This corresponds to a unique Friedrichs extension of $\mathcal{A}$. The coercivity estimate (4.7) now implies that $\mathcal{A}^{-1}$ is well defined. Therefore, calculating $\mathcal{A}^{-1} f$ via a standard procedure for the Sturm–Liouville operators, we obtain an integral equation with a Hilbert–Schmidt kernel $C(x, y) \in L^2(B \times B)$, cf. p. 250 in [2]. It then follows that there exists $\{\lambda_k\}$, an increasing sequence of the eigenvalues of $\mathcal{A}$, and the corresponding eigenfunctions $\{\psi_k\}$ form an orthonormal basis in $L^2(B)$ restricted to radial functions. Also, we have that λ_1 is simple and $\psi_1(r) > 0$ in B.

Case 2. $\alpha_+ < \alpha < \alpha^{**}$. In this case, it yields from (4.9) and (4.10) that both functions $\psi_\pm \in L^2(B)$, which corresponds to the "limit-circle" case of a singular endpoints $r = 0$, cf. [16]. For this case, by assuming (4.12) and using the uniform boundedness

$$w = S(r) - u \leqslant S(r) = 1 - r^{\frac{2+\alpha}{3}} \ll \psi_\pm(r) \quad \text{as } r \to 0, \tag{4.13}$$

we obtain a unique Friedrichs extension which again plays a special role. Indeed, setting $w = \psi_- W$, then the function W satisfies the equation

$$W_t = -\mathcal{B}W - \frac{1}{\psi_-} F(\psi_- W), \tag{4.14}$$

where $\mathcal{B}$ is the linear operator

$$\mathcal{B}W = -W_{rr} - \frac{\mu - 1}{r} W_r, \quad \mu = 2 - \sqrt{\Delta} < 2, \tag{4.15}$$

and (4.13) implies the boundary condition

$$W = 0 \quad \text{at } r = 0. \tag{4.16}$$

Moreover, the equation $\mathcal{B}\phi = 0$ admits linearly independent solutions $\phi_+ = \psi_+/\psi_-$ and $\phi_- = 1$, where the latter does not satisfy (4.16). Therefore, the endpoint $r = 0$ of $\mathcal{B}$ is now in the "limit-point" case (and can be treated as a regular one) for the operator (4.15) subject to the condition (4.16). Then in the similar way of Case 1, one can conclude that $\mathcal{B}^{-1}$ is a Hilbert–Schmidt operator. It then follows that there exists an increasing sequence of eigenvalues for $\mathcal{B}$, and the corresponding eigenfunctions form an orthonormal basis in the weight space $L^2_\rho(B)$, $\rho = r^{\mu - N}$, of the radial functions. Furthermore, we obtain the similar results for the operator $\mathcal{A}$.

2. For the case $N \geqslant 8$ and $\alpha = \alpha^{**}$, we again solve the homogeneous problem (4.8), which gives the following two linearly independent solutions

$$\psi_+ = r^{\frac{2-N}{2}} \quad \text{and} \quad \psi_- = r^{\frac{2-N}{2}} \log r,$$

where we apply (4.10) and the fact

$$N \geqslant 8 \text{ and } \alpha = \alpha^{**} \quad \Longleftrightarrow \quad \Delta = -8\alpha^2 - (24N - 16)\alpha + \left(9N^2 - 84N + 100\right) = 0.$$

The calculations show $\psi_\pm \in L^2(B)$. This again corresponds to the limit-circle case of the singular endpoints $r = 0$, cf. [16]. Similar to Case 2 in Lemma 4.1(1), one can deduce that the operator $\mathcal{A}$ has a unique self-adjoint Friedrichs extension with a purely discrete spectrum of simple eigenvalues $\sigma(B) = \{\cdots < \lambda_2 < \lambda_1\}$, and the orthonormal set of eigenfunction ψ_k is complete.

For this case, we now note that the operator $\mathcal{A}$ can be simplified into

$$\mathcal{A}\psi(r) = -\psi'' - \frac{N-1}{r}\psi' - \frac{(N-2)^2}{4}\psi.$$

Then the Hardy inequality implies that the operator $\mathcal{A}$ is non-positive (semi-bounded): $\mathcal{A}\psi \leqslant 0$ for all $\psi \in D(\mathcal{A}) = H^2(B) \cap H^1_0(B)$. Therefore, we have $\lambda_k \leqslant 0$ for any $k = 1, 2, \ldots$. Finally, if $\lambda_1 = 0$ then solving $\mathcal{A}\psi(r) = \lambda_1\psi(r) = 0$ yields that the unique solution $\psi_1 = \psi_- \notin \mathcal{D}(\mathcal{A})$, a contradiction. Therefore, it must have $\lambda_1 < 0$, and we are done. $\square$

4.2. Outer analysis

In this subsection, we now discuss that the asymptotic behavior of radial solutions in the outer region which is governed by the stable manifold of operator (4.2).

Lemma 4.2. *Assume $N \geqslant 8$ and let $w(r,t)$ be a solution of* (4.1). *Suppose that λ_1 and ψ_1 are the first eigenpair of operator $\mathcal{A}$.*

1. *If $0 \leqslant \alpha < \alpha^{**}$, then $\lambda_1 > 0$ and there exists a constant $C_1 = C_1(u_0) > 0$ such that as $t \to \infty$*

$$w(r,t) = C_1 e^{-\lambda_1 t}\psi_1(r)\big(1+o(1)\big) \quad \textit{uniformly in } \{\delta \leqslant r \leqslant 1\} \textit{ with } \delta > 0. \tag{4.17}$$

2. *If $\alpha = \alpha^{**}$, then $\lambda_1 < 0$ and there exists a constant $C_2 = C_2(u_0) > 0$ such that as $t \to \infty$*

$$w(r,t) = C_2 e^{\lambda_1 t}\psi_1(r)\big(1+o(1)\big) \quad \textit{uniformly in } \{\delta \leqslant r \leqslant 1\} \textit{ with } \delta > 0. \tag{4.18}$$

Proof. 1. Following Lemma 5.1 in [3], we set $w = e^{-\lambda_1 t}v$. Then v satisfies the equation

$$v_t = -\mathcal{A}v + \lambda_1 v - e^{\lambda_1 t}F\big(e^{-\lambda_1 t}v\big) \tag{4.19}$$

with initial data $v(x,0) = v_0 \equiv w_0$, where F is defined by (4.3). It follows from Lemma 4.1 that

$$(\mathcal{A}w, w) \geqslant \lambda_1 \|w\|_2^2 \quad \text{in } D(\mathcal{A}). \tag{4.20}$$

Therefore, multiplying (4.2) by w in $L^2(B)$, and using (4.20) and inequality (4.3), we obtain that

$$\frac{d}{dt}\|w\|_2^2 \leqslant -2\lambda_1 \|w\|_2^2,$$

which implies that $\int_B v^2(r,t)\,dt$ is non-increasing in t. Recall that one may assume $u(r,t) \geqslant 0$ to be symmetric and decreasing in r for all $t \geqslant 0$. Therefore, since $w = S(r) - u \leqslant S(r) = 1 - r^{\frac{2+\alpha}{3}}$ and (4.12) gives $\psi_1(r) \sim r^{\gamma_+}$ as $r \to 0$, we deduce that there exists a (small) constant A such that

$$v(r,t) \leqslant A\psi_1(r) \quad \text{in } B \times \mathbb{R}_+. \tag{4.21}$$

The last perturbation term in (4.19) is exponentially small in the sets $\{0 \leqslant v \leqslant C\}$, and hence it is integrable. We next prove that 0 does not belong to the ω-limit set $\omega(v_0)$ of the solution v.

In order to derive a lower bound of v, we substitute the upper bound (4.21) into (4.19) to get that $v \geqslant z$, where the function $z(r,t)$ solves the following linear parabolic equation

$$z_t = -\mathcal{A}z + \lambda_1 z - e^{\lambda_1 t}F\big(e^{-\lambda_1 t}A\psi_1(r)\big) \quad \text{in } B \times \mathbb{R}_+, \tag{4.22}$$

where z has the same initial data and boundary data as v. It follows from Lemma 4.1 that the solution z of (4.22) is given by the series

$$z(r,t) = \sum_k c_k(t)\psi_k(r), \tag{4.23}$$

where the coefficients $\{c_k(t)\}$ satisfy the dynamical system

$$c_1' = -e^{\lambda_1 t}\big\langle F\big(e^{-\lambda_1 t}A\psi_1\big), \psi_1\big\rangle, \tag{4.24}$$

$$c_k' = (\lambda_1 - \lambda_k)c_k - e^{\lambda_1 t}\big\langle F\big(e^{-\lambda_1 t}A\psi_1\big), \psi_k\big\rangle, \quad k = 2, 3, \ldots. \tag{4.25}$$

One can see from (4.3) and (4.11) that the right-hand sides in (4.24), (4.25) are well defined, and the scalar product terms are exponentially small. Specially, we split the integrals of

(4.24) and (4.25) into two parts, over B_{r_0} and over $B^c_{r_0}$, where $r_0 = r_0(t)$ is chosen such that $e^{-\lambda_1 t} A\psi_1(r_0(t)) = 1$. Similar to Lemma 5.1 in [3], one then deduce that

$$e^{\lambda_1 t}\big\langle F\big(e^{-\lambda_1 t} A\psi_1\big), \psi_k\big\rangle \to 0 \quad \text{as } t \to \infty. \tag{4.26}$$

Since $\lambda_1 - \lambda_k \leqslant \lambda_1 - \lambda_2 < 0$ for all $k = 2, 3, \ldots$, we conclude that the asymptotic behavior of z as $t \to \infty$ is governed by Eq. (4.24). Furthermore, (4.24) implies that the limit value $c_1(\infty)$ is strictly positive provided that A is not too large, a condition which is essential in view of the estimate (4.21).

Passing to the limit as $t \to \infty$, we now obtain that

$$\liminf_{t\to\infty} v(r,t) \geqslant \frac{1}{2} c_1(\infty)\psi_1(r), \tag{4.27}$$

which gives the lower bound of $v(r,t)$. Multiplying (4.19) by v_t in $L^2(B_1)$ and integrating it over t, one can prove that (4.19) admits an approximate Lyapunov function which is "almost" non-increasing on the evolution orbits. Further, it deduces from (4.21), (4.26) and (4.27) that the following integral convergence holds

$$\int_1^\infty \big\|v_t(s)\big\|_2^2\, ds < \infty. \tag{4.28}$$

Fix now a sequence $\{t_j \to \infty\}$. Passing to the limit in (4.19) as $t = t_j + s \to \infty$, in view of (4.28), the standard regularity results for uniformly parabolic equations yield that the limit set $\omega(v_0)$ of v satisfies

$$\omega(v_0) \subseteq \big\{v \geqslant 0\colon\ \mathcal{A}v = \lambda_1 v,\ v \in H_0^1(B)\big\}.$$

It then follows from (4.21) and (4.27) that

$$\omega(v_0) \subseteq \big\{v \geqslant 0\colon\ v = C\psi_1(r),\ 0 < C \leqslant A\big\}.$$

The uniqueness of limit points, i.e. $\omega(v_0) = \{C_0\psi_1\}$ with $C_0 > 0$, follows from the monotonicity: multiplying (4.19) by v and using (4.3) and $\psi_+ \in L^2(B)$, it yields that $\frac{d}{dt}\|v(t)\|_2^2 < 0$. This completes the proof of Lemma 4.2(1).

2. For the case where $N \geqslant 8$ and $\alpha = \alpha^{**}$, Lemma 4.1(2) already gives $\lambda_1 < 0$. Set $w = e^{\lambda_1 t} v$, then v satisfies the equation

$$v_t = -\mathcal{A}v - \lambda_1 v - e^{-\lambda_1 t} F\big(e^{\lambda_1 t} v\big)$$

with initial data $v(x,0) = v_0 \equiv w_0$, where F is defined by (4.3). The rest proof is completely similar to that for Lemma 4.2(1), and we leave the details to the interested reader. □

5. Asymptotic expansion of global solutions

In this section, we are ready to discuss the asymptotic expansion of the unique global solution u for (P) with $N \geqslant 8$. We shall study the cases $0 \leqslant \alpha < \alpha^{**}$ and $\alpha = \alpha^{**}$ in Sections 5.1 and 5.2, respectively.

5.1. Analytical expansion for the case $0 \leqslant \alpha < \alpha^{**}$

The main purpose of this subsection is to give a rigorous proof for Theorem 1.4 concerned with the asymptotic behavior for the case $N \geqslant 8$ and $0 \leqslant \alpha < \alpha^{**}$. We first observe the formal matching expansion of the unique global solution u for (P): using (3.12) and (3.7), and setting $r = \delta$ for $\delta \ll 1$, the difference $w = S(r) - u$ satisfies

$$w(\delta, t) \approx b_0\big(1 - \beta(t)\big)^{1 - \frac{3\gamma_+}{2+\alpha}} \delta^{\gamma_+}, \quad t \gg 1,$$

where $\beta(t)$ is as in (3.10), and while γ_+ is defined by (4.10). On the other hand, substituting (4.12) into (4.17) yields that at $r = \delta$,

$$w(\delta, t) \approx C_1 e^{-\lambda_1 t} \delta^{\gamma_+}\big(1 + o(1)\big) \quad \text{as } t \to \infty. \tag{5.1}$$

These lead to the following formally asymptotic equality

$$\ln \frac{1}{1 - \beta(t)} \approx \frac{\lambda_1(2 + \alpha)}{2 + \alpha + 3|\gamma_+|} t + O(1) \quad \text{as } t \to \infty,$$

which formally implies (1.10). Further, we have the following analytic proof of Theorem 1.4(1).

Theorem 5.1. *Suppose* $N \geqslant 8$ *and* $0 \leqslant \alpha < \alpha^{**}$*, and let* $\beta(t)$ *be as in* (3.10). *Then* $\beta(t)$ *satisfies*

$$\ln \frac{1}{1 - \beta(t)} = \frac{\lambda_1(2 + \alpha)}{2 + \alpha - 3\gamma_+} t + O(1) \quad \text{as } t \to \infty,$$

where $\lambda_1 > 0$ *is the first eigenvalue of operator* $\mathcal{A}$ *defined in* (4.2)*, and* γ_+ *satisfies*

$$\gamma_+ = \frac{1}{2}\Big[2 - N + \sqrt{-8\alpha^2 - (24N - 16)\alpha + \big(9N^2 - 84N + 100\big)}\,\Big] < 0.$$

Proof. We first claim that

$$\ln \frac{1}{1 - \beta(t)} \leqslant \frac{\lambda_1(2 + \alpha)}{2 + \alpha + 3|\gamma_+|} t + O(1) \quad \text{as } t \to \infty. \tag{5.2}$$

Indeed, it follows from (3.6), (3.7) and (3.19) that for a fixed positive $r \ll 1$,

$$u(r, t) \geqslant S(r) - b_0\big(1 - \beta(t)\big)^{1 - \frac{3\gamma_+}{2+\alpha}} r^{\gamma_+}\big(1 + o(1)\big) \quad \text{as } t \to \infty.$$

Comparing with (5.1), we conclude that

$$b_0\big(1 - \beta(t)\big)^{1 - \frac{3\gamma_+}{2+\alpha}} \geqslant a_1 C_0 e^{-\lambda_1 t}\big(1 + o(1)\big) \quad \text{as } t \to \infty,$$

which implies the estimate (5.2).

In order to complete the proof of Theorem 5.1, it now suffices to prove the following lower bound estimate

$$\ln \frac{1}{1-\beta(t)} \geqslant \frac{\lambda_1(2+\alpha)}{2+\alpha+3|\gamma_+|} t + O(1) \quad \text{as } t \to \infty. \tag{5.3}$$

For that, we set

$$\bar{w}(r,t) = C_0 e^{-\lambda_1(t-T)} \psi_1(r), \tag{5.4}$$

where $T \gg 1$ is chosen such that $w_0(r) = S(r) - u_0(r) \leqslant \bar{w}(r,0)$, and $\psi_1(r)$ is the first eigenfunction of $\mathcal{A}$. Direct calculations show that $\bar{w}(r,t)$ is a supersolution of (4.1) in $B \times (0,\infty)$. Hence, $\underline{u}(r,t) = S(r) - \bar{w}(r,t)$ is a subsolution of (P) in $B \times (0,\infty)$, i.e.,

$$1 - r^{\frac{2+\alpha}{3}} - \bar{w}(r,t) \leqslant u(r,t) \quad \text{in } B \times (T,\infty).$$

Since the maximum principle implies that $u'_r > 0$ in $B \times (0,\infty)$, we now conclude that

$$\begin{aligned} \frac{1}{1-\beta(t)} &= \frac{1}{1-u(0,t)} \geqslant \frac{1}{1-u(r,t)} \\ &\geqslant \sup_{r \in (0,1)} \frac{1}{r^{\frac{2+\alpha}{3}} + C_0 e^{-\lambda_1(t-T)} \psi_1(r)} \quad \text{as } t \to \infty. \end{aligned} \tag{5.5}$$

In view of (4.12), the supremum in (5.5) is attained at $r \approx e^{\frac{3\lambda_1}{3\gamma_+ - 2 - \alpha} t}$, which leads to the lower bound estimate (5.3), and we are done. $\square$

5.2. Formal expansion for the critical case $\alpha = \alpha^{**}$

In this subsection, we discuss the asymptotic expansion of global solution u for (P) in the critical case $N \geqslant 8$ and $\alpha = \alpha^{**}$. We first note that in this critical case, one cannot derive a similar upper bound estimate to (5.2) for $\beta(t)$ defined by (3.10). Indeed, even though one can obtain from (3.6), (3.8) and (3.19) that for a fixed positive $r \ll 1$,

$$u(r,t) \geqslant S(r) - b_0\big(1-\beta(t)\big)^{1+\frac{3(N-2)}{2(2+\alpha)}} r^{\frac{2-N}{2}} \left[-\frac{3}{2+\alpha} \ln\big(1-\beta(t)\big) + \ln r \right] \big(1+o(1)\big)$$
$$\text{as } t \to \infty,$$

and also, Lemma 4.1(2) and (4.19) give that for $\lambda_1 < 0$

$$w(\delta,t) = S(\delta) - u(\delta,t) \approx C_2 e^{\lambda_1 t} \delta^{\frac{2-N}{2}} \big(1+o(1)\big) \quad \text{as } t \to \infty,$$

these two estimates then lead to

$$b_0\big(1-\beta(t)\big)^{1+\frac{3(N-2)}{2(2+\alpha)}} \left[-\frac{3}{2+\alpha} \ln\big(1-\beta(t)\big) + \ln r \right] \geqslant C_2 e^{\lambda_1 t} \big(1+o(1)\big),$$

which cannot result in an upper bound estimate, due to the appearance of logarithmical term $\ln r$. However, similar to (5.3), we can obtain the following lower bound estimate which gives Theorem 1.4(2).

Lemma 5.2. *Suppose $N \geqslant 8$ and $\alpha = \alpha^{**}$, and let $\beta(t)$ be as in* (3.10). *Then $\beta(t)$ satisfies the following estimate*

$$\ln \frac{1}{1-\beta(t)} \geqslant \frac{2(2+\alpha)|\lambda_1|}{3N+2\alpha-2} t + C_1 \quad \text{as } t \to \infty, \tag{5.6}$$

where $\lambda_1 < 0$ is the first eigenvalue of operator $\mathcal{A}$ defined in (4.2).

Proof. Similar to the proof of (5.3), we can deduce that for sufficiently large T,

$$1 - r^{\frac{2+\alpha}{3}} - Ce^{\lambda_1(t-T)}\psi_1(r) \leqslant u(r,t) \quad \text{in } B \times (T,\infty),$$

where $\lambda_1 < 0$ and $\psi_1(r)$ are the first eigenpair of operator $\mathcal{A}$. This gives that

$$\frac{1}{1-\beta(t)} = \frac{1}{1-u(0,t)} \geqslant \frac{1}{1-u(r,t)} \geqslant \sup_{r\in(0,1)} \frac{1}{r^{\frac{2+\alpha}{3}} + Ce^{\lambda_1(t-T)}\psi_1(r)} \quad \text{as } t \to \infty. \tag{5.7}$$

Note that the supremum in (5.7) is attained at $r \approx e^{\frac{6\lambda_1}{3N+2\alpha-2}t}$, which then yields the estimate of (5.6). □

It should remark that the second term C_1 in the estimate (5.6) is not optimal. In the following, we apply formal asymptotic analysis to this problem, and we show that the optimal term is given by a logarithmically growing function, which is hard to be detected rigorously.

For applying formal asymptotic analysis, we now consider the inner problem of (P) as follows: similar to Section 3, for $t \gg 1$ we introduce the inner scalings

$$u(r,t) = \beta(t) + \big(1-\beta(t)\big)\Phi_0(\xi,t), \quad \xi = \big(1-\beta(t)\big)^{-\frac{3}{2+\alpha}} r, \tag{5.8}$$

where $\beta(t) = \max_r u(r,t) = u(0,t)$ remains to be determined. The proof of Lemma 3.1 gives that the leading-order balance of Φ_0 is then quasi-steady as $t \to \infty$, which is given by the following initial value problem

$$\Phi_0'' + \frac{N-1}{\xi}\Phi_0' + \frac{\lambda_* \xi^\alpha}{(1-\Phi_0)^2} = 0, \quad \xi \in (0,1), \qquad \Phi_0(0) = \Phi_0'(0) = 0, \tag{5.9}$$

such that Φ_0 corresponds to U_0 defined in Section 3. Setting $\Phi_0(\xi) = 1 - \Psi_0(\xi)$ yields that

$$\Psi_0'' + \frac{N-1}{\xi}\Psi_0' = \frac{\lambda_* \xi^\alpha}{\Psi_0^2}. \tag{5.10}$$

The far-field behavior of (5.10) is given by

$$\Psi_0 \sim \xi^{\frac{2+\alpha}{3}} + \mu(\xi) + \cdots \quad \text{as } \xi \to \infty,$$

where $\mu = \mu(\xi)$ satisfies

$$\xi^2\mu'' + (N-1)\xi\mu' + 2\lambda_*\mu = \xi^2\mu'' + (N-1)\xi\mu' + \frac{(N-2)^2}{4}\mu = 0 \quad \text{as } \xi \to \infty.$$

This shows that

$$\mu(\xi) = a\xi^{\frac{2-N}{2}} + b\xi^{\frac{2-N}{2}}\ln\xi + \cdots \quad \text{as } \xi \to \infty.$$

In terms of Φ_0, we thus have as $t \to \infty$

$$\Phi_0(\xi,t) \sim 1 - \xi^{\frac{2+\alpha}{3}} - \xi^{\frac{2-N}{2}}(A_0\ln\xi + B_0) + \cdots \quad \text{as } \xi \to \infty, \tag{5.11}$$

where the constants $A_0 > 0$ and B_0 are determined by the initial value data in (5.9).

We next consider the outer problem of (P). The leading-order outer expansion is $u(r,t) \sim 1 - r^{\frac{2+\alpha}{3}}$ as $t \to \infty$, which is given by the exact unique solution of the corresponding stationary problem matching the leading term in (5.11). Thus, we write

$$u(r,t) \sim 1 - r^{\frac{2+\alpha}{3}} - v(r,t) + \cdots \quad \text{as } t \to \infty,$$

and the linearized equation of v is determined by

$$v_t = \frac{1}{r^{N-1}}\big(r^{N-1}v_r\big)_r + \frac{(N-2)^2}{4r^2}v \quad \text{in } (0,1)\times\mathbb{R}_+, \qquad v(1,t) \equiv 0. \tag{5.12}$$

Because of the appearance of the logarithmic correction term in (5.11), we apply a moderated version of the method of separation of variables by writing

$$v(r,t) = e^{-\frac{4\nu_1^2}{(N-2)^2}t}\big[\gamma(t)\phi_0 + \gamma'(t)\phi_1 + \cdots\big],$$

where the constant ν_1 and the "slowly varying" function $\gamma(t)$ will be determined later. It then follows from (5.12) that

$$r^2\phi_0'' + (N-1)r\phi_0' + \left(\frac{(N-2)^2}{4} + \frac{4\nu_1^2 r^2}{(N-2)^2}\right)\phi_0 = 0,$$

$$r^2\phi_1'' + (N-1)r\phi_1' + \left(\frac{(N-2)^2}{4} + \frac{4\nu_1^2 r^2}{(N-2)^2}\right)\phi_1 = r^2\phi_0.$$

Writing $\phi_0 = r^{\frac{2-N}{2}}\sigma_0$ and $\phi_1 = r^{\frac{2-N}{2}}\sigma_1$ yields that

$$r^2\sigma_0'' + r\sigma_0' + \frac{4\nu_1^2 r^2}{(N-2)^2}\sigma_0 = 0, \tag{5.13}$$

$$r^2\sigma_1'' + r\sigma_1' + \frac{4\nu_1^2 r^2}{(N-2)^2}\sigma_1 = r^2\sigma_0. \tag{5.14}$$

Here (5.13) is the zeroth-order Bessel's equation. Since (5.11) yields, in outer variables, the matching condition

$$v(r,t) \sim \bigl(1-\beta(t)\bigr)^{1+\frac{3(N-2)}{2(2+\alpha)}} r^{\frac{2-N}{2}} \left(-\frac{3A_0}{2+\alpha}\ln\bigl(1-\beta(t)\bigr) + A_0 \ln r + B_0\right) \quad \text{as } t \to \infty, \tag{5.15}$$

the leading-order matching now requires that the generic form of the outer solution is $\sigma_0(r) = J_0(\frac{2\nu_1}{N-2}r)$ with

$$\gamma(t) e^{-\frac{4\nu_1^2}{(N-2)^2}t} \sim -\frac{3A_0}{2+\alpha}\bigl(1-\beta(t)\bigr)^{1+\frac{3(N-2)}{2(2+\alpha)}} \ln\bigl(1-\beta(t)\bigr) \quad \text{as } t \to \infty. \tag{5.16}$$

Hence, ν_1 is chosen to be the first zero of the zeroth-order Bessel function: $J_0(\frac{2\nu_1}{N-2}) = 0$. We now denote $K_1(r) = J_1(\frac{2\nu_1}{N-2}r)$ the solution of the first-order Bessel's function. In view of the recurrence relations

$$\sigma_0' = -K_1, \qquad K_1' = \sigma_0 - \frac{1}{r}K_1,$$

it then follows from (5.13) and (5.14) that

$$r\bigl(\sigma_0\sigma_1' - \sigma_1\sigma_0'\bigr) = -\int_r^1 s\sigma_0^2(s)\,ds = -\frac{1}{2}J_1^2\left(\frac{2\nu_1}{N-2}\right) + \frac{1}{2}r^2\left[J_0^2\left(\frac{2\nu_1 r}{N-2}\right) + J_1^2\left(\frac{2\nu_1 r}{N-2}\right)\right].$$

Since $\sigma_0(0) = \text{const}$, this identity leads to

$$\sigma_1(r) = -\frac{1}{2}J_1^2\left(\frac{2\nu_1}{N-2}\right)\ln r + O(1) \quad \text{as } r \to 0.$$

Matching with the $\ln r$ term in (5.15) requires that

$$-\frac{1}{2}J_1^2\left(\frac{2\nu_1}{N-2}\right)\gamma'(t) e^{-\frac{4\nu_1^2}{(N-2)^2}t} \sim A_0\bigl(1-\beta(t)\bigr)^{1+\frac{3(N-2)}{2(2+\alpha)}}. \tag{5.17}$$

Therefore, we deduce from (5.16) and (5.17) that, by writing

$$\ln\frac{1}{1-\beta(t)} \sim \frac{8(2+\alpha)\nu_1^2}{(3N+2\alpha-2)(N-2)^2}t + \alpha_1(t) + \cdots \quad \text{as } t \to \infty, \tag{5.18}$$

$\gamma(t)$ is determined by

$$\gamma(t) \sim \frac{24A_0\nu_1^2 t}{(3N+2\alpha-2)(N-2)^2}\exp\left[-\left(1+\frac{3(N-2)}{2(2+\alpha)}\right)\alpha_1(t)\right],$$
$$\gamma'(t) \sim -\frac{2A_0}{J_1^2(\frac{2\nu_1}{N-2})}\exp\left[-\left(1+\frac{3(N-2)}{2(2+\alpha)}\right)\alpha_1(t)\right].$$

And hence,

$$\gamma(t) \sim \gamma_\infty t^{-\frac{(3N+2\alpha-2)(N-2)^2}{12J_1^2(\frac{2\nu_1}{N-2})\nu_1^2}} + \cdots \quad \text{as } t \to \infty$$

for some $\gamma_\infty > 0$ depending only on the initial data. This leads to

$$\alpha_1(t) \sim \frac{2(2+\alpha)}{3N+2\alpha-2}\left(1 + \frac{(3N+2\alpha-2)(N-2)^2}{12J_1^2(\frac{2\nu_1}{N-2})\nu_1^2}\right)\ln t \\ + \frac{2(2+\alpha)}{3N+2\alpha-2}\ln\frac{24A_0\nu_1^2}{(3N+2\alpha-2)(N-2)^2\gamma_\infty} + \cdots \quad \text{as } t \to \infty. \tag{5.19}$$

Finally, we conclude from (5.18) and (5.19) that

$$\ln\frac{1}{1-\beta(t)} \sim \frac{8(2+\alpha)\nu_1^2}{(3N+2\alpha-2)(N-2)^2}t + \frac{2(2+\alpha)}{3N+2\alpha-2}\left(1 + \frac{(3N+2\alpha-2)(N-2)^2}{12J_1^2(\frac{2\nu_1}{N-2})\nu_1^2}\right)\ln t \\ + \frac{2(2+\alpha)}{3N+2\alpha-2}\ln\frac{24A_0\nu_1^2}{(3N+2\alpha-2)(N-2)^2\gamma_\infty} + \cdots \quad \text{as } t \to \infty. \tag{5.20}$$

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Appendix A. Ordering properties of stationary solutions

In this appendix, we discuss an ordering property of weak-$H_0^1(\Omega)$ stationary solutions for (P), which is applied in Theorem 2.6. For more generality, we focus on the stationary solutions of (1.1) in the weak sense, i.e., we consider weak-$H_0^1(\Omega)$ solutions of

$$\begin{cases} -\Delta u = \dfrac{\lambda f(x)}{(1-u)^2}, & x \in \Omega, \\ 0 < u < 1, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag*{$(S)_\lambda$}$$

Let u be any weak solution of $(S)_\lambda$, we first consider weak solutions of the problem

$$\begin{cases} -\Delta v = g(x), & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases} \tag{A.1}$$

where $g(x) = \frac{\lambda f(x)}{[1-u(x)]^2}$. Let $G(x,\xi,\Omega)$ be Green's function of Laplace operator, with $G(x,\xi,\Omega) = 0$ on $\partial\Omega$, it is then clear that $u(x)$ and

$$(Fu)(x) = \int_{\Omega} \frac{\lambda f(\xi)}{[1-u(\xi)]^2} G(x,\xi,\Omega)\,d\xi \quad \text{in } \Omega \tag{A.2}$$

are two solutions of (A.1). However, one can note that for any $K(x) \in H_0^1(\Omega)$,

$$\int_{\Omega} g(x)K(x)\,dx = \int_{\Omega} \frac{\lambda f(x)K(x)}{[1-u(x)]^2}\,dx = -\int_{\Omega} K\,\Delta u\,dx = \int_{\Omega} \nabla K \nabla u\,dx < \infty,$$

which implies that $g(x) \in H_0^{-1}(\Omega)$. Therefore, (A.1) has a unique solution, which further gives that $u \equiv Fu$ on Ω.

Since $u \equiv Fu$ on Ω holds for any weak solution u of $(S)_\lambda$, in the following we adopt Fujita's argument [5], which was used only for classic solutions of elliptic equations with exponential nonlinearities, to derive the ordering property of weak solutions for singular elliptic equations $(S)_\lambda$. For convenience, we write $w_1 \preccurlyeq w_2$ on Ω if there exists $\gamma > 0$ such that $\gamma\rho \leqslant w_2 - w_1$ on Ω, where $\rho = \rho(x)$ is the distance from x to $\partial\Omega$. Note that if $w_1 \leqslant w_2$ are two different solutions of $(S)_\lambda$, then E. Hopf's maximum principle gives $w_1 \preccurlyeq w_2$ on Ω. Using this denotation, we now establish the following ordering property in the weak sense.

Proposition A.1. *Suppose that $u(x)$ and $v(x)$ are two different solutions of $(S)_\lambda$ satisfying $u(x) \leqslant v(x)$ on Ω, where $v(x)$ may be singular. If $u(x)$ is a classic solution, then it must have $u \equiv u_\lambda$ on Ω, where u_λ is the unique minimal solution of $(S)_\lambda$.*

Proof. On the contrary, suppose that $(S)_\lambda$ has three different solutions $u_\lambda \leqslant u \leqslant v$ on Ω. Then E. Hopf's maximum principle gives $u_\lambda \preccurlyeq u \preccurlyeq v$ on Ω, and hence there exist two constants $0 < \gamma_1, \gamma_2 < 1$ such that

$$\gamma_1(u - u_\lambda) \leqslant v - u, \qquad \gamma_2(v - u) \leqslant u - u_\lambda.$$

We first establish the following two claims.

Claim 1. *Let $\beta \in (0,1)$ be any positive constant, and define*

$$\beta_n = (\gamma_1\gamma_2)^n\beta, \qquad a_n = (1-\beta_n)u + \beta_n v, \qquad b_n = u + \beta_n\gamma_1(u - u_\lambda), \quad n = 0,1,2,\ldots.$$

Then there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of solutions φ_n for $(S)_\lambda$ satisfying

$$a_n \leqslant b_{n-1} \leqslant \varphi_n \leqslant a_{n-1}, \quad n = 1,2,\ldots. \tag{A.3}$$

Furthermore, the sequence $\{\varphi_n\}_{n=1}^{\infty}$ uniformly converges to $\varphi_0 := u$.

Since the direct calculation gives $b_{n-1} - a_n = \beta_{n-1}\gamma_1[(u - u_\lambda) - \gamma_2(v - u)] \geqslant 0$ for any $n \geqslant 1$, in the following we need only to prove that there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of solutions φ_n for $(S)_\lambda$ satisfying

$$b_{n-1} \leqslant \varphi_n \leqslant a_{n-1}, \quad n = 1, 2, \ldots. \tag{A.4}$$

We first prove (A.4) for $n = 1$. To this end, we define $\{v^k\}_{k=0}^{\infty}$ and $\{w^k\}_{k=0}^{\infty}$ satisfying

$$v^0 = a_0, \qquad v^{k+1} = Fv^k, \quad k = 0, 1, 2, \ldots, \tag{A.5}$$

$$w^0 = b_0, \qquad w^{k+1} = Fw^k, \quad k = 0, 1, 2, \ldots, \tag{A.6}$$

respectively. When $k = 0$, in view of (A.2), the convexity of $\frac{1}{(1-u)^2}$ gives that

$$v^1 = Fv^0 \leqslant (1 - \beta)Fu + \beta Fv = (1 - \beta)u + \beta v = v^0 \quad \text{in } \Omega. \tag{A.7}$$

Since Fv^i is monotone with respect to the order relation of v^i, we now deduce that $\{v^k\}_{k=0}^{\infty}$ form a decreasing sequence. Similarly, one can prove that $\{w^k\}_{k=0}^{\infty}$ form an increasing sequence. On the other hand, since $v^0 - w^0 = a_0 - b_0 = \beta[(v - u) - \gamma_1(u - u_\lambda)] \geqslant 0$, the iteration of (A.5) shows that $v^k \geqslant w^k$ for all $k \geqslant 0$. So we have

$$w^0 \leqslant w^1 \leqslant \cdots \leqslant w^k \leqslant \cdots \leqslant v^k \leqslant \cdots \leqslant v^1 \leqslant v^0.$$

Therefore, v^k converges to some function φ_1 with $b_0 = w^0 \leqslant \varphi_1 \leqslant v^0 = a_0$, where φ_1 is a solution of $(S)_\lambda$ in view of (A.5). This proves (A.4) for $n = 1$.

Similarly, for any $n \geqslant 2$ one can obtain that there exists a solution φ_n of $(S)_\lambda$ satisfying (A.4). Finally, note from (A.3) that the sequence $\{\varphi_n\}_{n=1}^{\infty}$ uniformly converges to $\varphi_0 := u$. This completes the proof of Claim 1.

Claim 2. *Let $\{\varphi_n\}_{n=0}^{\infty}$ be as in Claim 1, then for any $n \geqslant 0$, the linearized boundary value problem*

$$\begin{cases} -\Delta w = \dfrac{2\lambda f(x)}{(1 - \varphi_n)^3} w, & x \in \Omega, \\ w = 0, & x \in \partial\Omega, \end{cases} \tag{A.8}$$

has a non-trivial solution.

We first prove Claim 2 for $n = 0$. On the contrary, suppose that (A.8) has only a trivial solution. Then the self-adjoint operator $\mathcal{H}_0$ in $L^2(\Omega)$

$$\begin{cases} \mathcal{H}_0 u = \Delta u + \dfrac{2\lambda f(x)}{(1 - \varphi_0)^3} u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

is one-to-one. Therefore, $\mathcal{H}_0^{-1}$ exists and it is a bounded operator, which is denoted by R_0 with L^∞-norm. If we can prove that the solutions of $(S)_\lambda$ must be unique in a certain neighborhood of φ_0, then we reach a contradiction with Claim 1.

It now suffices to prove the uniqueness of solutions for $(S)_\lambda$ in a certain neighborhood of φ_0. Indeed, since $\varphi_0 := u$ is a classic solution, we can fix $0 < \delta \leqslant \frac{1-\|\varphi_0\|_\infty}{2}$, and determine a positive constant $M = M(\delta)$ such that $\frac{3\lambda f(x)}{(1-t)^4} \leqslant M$ holds for $0 \leqslant t \leqslant \|\varphi_0\|_\infty + \delta$. Further, $\delta > 0$ can be chosen sufficiently small such that $\delta < 1/(M\|R_0\|)$. For such a choice of δ, we now suppose that U is any solution of $(S)_\lambda$ satisfying $|U - \varphi_0| < \delta$ in Ω. Then for $w = U - \varphi_0$, we have

$$\mathcal{H}_0 w = \Delta w + \frac{2\lambda f(x) w}{(1-\varphi_0)^3} = -\frac{\lambda f(x)}{[1-(w+\varphi_0)]^2} + \frac{\lambda f(x)}{(1-\varphi_0)^2} + \frac{2\lambda f(x) w}{(1-\varphi_0)^3}.$$

And hence

$$\begin{aligned}|w| &= \left| R_0\left[-\frac{\lambda f(x)}{[1-(w+\varphi_0)]^2} + \frac{\lambda f(x)}{(1-\varphi_0)^2} + \frac{2\lambda f(x) w}{(1-\varphi_0)^3}\right]\right| \leqslant \|R_0\| \left| \frac{3\lambda f(x) w^2}{(1-\|\varphi_0\|_\infty - \delta)^4} \right| \\ &\leqslant M\|R_0\||w|^2 < |w|,\end{aligned}$$

which implies $w \equiv 0$, i.e., $U \equiv \varphi_0$ on Ω. This proves Claim 2 for $n = 0$.

The above analysis shows that if $u_\lambda \leqslant \varphi_0 := u \leqslant v$ in Ω, we then have Claim 2 for $n = 0$. If one replaces $u_\lambda \leqslant u \leqslant v$ by $\varphi_{n+1} \leqslant \varphi_n \leqslant \varphi_{n-1}$, then a similar proof as above gives Claim 2 for any $n \geqslant 1$, and we are done.

We are now ready to complete the proof of Proposition A.1. We introduce the self-adjoint operator $\mathcal{H}_k$ in $L^2(\Omega)$

$$\begin{cases} \mathcal{H}_k u = \Delta u + \dfrac{2\lambda f(x)}{(1-\varphi_k)^3} u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $k = 0, 1, 2, \ldots$. Then Claim 2 implies that 0 is an eigenvalue of any operator $\mathcal{H}_k$. If we put $V_k = \mathcal{H}_k - \mathcal{H}_0$, then V_k is a multiplication by

$$q_k = \frac{2\lambda f(x)}{(1-\varphi_k)^3} - \frac{2\lambda f(x)}{(1-\varphi_0)^3},$$

and hence V_k is a positive definite operator. Moreover, we have

$$m_0 \beta_k f(x)(v-u) \leqslant q_k \leqslant m_1 \beta_{k-1} f(x)(v-u), \tag{A.9}$$

where m_0 and m_1 are positive constants satisfying $m_0 \leqslant \frac{6\lambda}{(1-t)^4} \leqslant m_1$ for $0 \leqslant t \leqslant \|\varphi_k\|_\infty$. By means of (A.9), we can estimate eigenvalues of $\mathcal{H}_k$ close to 0 with the aid of perturbation theory of eigenvalues. In particular, applying a theorem in [12] for estimating eigenvalues from below, one can shows that if $\mathcal{H}_0$ has 0 as an eigenvalue (simple or degenerate), then for large k this eigenvalue is moved to the right by a positive perturbation V_k. Thus, 0 cannot be an eigenvalue of $\mathcal{H}_k$ for large k, which contradicts Claim 2. This completes the proof of Proposition A.1. □

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