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# Boundedness Relations in Linear Semi-infinite Programming

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This paper gives theorems on the boundedness of the feasible and the optimal solutions sets of a dual pair of linear semi-infinite programs (in Haar's duality). It also provides conditions for the boundedness of the primal slack variables and dual structural variables. 

## 1. INTRODUCTION

We shall consider the following dual pair of problems in Semi-Infinite Linear Programming:

$$\operatorname{Inf} c'x \quad \text{s.t. } a_t'x \geq b_t, \qquad t \in T, \qquad x \in R^n \tag{P}$$

and

$$\operatorname{Sup} \sum_{t \in T} \lambda_t b_t \quad \text{st. } \sum_{t \in T} \lambda_t a_t = c, \qquad \lambda \in R_+^{(T)}, \tag{D}$$

where $R_+^{(T)} = \{\lambda\colon\ T \to R_+/\operatorname{supp}\lambda \text{ is finite}\}$, $\operatorname{supp}\lambda = \{t \in T/\lambda_t > 0\}$ and $T \neq \varnothing$.

The feasible sets of (P) and (D) will be represented by $F$ and $\Lambda$, and their values by $v(P)$ and $v(D)$, respectively. The optimal set of (P) will be denoted $F^* = \{x \in F/c'x = v(P)\}$, whereas the optimal set of (D) will be denoted $\Lambda^* = \{\lambda \in \Lambda/\sum_{t \in T}\lambda_t b_t = v(D)\}$.

In the case of $T$ being a finite set, (P) and (D) become a dual pair in ordinary LP. In that case boundedness relations between both problems have been established by Clark [2], whose theorem has been extended by Duffin [3] to convex programs, Williams [18, 19] and Nozicka [16]. The



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main results in this paper provide different characterizations of the boundedness of the sets $F$, $F^*$, $\Lambda$, and $\Lambda^*$, as well as some relations between them. Obviously, we must define what a bounded set in $R_+^{(T)}$ is.

We say that $\Gamma \subset R_+^{(T)}$, $\Gamma \neq \varnothing$, is *bounded* if it is contained in some generalized cube, i.e., if there exists a $M > 0$ such that $\lambda_t \leq M$ for all $t \in T$ and for all $\lambda \in \Gamma$. Other concepts generalizing the bounded sets in a finite-dimensional euclidean space are available in order to generalize the known boundedness relations of Linear Programming to Semi-Infinite Programming (see, e.g., the papers due to Kortanek and Strojwas [14, 15]), but the uniform boundedness considered throughout this paper results suitable enough for our purpose.

We can advance that the boundedness conditions for the dual sets usually require some assumptions concerning to the linear representation of $F$.

We represent by $\sigma = \{a_t'x \geq b_t,\ t \in T\}$ the constraints system of (P) and by $\sigma_0 = \{a_t'x \geq 0,\ t \in T\}$ the corresponding homogeneous system. The feasible set of $\sigma_0$ will be denoted $F_0$ (obviously, $O_n \in F_0$). We associate with $\sigma$ the convex cone generated by $\{a_t,\ t \in T\}$, $M_n(\sigma)$, and by $\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T;\ \begin{pmatrix} O_n \\ -1 \end{pmatrix}\right\}$, $K(\sigma)$.

We say that $\sigma$ is Farkas–Minkowski (FM) if $K(\sigma)$ is closed and $F \neq \varnothing$. It is easily proved that $\sigma_0$ is FM if, and only if, $M_n(\sigma)$ is closed. We say that a consistent system is *compact* if there exists a function $\alpha_t$: $T \to R_+/\{0\}$ such that $\left\{\alpha_t \begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T\right\}$ is a compact set. The relations between these two classes of systems have been analyzed in [6].

The first paper extending to semi-infinite programs the theorems of Clark, Williams, and Nozicka is due to Eckhardt [4], who imposes a too hard general assumption: $\sigma_0$ must be normal, i.e., $\{a_t,\ t \in T\}$ is a closed subset of the unit sphere in $R^n$ and $M_n(\sigma)$ is closed (the closedness condition for $\{a_t,\ t \in T\}$ is implicitly used in the proof of some results). We shall generalize Eckhardt's theorems and give others completely new.

Now let us introduce a bit of notation. Given a consistent system $\sigma$, we say that $t \in T$ is *unstable* if $a_t'x = b_t$ for all $x \in F$. The set of all the unstable indices will be denoted by $I$. Similarly, $I_0$ will represent the set of unstable indices in $\sigma_0$. The inequality $O_n'x \geq 0$ will be called the *trivial inequality*.

Given a set $X \subset R^p$, $X \neq \varnothing$, we denote by cl $X$, int $X$, bd $X$, rb $X$, conv $X$, $K(X)$, $L(X)$, $X^\perp$, $X^0$, and dim $X$ the closure of $X$, the interior of $X$, the boundary and relative boundary of $X$, the convex hull of $X$, the convex cone generated by $X$, the linear subspace generated by $X$, the linear subspace of all the vectors which are orthogonal to $X$, the positive polar cone of $X$, and the dimension of $X$, respectively. Moreover, rank $X =$ dim $L(X)$. We always consider in $R^p$ the euclidean norm: $\|x\| = \{\sum_{i=1}^p (x_i)^2\}^{1/2}$.

The following theorems, concerning (P) and (D) will be used in what follows.

THEOREM 1.1. *Let us assume that* $F \neq \emptyset$. *Then, the inequality* $a'x \geq b$ *is a consequence of* $\sigma$ *if, and only if,* $\binom{a}{b} \in \operatorname{cl} K(\sigma)$.

This result constitutes a specification of Theorem 2 in [20]. A direct proof of this generalized Farkas Theorem can be found in [9, Theorem 2.1].

THEOREM 1.2. *If either* $\operatorname{conv}\{a_t,\ t \in T\}$ *or* $M_n(\sigma)$ *is closed, then one, and only one, of the following propositions holds*:

(I) $\sigma_0$ *has strict solutions.*

(II) $O_n \in \operatorname{conv}\{a_t,\ t \in T\}$.

This generalization of Gordan's Alternative Theorem is proved in [10, Theorem 1.2].

THEOREM 1.3. *If* $\sigma$ *is* FM, *then the following formula holds*:

$$\dim F = n - \operatorname{rank}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in I\right\}.$$

This theorem is proved in [7, Theorem 3.2].

THEOREM 1.4. *If* $c \in \operatorname{ri} M_n(\sigma)$ *and* $F \neq \emptyset$, *then* (P) *is solvable* (*i.e.*, $F^* \neq \emptyset$) *and* $v(\mathrm{P}) = v(\mathrm{D})$.

(See [8, Lemma 2.3].)

## 2. BOUNDED FEASIBLE SETS

Let us denote $\Lambda_0 = \{\lambda \in R_+^{(T)} / \sum_{t \in T} \lambda_t a_t = O_n\}$.

THEOREM 2.1. *If* $F \neq \emptyset$, *the following propositions are equivalent to each other*:

(I) $F$ *is bounded.*

(II) $F_0 = \{O_n\}$.

(III) $M_n(\sigma) = R^n$.

(IV) *There exists a finite subsystem of* $\sigma$ *whose feasible set is bounded.*

*Moreover, if* $\operatorname{rank}\{a_t,\ t \in T\} = n$, *the following propositions are also equivalent to the preceding ones*:

(V) $I_0 = T$.

(VI) $\bigcup_{\lambda \in \Lambda_0} \operatorname{supp} \lambda = T$.

*Proof.* We shall prove (I) → (II) → (III) → (IV), whereas (IV) → (I) is trivial. When the additional assumption holds, we shall prove (III) → (VI) → (V) → (II).

(I) → (II) If $F$ is bounded, $O^+F = \{O_n\}$. But $O^+F = F_0$.

(II) → (III) By Theorem 1.1, $(O^+F)^0 = \text{cl}\, M_n(\sigma)$. Consequently, $\text{int}\, M_n(\sigma) = \text{int}\,\text{cl}\, M_n(\sigma) = R^n$.

(III) → (IV) If $\{e^1, \dots, e^n\}$ is the canonical basis in $R^n$, we can write $e^i = \sum_{t \in T} \lambda_t^i a_t$ and $-e^i = \sum_{t \in T} \gamma_t^i a_t$, for some $\lambda^i$, $\gamma^i \in R_+^{(T)}$. Let us consider the following finite subsystem of $\sigma$,

$$\sigma' := \left\{ a_t' x \geq b_t,\ t \in \bigcup_{i=1}^{n} \{\text{supp}\, \lambda^i \cup \text{supp}\, \gamma^i\} \right\}.$$

If $F'$ is the feasible set of $\sigma'$, we have $(O^+F')^0 = \text{cl}\, M_n(\sigma') = R_n$. Therefore, $O^+F' = \{O_n\}$, and $F'$ is bounded.

From now on we suppose $\text{rank}\{a_t,\ t \in T\} = n$.

(III) → (VI) Given $s \in T$, we know that $-a_s \in M_n(\sigma)$. Hence $-a_s = \sum_{t \in T} \lambda_t a_t$ for some $\lambda \in R_+^{(T)}$. Defining

$$\gamma_t = \begin{cases} 1 + \lambda_s, & t = s \\ \lambda_t, & t \in T/\{s\}, \end{cases}$$

we have $s \in \text{supp}\, \gamma$, $\gamma \in \Lambda_0$.

(VI) → (V) It is a consequence of the inclusion $\bigcup_{\lambda \in \Lambda_0} \text{supp}\, \lambda \subset I_0$. In fact, if $s \in \text{supp}\, \lambda$, and $\lambda \in \Lambda_0$, then $-a_s \in K\{a_t,\ t \in T/\{s\}\}$. Hence $a_s' x = 0$ is a consequence of $\sigma_0$, i.e., $s \in I_0$.

(V) → (II) If $y \in F_0$, then $a_t' y = 0$ for all $t \in T$ (since $I_0 = T$). Therefore $y \in L\ \{a_t,\ t \in T\}^\perp = \{O_n\}$ and the conclusion follows. □

THEOREM 2.2. *Let $\sigma_0$ be either* FM *or compact and $O_n \notin \text{cl}\{a_t,\ t \in T\}$. If $\Lambda \neq \emptyset$, the following propositions are equivalent to each other*:

(I) $\Lambda$ *is bounded.*

(II) $\Lambda_0 = \{0\}$.

(III) $M_n(\sigma)$ *is a pointed cone.*

(IV) $\sigma_0$ *has strict solutions.*

(V) $I_0 = \emptyset$.

(VI) $\dim F_0 = n$.

*Proof.* We shall demonstrate (I) → (II) → (III) → (IV) → (V) → (VI) → (II) → (I).

(I) → (II) Given $\lambda \in \Lambda$ and $\gamma \in \Lambda_0$, it can be easily verified that $\lambda + \rho\gamma \in \Lambda$ for all $\rho \geq 0$. If $\gamma_s > 0$ for some $s \in T$, then $\lim_{\rho \to +\infty}(\lambda_s + \rho\gamma_s) = +\infty$, contradicting the assumption.

(II) → (III) Let us assume that $M_n(\sigma)$ contains the one-dimensional subspace generated by $a \neq O_n$. In that case there is a $\lambda \in R_+^{(T)}$ and a $\gamma \in R_+^{(T)}$ such that

$$a = \sum_{t \in T} \lambda_t a_t \tag{2.1}$$

and

$$-a = \sum_{t \in T} \gamma_t a_t. \tag{2.2}$$

The addition of (2.1) and (2.2) gives $\Sigma_{t \in T}(\lambda_t + \gamma_t)a_t = O_n$, i.e., $\lambda + \gamma \in \Lambda_0/\{0\}$.

(III) → (IV) First we assume that $\sigma_0$ is FM. If we had $O_n \in \text{conv}\{a_t, t \in T\}$, it would be $O_n = \Sigma_{t \in T}\lambda_t a_t$, $\Sigma_{t \in T}\lambda_t = 1$, for some $\lambda \in R_+^{(T)}$. Taking $s \in \text{supp}\,\lambda$ we would have $L\{a_s\} \subset M_n(\sigma)$, with $a_s \neq O_n$, contradicting (III). Therefore $O_n \notin \text{conv}\{a_t, t \in T\}$ and (IV) follows from Theorem 1.2.

Now let us assume $\sigma_0$ compact. Let $\alpha_t$: $T \to R_+/\{0\}$ be such that $\{\alpha_t a_t, t \in T\}$ is compact. Reasoning as above, one has $O_n \notin \text{conv}\{\alpha_t a_t, t \in T\}$ and, by Theorem 1.2, $\{(\alpha_t a_t)'x > 0, t \in T\}$ has at least a solution, which will be a strict solution of $\sigma_0$.

(IV) → (V) Is trivial.

(V) → (VI) The convex sets $F_0$ and $H_s = \{x \in R^n/a_s'x \geq 0\}$ satisfy the relations $F_0 \subset \text{cl}\, H_s$ and $F_0 \not\subset \text{rb}\, H_s$ (since $s \notin I_0$). By a well-known result [17, Corollary 6.5.2], we have $\text{ri}\, F_0 \subset \text{ri}\, H_s = \{x \in R^n/a_s'x > 0\}$. Therefore, if $x^0 \in \text{ri}\, F_0 \neq \emptyset$, then $a_t'x^0 > 0$ for all $t \in T$. If we assume $\sigma_0$ compact and $\alpha_t$: $T \to R_+/\{0\}$ to be such that $\{\alpha_t a_t, t \in T\}$, is a compact set, it can be verified that $F_0$ contains the open ball with centrum in $x^0$ and with radius $\xi \cdot \eta^{-1}$, where $\xi = \min_{t \in T}(\alpha_t a_t'x^0) > 0$ and $\eta = \max_{t \in T}\|\alpha_t a_t\| > 0$. On the other hand, if $\sigma_0$ is FM the result follows by applying Theorem 1.3 to $\sigma_0$.

(VI) → (II) As $\sigma_0$ does not contain the trivial inequality, if $x^0 \in \text{int}\, F_0$, then $a_t'x^0 > 0$ for all $t \in T$. Let $\lambda \in \Lambda_0$ be arbitrarily chosen. The inner product of $x^0$ by $\Sigma_{t \in T}\lambda_t a_t = O_n$ yields $\Sigma_{t \in T}\lambda_t(a_t'x^0) = 0$, which implies $\lambda_t = 0$ for all $t \in T$. Thus, $\Lambda_0$ is reduced to the null function.

(II) → (I) Let us suppose $\sigma_0$ FM and $O_n \notin \text{cl}\{a_t, t \in T\}$. Since (II) → (IV) we can take a point $x^0 \in R^n$ such that $a_t'x^0 > 0$ for all $t \in T$. If $\xi > 0$ where a lower bound of the scalar set $\{a_t'x^0, t \in T\}$ then, for every $\lambda \in \Lambda$, it would be $c'x^0 = \Sigma_{t \in T}\lambda_t\{a_t'x^0) \geq \xi(\Sigma_{t \in T}\lambda_t)$. Hence $\xi^{-1}(c'x^0)$ would be a uniform upper bound of $\Lambda$. Thus, we must suppose the existence of a sequence $\{t_r\} \subset T$ such that $\lim_{r \to \infty} a_{t_r}'x^0 = 0$.

Two cases can arise: either $\{a_{t_r}\}$ contains a convergent subsequence or $\lim_{r \to \infty}\|a_{t_r}\| = \infty$. In the first case we can denote the subsequence as the whole sequence. Let $a = \lim_{r\to\infty} a_{t_r}$. Clearly, $a \in M_n(\sigma)/\{O_n\}$, by the assumptions. Therefore, there is a $\lambda \in R_+^{(T)}$ such that $a = \sum_{t\in T}\lambda_t a_t$, $\operatorname{supp}\lambda \neq \varnothing$. Multiplying by $x^0$, we obtain $a'x^0 = \sum_{t\in T}\lambda_t(a_t'x^0) > 0$, whereas $a'x^0 = \lim_{r\to\infty} a_{t_r}'x^0 = 0$ by continuity.

In the second case, we define $u^r = \|a_{t_r}\|^{-1}a_{t_r}$; the sequence $\{u^r\}$ contains a convergent subsequence at which we can apply the reasoning above.

In both cases we get a contradiction. Hence $\Lambda$ is bounded.

Finally we suppose $\sigma_0$ is compact, instead of being FM. Let $\alpha_t$: $T \to R_+/\{0\}$ be such that $\{\alpha_t a_t,\ t \in T\}$ is compact. If $\Lambda$ were not bounded there would be a sequence $\{\lambda^r\} \subset \Lambda$ such that $\lambda_{t_r}^r \geq r$ for some $t_r \in T$, $r = 1, 2, \dots$ . Let us denote $s(r) = \sum_{t\in T}\alpha_t^{-1}\lambda_t^r > 0$. Again, two cases can arise, $\{s(r)\}$ being bounded or unbounded.

In the first case, there is some $M > 0$ such that $\alpha_{t_r}^{-1}r \leq \sum_{t\in T}\alpha_t^{-1}\lambda_t^r \leq M$. Hence, $\lim_{r\to\infty}\alpha_{t_r} = +\infty$ which, in turn, implies $\lim_{r\to\infty}\|a_{t_r}\| = 0$, because of the boundedness of $\{\alpha_t a_t,\ t \in T\}$. Therefore, $\lim_{r\to\infty} a_{t_r} = O_n$, contradicting the hypothesis.

In the second case we can write, without loss of generality, $\lim_{r\to\infty} s(r) = \infty$. Since $s(r)^{-1}c = s(r)^{-1}\{\sum_{t\in T}(\alpha_t^{-1}\lambda_t^r)\alpha_t a_t\} \in \operatorname{cl}\operatorname{conv}\{\alpha_t a_t,\ t \in T\}$, and $\{\alpha_t a_t,\ t \in T\}$ is compact, we have, at the limit, $O_n \in \operatorname{conv}\{\alpha_t a_t,\ t \in T\}$, which contradicts the assumption (II).

This completes the proof. □

We have proved the chain of implications from (I)–(VI) under the assumption of $\sigma_0$ being FM or compact and $a_t \neq O_n$ for all $t \in T$. On the other hand, if we take an arbitrary point $x^0 \in F$, we have $x^0 + F_0 = x^0 + O^+F \subset F$. Therefore $\dim F_0 \leq \dim F$ and we get the following result:

COROLLARY 2.2.1. *If* $F \neq \varnothing \neq \Lambda$ *and* $\sigma_0$ *is* FM *or compact, then* $\Lambda$ *bounded* $\to \dim F = n$.

Comparing the conditions (III) in Theorems 2.1 and 2.2 we conclude the impossibility of the simultaneous boundedness of $F$ and $\Lambda$, for the particular class of systems considered in Theorem 2.2. However, this relation is always valid, since we proved (I) $\to$ (II) $\to$ (III) in the last theorem without making use of any assumption. Therefore, we get the following generalization of Clark's Theorem to Semi-Infinite Linear Programming:

THEOREM 2.3. *Both $F$ and $\Lambda$ cannot be bounded.*

*Remarks.*

(1) Most conditions for the boundedness of $F$ given in Theorem 2.1 are well known (asserted in [20], and proved in [7], although the proofs given in this work are more direct). The purpose of Theorem 2.1 is to underline the symmetry between primal and dual conditions, given in Theorem 2.2.

(2) The full-rank assumption for $\{a_t,\ t \in T\}$ is not superfluous in order to get (V) and (VI) in Theorem 2.1 equivalent to the remaining conditions.

EXAMPLE 2.1. Consider the system $\sigma = \{x_2 \geq 0,\ -x_2 \geq 0\}$ in $R^2$. Clearly (V) and (VI) hold, whereas (I)–(IV) fail.

(3) The assumptions on $\sigma$ in Theorem 2.2 are not superfluous:

EXAMPLE 2.2. (P) $\operatorname{Inf} x_2$ s.t. $tx_1 + t^2x_2 \geq 0$, $t \in [-1, 1]$. (I) fails: Take $\lim_{t \to 0^+}$ in $\binom{0}{1} = (1/2t^2)\binom{t}{t^2} + (1/2t^2)\binom{-t}{t^2}$, $t \in ]0, 1[$. It can also be proved that (II), (IV), and (V) also fail (since $\sigma$ contains the trivial inequality $O_n'x \geq 0$), whereas (III) holds.

(4) The assumption "$\sigma_0$ is FM or compact" does not guarantee the equivalence between the six propositions considered in Theorem 2.2.

EXAMPLE 2.3. (P) $\operatorname{Inf} x_1 + x_2$ s.t. $tx_1 + tx_2 \geq 0$, $t \in ]0, 1]$. Here $\sigma_0$ is FM and compact (take $\alpha_t = t^{-1}$). If we define $\lambda^r \in R_+^{(T)}$ such that

$$\lambda_t^r = \begin{cases} r, & t = r^{-1} \\ 0, & t \in ]0,1]/\{r^{-1}\}, \end{cases}$$

it can be shown that $\{\lambda^r\} \subset \Lambda$. Hence (I) fails, whereas the remaining properties hold.

(5) Eckhardt [4] provided the extension of Clark's Theorem [2] to those linear semi-infinite programs whose feasible set is normally represented. However, Karney [13] proved that the property also applies in Convex Semi-Infinite Programming, whichever the primal problem is.

(6) The converse of Corollary 2.2.1 is false, even for finite systems.

EXAMPLE 2.4. (P) $\operatorname{Inf} x$ s.t. $x \geq -1$, $-x \geq -1$. In spite of the full dimensionality of $F$, $\Lambda$ is not bounded.

## 3. ON THE BOUNDEDNESS OF PRIMAL AND DUAL VARIABLES

The purpose of this section is to provide conditions for the boundedness of the functions $x \to a_t'x$ on $F$ (and $F^*$) and $\lambda \to \lambda_t$ on $\Lambda$ (and $\Lambda^*$). These functions are called in the literature [18, 19] primal slack variables (we consider $a_t'x$ instead of $a_t'x - b_t$ since the addition of $-b_t$ does not modify the boundedness of the function) and dual structural variables, respectively. It is clear that both functions are lower bounded, for all $t \in T$. Therefore we must only analyze the existence of upper bounds.

THEOREM 3.1. *Let* $\sigma_0$ *be* FM. *If* $F \neq \emptyset \neq \Lambda$, *the following statements are equivalent*:

(I) $s \in I_0$.

(II) $x \to a_s'x$ *is bounded on* $F$.

(III) $\lambda \to \lambda_s$ *is not bounded on* $\Lambda$.

*Proof.* We shall prove (I) → (III) → (II) → (I).

(I) → (III) Given $s \in I_0$, the inequality $-a_s'x \geq 0$ is a consequence of $\sigma_0$. Hence, by Theorem 1.1 and the assumption on $\sigma_0$, $-a_s \in M_n(\sigma)$, which proves the existence of some $\gamma \in \Lambda_0$ such that $\gamma_s > 0$. Choosing a $\lambda \in \Lambda$ arbitrarily, $\lambda + \rho\gamma \in \Lambda$ for all $\rho \geq 0$. Therefore (III) holds.

(III) → (II) let $\{\lambda^r\} \subset \Lambda$ be such that $\lim_{r \to \infty} \lambda_s^r = \infty$. From the identity $-a_s = \sum_{t \in T/\{s\}} (\lambda_s^r)^{-1} \cdot \lambda_t^r a_t - (\lambda_s^r)^{-1} c$ we obtain $-a_s \in \operatorname{cl} M_n(\sigma) = M_n(\sigma)$ at the limit. Therefore the following dual problems are consistent:

$$\operatorname{Inf}(-a_s)'x \quad \text{s.t. } a_t'x \geq b_t, \qquad t \in T \tag{$P_s$}$$

and

$$\operatorname{Sup} \sum_{t \in T} \lambda_t b_t \quad \text{s.t. } \sum_{t \in T} \lambda_t a_t = -a_s. \tag{$D_s$}$$

In that case $v(P_s)$ (as well as $v(D_s)$) is finite, i.e., (II) holds.

(II) → (I) If $\alpha$ is a bound for $x \to a_s'x$ on $F$, then $-a_s'x \geq -\alpha$ is a consequence of $\sigma_0$. Hence $-\binom{a_s}{\alpha} \in \operatorname{cl} K(\sigma)$ and $-a_s \in \operatorname{cl} M_n(\sigma)$, which proves that $s \in I_0$. □

COROLLARY 3.1.1. *Let* $\sigma_0$ *be* FM *and* $F \neq \emptyset \neq \Lambda$. *The following relations hold.*

(I) *If* $\Lambda$ *is bounded, then none of the functions* $x \to a_s'x$ *are bounded on* $F$. *The converse statement holds when* $T$ *is finite.*

(II) *If* $F$ *is bounded, then none of the functions* $\lambda \to \lambda_s$ *are bounded on* $\Lambda$. *The converse statement holds when rank*$\{a_t, t \in T\} = n$.

*Proof.* The direct part (in both propositions) follows from the equivalence between (II) and (III) in Theorem 3.1. We now show the converse statements:

(I) If $x \to a_s'x$ is not bounded, then there is a $M_s > 0$ such that $\lambda_s \leq M_s$ for all $\lambda \in \Lambda$. Let $M = \max_{s \in T} M_s$. It is clear that $M$ is a uniform bound for $\Lambda$.

(II) If $\lambda \to \lambda_s$ is not bounded, for all $s \in T$, then $I_0 = T$ (Theorem 3.1) and we can apply Theorem 2.1. □

THEOREM 3.2. *Let* $\sigma$ *be* FM *and* $\Lambda \neq \varnothing$. *It holds*:

(I) $\Lambda^* \neq \varnothing$, *and*

(II) $s \in I$ *if, and only if*, $\lambda \to \lambda_s$ *is not bounded on* $\Lambda^*$.

*Proof.* (I) Since (P) and (D) are consistent and $\sigma$ is FM, we can apply the Haar's Duality Theorem (proved in [1]). Hence (D) is solvable and $v(\mathrm{P}) = v(\mathrm{D})$.

(II) Let us consider the following problem:

$$\operatorname{Inf} c'x + v(\mathrm{P})x_{n+1} \quad \text{s.t. } a_t'x + b_t x_{n+1} \geq 0, \qquad t \in T, \qquad (\hat{\mathrm{P}})$$
$$O_n'x - x_{n+1} \geq 0.$$

We denote by $(\hat{\mathrm{D}})$ its dual problem, by $\hat{F}$ and $\hat{\Lambda}$ their respective feasible sets and by $\hat{I}_0$ the set of unstable indices in the constraints system of $(\hat{\mathrm{P}})$, $\hat{\sigma}$. It can be easily shown that $\hat{F} \neq \varnothing \neq \hat{\Lambda}$ and $\hat{\sigma}_0$ is FM. On the other hand, given any $\lambda \in R_+^{(T)}$ and any $\mu \geq 0$, we have

$$(\lambda, \mu) \in \hat{\Lambda} \leftrightarrow \sum_{t \in T} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} = \begin{pmatrix} c \\ v(D) + \mu \end{pmatrix} \leftrightarrow \lambda \in \Lambda^* \text{ and } \mu = 0.$$

We therefore get from Theorem 3.1 that $s \in \hat{I}_0$ if, and only if, $(\lambda, \mu) \to \lambda_s$ is not bounded on $\hat{\Lambda}$. But $\hat{I}_0 = I$ (observe that $K(\sigma) = M_n(\hat{\sigma})$) and $(\lambda, \mu) \to \lambda_s$ is bounded on $\hat{\Lambda}$ if, and only if, $\lambda \to \lambda_s$ is bounded on $\Lambda^*$. □

In order to carry out the proof of the dual result of Theorem 3.2 we need the following:

LEMMA 3.3. *Let* $M$ *be a convex cone and* $c \in L(M)$. *Then* $K(M \cup \{-c\}) = L(M)$ *if, and only if*, $c \in \operatorname{ri} M$.

*Proof.* For the sake of brevity we denote $K = K(M \cup \{-c\})$ and $L = L(M)$.

First suppose $c \in \operatorname{ri} M$. Let $\xi > 0$ such that $\{x \in L / \|x - c\| \leq \xi\} \subset M$. Given an arbitrary point $y \in L$, $y \neq c$, we have $x := c + \xi\|y - c\|^{-1}(y - c) \in M$ and, consequently, $y = c + \xi^{-1}\|y - c\|(x - c) \in K$. Hence $L \subset K$, whereas $K \subset L$ is trivial.

Suppose now $c \notin \operatorname{ri} M$. Two cases can arise:

(i) $c \in \operatorname{rb} M$. Since there is a proper supporting hyperplane to $M$ in $c$, let $z \in R^n$ be such that $z'c = 0$, $z'x \geq 0$ for all $x \in M$ and $z'x^0 > 0$ for some $x^0 \in M$.

(ii) $c \notin \operatorname{rb} M$. It can be shown the existence of a $z \in R^n$ such that $z'c < 0$ and $z'x \geq 0$, for all $x \in M$.

We get in both cases the relations $K \subset (x \in L / z'x \geq 0\} \subsetneq L$ (recall that $c \in L$). Thus we reach the desired conclusion. □

THEOREM 3.4. *Let* $c \in \operatorname{ri} M_n(\sigma)$ *and* $F \neq \varnothing$. *Then the following holds*:

(I) $F^* \neq \varnothing$.

(II) $s \in \bigcup_{\lambda \in \Lambda} \operatorname{supp} \lambda$ *if, and only if,* $x \to a_s'x$ *is bounded on* $F^*$.

*Proof.* (I) It is a consequence of Theorem 1.4. Moreover, $v(\mathrm{P}) = v(\mathrm{D})$.
(II) Consider the problem

$$\operatorname{Inf} c'x \quad \text{s.t. } a_t'x \geq b_t, \quad t \in T, \qquad -c'x \geq -v(\mathrm{P}) \qquad (\hat{\mathrm{P}})$$

whose dual problem will be denoted $(\hat{\mathrm{D}})$, whereas $\hat{\sigma}$, $\hat{\sigma}_0$, $\hat{F}$, $\hat{\Lambda}$, and $\hat{I}_0$ will denote the constraints system of $(\hat{\mathrm{P}})$, its homogeneous associated system, the primal and the dual feasible sets and the set of unstable indices in $\hat{\sigma}_0$, respectively.

It is clear that $\hat{F} = F^*$. Hence, by the assumption on $c$, we find that $\hat{F} \neq \varnothing \neq \hat{\Lambda}$. Moreover, $M_n(\hat{\sigma}) = K\{M_n(\sigma) \cup \{-c\}\}$ is closed, since it is a linear subspace (by Lemma 3.3).

Suppose now $s \in \operatorname{supp} \lambda$, $\lambda \in \Lambda$. As $\sum_{t \in T} \lambda_t a_t + 1(-c) = O_n$ and $\lambda_s > 0$, we can assert the existence of $a$ $\gamma \in \hat{\Lambda}_0$ such that $\gamma_s > 0$, which proves $s \in \hat{I}_0$. We therefore get from Theorem 3.1 (applied to $(\hat{\mathrm{P}})$) that $x \to a_s'x$ is bounded on $\hat{F} = F^*$. Conversely, if $x \to a_s'x$ is bounded on $F^* = \hat{F}$, then $\lambda \to \lambda_s$ is not bounded on $\hat{\Lambda}$. Hence, there is a $\lambda \in R_+^{(T)}$ and a $\mu \geq 0$ such that $\sum_{t \in T} \lambda_t a_t + \mu(-c) = c$ and $\lambda_s > 0$. If $\mu > 0$, we can take $(1 + \mu)^{-1}\lambda \in \Lambda$. Otherwise, $\lambda \in \Lambda$. Consequently, $s \in \bigcup_{\lambda \in \Lambda} \operatorname{supp} \lambda$. This completes the proof. □

*Remarks.*

(1) Analyzing the proof of Theorem 3.1 one observes that (III) → (II) → (I) have been proved without making use of the assumption on $\sigma_0$ which is, however, necessary in order to guarantee the equivalence.

EXAMPLE 3.1. Consider $\sigma = \{-x_1 \geq 0;\ tx_1 - t^2 x_2 \geq 0,\ t \in ]0,1]\}$ and $c = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

The first inequality is unstable in $\sigma_0 = \sigma$ and the mapping $x \to -x_1$ is bounded on $F$. However, $\Lambda$ contains an only function. Therefore (III) fails.

(2) The converse statement of (I) and (II) in Corollary 3.1.1 can fail, if the respective assumptions are violated.

EXAMPLE 3.2.

$$\operatorname{Inf} x \quad \text{s.t. } \frac{1}{r} x \geq \frac{1}{r}, \qquad r = 1, 2, \ldots . \qquad (\mathrm{P})$$

None of the functions $x \to x/r$ are bounded on $F = [1, +\infty[$. However,

$\{\lambda^r\} \subset \Lambda$, if we define

$$\lambda_t^r = \begin{cases} r, & t = r \\ 0, & t \neq r \end{cases},$$

being $\lim_{r \to \infty} \lambda_t^r = \infty$.

EXAMPLE 3.3. Consider the problem

$$\operatorname{Inf} x_2 \quad \text{s.t. } x_2 \geq 0,\ -x_2 \geq 0 \qquad \text{in } R^2. \tag{P}$$

Since $\Lambda = \left\{ \begin{pmatrix} 1 + \lambda_2 \\ \lambda_2 \end{pmatrix} / \lambda_2 \geq 0 \right\}$, none of the functions $\lambda \to \lambda_1$ and $\lambda \to \lambda_2$ are bounded on $\Lambda$. However, $F = R \times \{0\}$ is not bounded.

(3) The FM assumption for $\sigma$ in Theorem 3.2 is not superfluous.

EXAMPLE 3.4.

$$\operatorname{Inf} x \quad \text{s.t. } tx \geq -t^2, \qquad t \in [-1, 1].$$

We have $v(\mathrm{P}) = 0$. As $\{\lambda^r\} \subset \Lambda$, if we take

$$\lambda_t^r = \begin{cases} r, & t = r^{-1} \\ 0, & t \in [-1,1] / \{r^{-1}\} \end{cases},$$

and $\lim_{r \to \infty} \Sigma_{t \in T} \lambda_t^r b_t = 0$, also $v(\mathrm{D}) = 0$. However, $\Lambda^* = \varnothing$. Hence we cannot substitute $\sigma$ by $\sigma_0$ in the hypothesis.

(4) The condition $c \in \operatorname{ri} M_n(\sigma)$ cannot be substitute, in Theorem 3.4, by the weaker assumption $c \in M_n(\sigma)$.

EXAMPLE 3.5.

$$\begin{aligned} \operatorname{Inf} x_1 \quad \text{s.t. } & x_1 \geq -1 & (t = 0), \\ & \frac{1}{t} x_1 + t x_2 \geq 1, \ t \in ]0, 1]. & \end{aligned} \tag{P}$$

It can be shown that (P) and (D) are consistent, $v(\mathrm{P}) = 0$ and $v(\mathrm{D}) = -1$. Therefore $F^* = \varnothing$, i.e., (I) fails.

(5) Theorems 3.1, 3.2, and 3.4 are related with Theorems 5.1, 5.2, and 5.3 in [4].

## 4. BOUNDED OPTIMAL SETS

The major topic of this section is the characterization of the problems for which $F^*$ (or $\Lambda^*$) is bounded. As in Section 2 we will obtain two theorems giving symmetric (but not opposite) conditions for the boundedness of both sets. One of them is related with the finite subproblems of (P). So, a bit of notation will be useful.

Given a finite subset $S$ of $T$, we denote by $(P_S)$ the problem

$$\operatorname{Inf} c'x \quad \text{s.t. } a_t'x \geq b_t, \qquad t \in S. \tag{$P_S$}$$

Similarly, all the elements associated with $(P_S)$ will be distinguished from the corresponding to (P) by the subindex $S$.

THEOREM 4.1. *If $F \neq \varnothing$, the following statements are equivalent*:

(I) *$F^*$ is bounded.*

(II) $\operatorname{rank}\{a_t, t \in T\} = n$ *and* $\bigcup_{\lambda \in \Lambda} \operatorname{supp} \lambda = T$.

(III) *There exists a finite set $S \in T$ such that $F_S^*$ is bounded.*

(IV) $c \in \operatorname{int} M_n(\sigma)$.

*Proof.* First we will prove (I) ↔ (IV), equivalence which will be useful to show the following chain of implications: (I) → (II) → (III) → (I).

The proof of (I) ↔ (IV) will be based upon the solvability of (P) (Theorem 1.4) in both cases. Thus we can consider the system

$$\hat{\sigma} = \{a_t'x \geq b_t,\ t \in T;\ -c'x \geq -v(P)\},$$

whose feasible set is $F(\hat{\sigma}) = F^*$.

(I) → (IV) Since $F(\hat{\sigma})$ is bounded, $M_n(\hat{\sigma}) = R^n$ (Theorem 2.1), i.e.,

$$K\{M_n(\sigma) \cup \{-c\}\} = R^n \tag{4.1}$$

On the other hand, $\begin{pmatrix} c \\ v(P) \end{pmatrix} \in \operatorname{cl} K\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T;\ \begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}$ (Theorem 1.1). Hence $c \in \operatorname{cl} M_n(\sigma)$, so that $-c \in L\{M_n(\sigma)\}$. Therefore

$$K\{M_n(\sigma) \cup \{-c\}\} \subset L\{M_n(\sigma)\}. \tag{4.2}$$

From (4.1) and (4.2) we get $K\{M_n(\sigma) \cup \{-c\}\} = L\{M_n(\sigma)\}$ and, by Lemma 3.3, $c \in \operatorname{ri} M_n(\sigma)$, whereas $L\{M_n(\sigma)\} = R^n$ proves that $\operatorname{int} M_n(\sigma) \neq \varnothing$.

(IV) → (I) Since $c \in \operatorname{ri} M_n(\sigma)$, we have $M_n(\hat{\sigma}) = K\{M_n(\sigma) \cup \{-c\}\} = L\{M_n(\sigma)\}$ (Lemma 3.3). As $L\{M_n(\sigma)\}\} = R^n$ we reach the desired conclusion by applying Theorem 2.1 to $\hat{\sigma}$.

(I) → (II) Since $M_n(\hat{\sigma}) = R^n$ and $c \in L\{a_t, t \in T\}$ (by (IV)), it is clear that $\operatorname{rank}\{a_t, t \in T\} = n$. On the other hand $x \to a_s'x$ is bounded on $F^*$, for all $s \in T$. The aimed conclusion follows from Theorem 3.4.

(II) → (III) Let $\{a_{t_1}, \ldots, a_{t_n}\}$ be a basis of $R^n$. We assume the existence of some $\lambda^i \in \Lambda$ such that $\lambda_{t_i}^i > 0$, for all $i = 1, \ldots, n$. If we define $\gamma = (1/n)\sum_{i=1}^n \lambda_i$, we have $c = \sum_{t \in T} \gamma_t^i a_t$, with $\operatorname{rank}\{a_t, t \in \operatorname{supp} \lambda\} = n$.

Hence $c \in \operatorname{int} K\{a_t,\ t \in \operatorname{supp} \gamma\}$ (as a consequence of a well-known characterization of the interior points of a convex set). Therefore $c \in \operatorname{int} M_n(\sigma_S)$, for $S = \operatorname{supp} \lambda$. Applying (IV) → (I) to ($P_S$) we conclude that $F_S^*$ is bounded.

(III) → (I) Let $S$ be a finite subset of $T$ such that $F_S^*$ is a bounded set. Applying (I) → (IV) to ($P_S$) we get $c \in \operatorname{int} M_n(\sigma_S) \subset \operatorname{int} M_n(\sigma)$ and we can apply (IV) → (I) to (P). Therefore $F^*$ is bounded. □

THEOREM 4.2. *Let* $\sigma$ *be* FM *and* $O_{n+1} \notin \operatorname{cl}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T\right\}$. *If* $\Lambda \neq \varnothing$, *the following statements are equivalent to each other*:

(I) $\Lambda^*$ *is bounded.*

(II) $I \neq \varnothing$.

(III) *For all finite* $S \subset T$ *it holds* $\Lambda_S \neq \varnothing \to \Lambda_S^*$ *is bounded.*

(IV) $\dim F = n$.

*Proof.* We will prove (I) → (II) → (IV) → (I) and the equivalence between (II) and (III).

(I) → (II) Since all the functions $\lambda \to \lambda_s$ are bounded on $\Lambda^*$, we have $I = \varnothing$ (Theorem 3.2).

(II) → (IV) It is an immediate consequence of Theorem 1.3.

(IV) → (I) A contradiction will be obtained by assuming that $\Lambda^*$ is not bounded. In that case, for every $r \in N$ there is some $\lambda^r \in R_+^{(T)}$ and a $t_r \in T$ such that

$$\sum_{t \in T} \lambda_t^r \begin{pmatrix} a_t \\ b_t \end{pmatrix} = \begin{pmatrix} c \\ v(D) \end{pmatrix} \tag{4.3}$$

and

$$\lambda_{t_r}^r \geq r. \tag{4.4}$$

As $\operatorname{int} F \neq \varnothing$ there is some $x^0 \in F$ and $\xi > 0$ such that $a_t'(x^0 + \xi u) \geq b_t$, for all $t \in T$ and for all $u \in R^n$, $\|u\| = 1$.

The inner product of (4.3) by $\begin{pmatrix} x^0 + \xi u \\ -1 \end{pmatrix}$, taking into account (4.4), yields the relation

$$0 \leq r\left\{a_{t_r}'(x^0 + \xi u) - b_{t_r}\right\} \leq c'(x^0 + \xi u) - v(D) \leq c'x^0 - v(D) + \xi\|c\|. \tag{4.5}$$

Particularly, if we take $u = \|a_{t_r}\|^{-1} a_{t_r}$ in (4.5), we obtain

$$0 \leq r\left\{a_{t_r}' x^0 - b_{t_r} + \xi\|a_{t_r}\|\right\} \leq c'x^0 - v(D) + \xi\|c\| \tag{4.6}$$

which also holds when $a_{t_r} = O_n$, as it can be directly observed.

But (4.6) requires $\lim_{r\to\infty}(a'_{t_r}x^0 - b_{t_r}) = \lim_{r\to\infty}\|a_{t_r}\| = 0$, i.e., $\lim_{r\to\infty}\begin{pmatrix} a_{t_r} \\ b_{t_r}\end{pmatrix} = O_{n+1}$ contradicting the hypothesis.

(II) → (III) Let $S$ be a finite subset of $T$ such that $\Lambda_S \neq \emptyset$. As $I_S = \emptyset$, we can apply Theorem 3.2 to conclude the boundedness of all the functions $\lambda \to \lambda_t$, $t \in S$, on $\Lambda_S^*$. The maximal bound of these functions is then the uniform bound of $\Lambda_S^*$ we were looking for.

(III) → (II) Let us suppose the existence of some index $s \in I$. In that case

$$-\begin{pmatrix} a_s \\ b_s \end{pmatrix} = \sum_{t\in T} \gamma_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \mu \begin{pmatrix} O_n \\ -1 \end{pmatrix}, \tag{4.7}$$

for some $\gamma \in R_+^{(T)}$ and $\mu \geq 0$ (by Theorem 1.1).

Let us consider an arbitrary $\lambda \in \Lambda$ and define $S = \operatorname{supp}\gamma \cup \operatorname{supp}\lambda \cup \{s\}$. As $c = \sum_{t\in\operatorname{supp}\lambda}\lambda_t a_t + \sum_{t\in\operatorname{supp}\gamma}\gamma_t a_t + a_s$, it is clear that $\Lambda_S \neq \emptyset$. Hence $\Lambda_S^*$ is bounded and, again by Theorem 3.2, $I_S = \emptyset$. But, from (4.7), $s \in I_S$, which constitutes a contradiction. □

COROLLARY 4.2.1. *Let $\sigma$ be a* FM *system. If $\Lambda^*$ is bounded, then* $\dim F = n$. *Conversely, if* $\dim F = n$, *$T$ is finite and $\sigma$ does not contain the trivial inequality, then $\Lambda^*$ is either empty or bounded.*

*Proof.* Analyzing the proof of Theorem 4.2 it can be observed that the chain of implications (I) → (II) → (IV) has been proved under the only hypothesis of $\sigma$ being FM. Thus the direct statement holds.

Let us suppose now $\dim F = n$ and $\Lambda \neq \emptyset$. As $\begin{pmatrix} a_t \\ b_t \end{pmatrix} \neq O_{n+1}$ for all $t \in T$, one has $I = \emptyset$ (Theorem 1.3). By Theorem 3.2 there exists a bound, $M_t > 0$, for all the functions $\lambda \to \lambda_t$ on $\Lambda^*$. Therefore $\max_{t\in T} M_t$ is a uniform bound of $\Lambda^*$. □

*Remarks.*

(1) Some of the conditions given in Theorem 4.1 appear in the literature, although not related with the boundedness of $F^*$: see the Duality Theorem of Isii [11] (other proofs can be found in [12] and [5]), as well as the description of the simplex algorithm given by Glashoff and Gustafson [5].

(2) It can be observed that, if $F^* \neq \emptyset$, then $F^* = F^* + L\{a_t, t \in T\}^\perp$. That is why the boundedness of $F^*$ requires the maximality of $\operatorname{rank}\{a_t, t \in T\}$.

(3) None of the assumptions in Theorem 4.2 are superfluous.

EXAMPLE 4.1.

$$\operatorname{Inf} x \quad \text{s.t. } tx \geq 0, \qquad t \in ]0,1]. \tag{P}$$

Since $v(\mathrm{D}) = 0$, $r\begin{pmatrix} r^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ v(\mathrm{D}) \end{pmatrix}$, $r = 1, 2, \ldots$. Thus (I) fails, whereas (II), (III), and (IV) hold. However, $\sigma$ is FM.

EXAMPLE 4.2.

$$\operatorname{Inf} x_1 \quad \text{s.t. } tx_1 + (1 - t^2)x_2 \geq 0, \qquad t \in ]-1,1]. \tag{P}$$

It is easily shown that the only element in $\Lambda^*$ is

$$\lambda_t = \begin{cases} 1, & t = 1 \\ 0, & t \in ]-1,1[. \end{cases}$$

Therefore (I) holds, whereas (II) fails. However,

$$O_3 \notin \operatorname{cl}\left\{ \begin{pmatrix} t \\ 1 - t^2 \\ 0 \end{pmatrix}, t \in ]-1,1] \right\}.$$

(4) The assumptions in the converse statement in Corollary 4.2.1 are not superfluous.

EXAMPLE 4.3.

$$\operatorname{Inf} x \quad \text{s.t. } (r^{-1})x \geq r^{-1}, \qquad r = 1, 2, \ldots. \tag{P}$$

Here $\Lambda^* = \Lambda$ is not bounded, although $\dim F = 1$. Obviously, the assumption of finiteness for $T$ is violated in this example.

EXAMPLE 4.4.

$$\operatorname{Inf} x \quad \text{s.t. } x \geq 0, \qquad 0x \geq 0.$$

Once again $\Lambda^*$ is not bounded, whereas $\dim F = 1$. In this example $\sigma$ contains the trivial inequality.

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