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Applied Mathematics Letters 16 (2003) 675–681

Applied Mathematics Letters

www.elsevier.com/locate/aml

The Atkinson-Wilcox Expansion Theorem for Electromagnetic Chiral Waves

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(Received and accepted July 2002)

Communicated by Dr. J. R. Ockendon

Abstract—Consider the problem of scattering of a time-harmonic electromagnetic wave by a three-dimensional bounded and smooth obstacle. The infinite space outside the obstacle is filled by a homogeneous isotropic chiral medium. In the region exterior to a sphere that includes the scatterer, any solution of the generalized Helmholtz's equation that satisfies the Silver-Müller radiation condition has a uniformly and absolutely convergent expansion in inverse powers of the radial distance from the center of the sphere. The coefficients of the expansion can be determined from the leading coefficient, "the radiation pattern", by a recurrence relation. © 2003 Elsevier Science Ltd. All rights reserved.

Keywords—Chiral media, Chiral radiation function, Dyadic Green function.

1. INTRODUCTION

The Sommerfeld radiation condition completely characterizes the behavior of the scattered field at infinity. Atkinson in [1] proved that the asymptotic condition of Sommerfeld can be replaced by a uniformly and absolutely convergent series representation of the scattered wave outside the smallest sphere that contains the scatterer. This is an expansion in inverse powers of the radial distance and its leading term recovers the Sommerfeld radiation condition. Wilcox [2,3] extended Atkinson's theorem for the electromagnetic scattering. He proved that all the coefficients in the Atkinson's expansion can be recurrently recovered by the leading coefficient, known as the "radiation pattern" [4,5].

Other extensions to elasticity and thermoelasticity were obtained by Dassios [6] and Cakoni and Dassios [7].

The purpose of this work is to state and prove a version of Atkinson-Wilcox's expansion theorem in the theory of scattering of a time-harmonic electromagnetic wave, in an unbounded homogeneous, isotropic chiral environment (exterior to the scatterer).

Chiral media support two kinds of electromagnetic waves, the left-circularly polarized (LCP) wave and the right-circularly polarized (RCP) wave [8–10]. This phenomenon, known as *optical*

This research was supported by the Research Committee of the University of Athens (Grants 70/4/3408 and 70/4/4159).

PII: S0893-9659(03)00066-1

activity, can be modelled using appropriate constitutive relations; we use the Drude-Born-Fedorov relations [9].

Lakhtakia in [10] has obtained the Beltrami fields $\mathbf{Q}_1$ and $\mathbf{Q}_2$ in series expansions, and hence the scattered electric field, by using spherical harmonic functions. More precisely, from (2a), (15a,b), and (18a,b) in [10] we can see that

$$\mathbf{E}^{Sc}(\mathbf{r}) = \sum_{\nu} A_\nu \left[\mathbf{M}_\nu^{(3)}(\gamma_1\mathbf{r}) + \mathbf{N}_\nu^{(3)}(\gamma_1\mathbf{r})\right] + \sum_{\nu} B_\nu \left[\mathbf{M}_\nu^{(3)}(\gamma_2\mathbf{r}) + \mathbf{N}_\nu^{(3)}(\gamma_2\mathbf{r})\right],$$

where A_ν, B_ν are unknown coefficients and $\mathbf{M}_\nu^{(3)}$, $\mathbf{N}_\nu^{(3)}$ the well-known vector spherical wave functions.

In Theorem 1 below, we obtain $\mathbf{E}^{Sc}$ as series expansion in powers of $1/r$, where the vectorial coefficients $\mathbf{F}_n^A$ can be obtained by the recursion relations (3.18) and (3.19).

The advantages of our approach are the following: first, we can easily obtain from the recursion relations that " if the radiation pattern is zero then the scattered electric field is also zero" (see [5, pp. 75,117]), which is a very useful property for the study of inverse scattering problems; second, expansion (3.17) replaces the radiation condition with an exact boundary condition on any sphere surrounding the scatterer, and in this respect it can be useful for numerical evaluation of the scattered electric field (see [11, pp. 45,60]).

Note that the first term in our expansion (3.17) is the radiation pattern which Lakhtakia obtained in (31a,b) of [10], by using spherical harmonic functions.

In Section 2, we formulate the electromagnetic scattering problem and in Section 3 we state and prove the appropriate expansion theorem and the recurrence relations for the coefficients. Finally, in Section 3 we describe the reduction to the achiral case.

2. THE SCATTERING PROBLEM

Let $\bar{\Omega}$ be a compact subset of $\mathbb{R}^3$, with a smooth boundary S. This is referred to as the scatterer. The exterior field of propagation $\mathbb{R}^3 \setminus \bar{\Omega}$ is occupied by a homogeneous, isotropic chiral medium of electric permittivity ε, magnetic permeability μ, and chirality measure β. For chirality, we use the Drude-Born-Fedorov constitutive relations [9]. The propagation of a time-harmonic electromagnetic wave $(\mathbf{E}, \mathbf{H})$ in the chiral medium $\mathbb{R}^3 \setminus \bar{\Omega}$ is governed by the modified Maxwell equations (in the frequency domain form) [9],

$$\nabla \times \mathbf{E} - \gamma^2\beta\mathbf{E} - i\omega\mu\left(\frac{\gamma}{\kappa}\right)^2 \mathbf{H} = \mathbf{0}, \tag{2.1a}$$

$$\nabla \times \mathbf{H} - \gamma^2\beta\mathbf{H} + i\omega\varepsilon\left(\frac{\gamma}{\kappa}\right)^2 \mathbf{E} = \mathbf{0}, \tag{2.1b}$$

where ω is the angular frequency imposed by the assumed time-harmonic dependence, $\kappa = \omega\sqrt{\mu\varepsilon}$ is simply a shorthand notation and does not represent any wavenumber inside the chiral medium, and the parameter γ is given by $\gamma^2 = \kappa^2(1 - \kappa^2\beta^2)^{-1}$.

From (2.1a),(2.1b) above, we can easily obtain

$$\nabla \times \nabla \times \mathbf{U} - 2\gamma^2\beta\nabla \times \mathbf{U} - \gamma^2\mathbf{U} = \mathbf{0}, \tag{2.2}$$

a Helmholtz-like differential equation for $\mathbf{U} = \mathbf{E}$ or $\mathbf{H}$.

The total field $\mathbf{U}^t = \mathbf{E}^t$ or $\mathbf{H}^t$ in $\mathbb{R}^3 \setminus \bar{\Omega}$ can be expressed as

$$\mathbf{U}^t = \mathbf{U}^i + \mathbf{U}^s, \qquad \text{in } \mathbb{R}^3 \setminus \bar{\Omega}, \tag{2.3}$$

where $\mathbf{U}^i = \mathbf{E}^i$ or $\mathbf{H}^i$ is the incident (electric or magnetic) field and $\mathbf{U}^s = \mathbf{E}^s$ or $\mathbf{H}^s$ is the scattered field, and it is assumed to satisfy the Silver-Müller radiation condition [5,11,12]

$$\sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{x}} \times \mathbf{H}^s(\mathbf{x}) + \mathbf{E}^s(\mathbf{x}) = o\left(\frac{1}{|\mathbf{x}|}\right), \qquad |\mathbf{x}| \to \infty \tag{2.4}$$

uniformly over all directions $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, or (in view of (2.1a),(2.1b)) equivalently,

$$\hat{\mathbf{x}} \times \nabla \times \mathbf{U}^s(\mathbf{x}) - \gamma\beta^2 \hat{\mathbf{x}} \times \mathbf{U}^s(\mathbf{x}) + i\frac{\gamma^2}{\kappa}\mathbf{U}^s(\mathbf{x}) = o\left(\frac{1}{|\mathbf{x}|}\right), \qquad |\mathbf{x}| \to \infty \tag{2.5}$$

uniformly over all directions $\hat{\mathbf{x}}$.

On the surface of the scatterer, any kind of boundary or transmission conditions that secures well-posedness can hold.

The direct scattering problem is the following. Given an incident field $\mathbf{U}^i$ and boundary conditions on S, find the scattered field $\mathbf{U}^s$ which satisfies (2.2) and (2.4).

The solution of this problem is given by the exterior integral representation of the scattered field [8]

$$\begin{aligned} \mathbf{U}^s(\mathbf{x}) = -2\gamma^2\beta \int_S \tilde{\mathcal{B}}(\mathbf{x},\mathbf{x}') \cdot \left[\hat{\mathbf{n}} \times \mathbf{U}^t(\mathbf{x}')\right] ds(\mathbf{x}') \\ + \int_S \left\{\tilde{\mathcal{B}}(\mathbf{x},\mathbf{x}') \cdot \left(\hat{\mathbf{n}} \times \left[\nabla \times \mathbf{U}^t(\mathbf{x}')\right]\right) + \left[\nabla_x \times \tilde{\mathcal{B}}(\mathbf{x},\mathbf{x}')\right] \cdot \left(\hat{\mathbf{n}} \times \mathbf{U}^t(\mathbf{x}')\right)\right\} ds(\mathbf{x}') \end{aligned} \tag{2.6}$$

for every $\mathbf{x} \in \mathbb{R}^3 \setminus \bar{\Omega}$, where $S := \vartheta\Omega$ and $\tilde{\mathcal{B}}(\mathbf{x},\mathbf{x}')$ is the infinite medium dyadic (i.m.d.) Green's function [8].

The i.m.d. Green's function $\tilde{\mathcal{B}}(\mathbf{x},\mathbf{x}')$ can be expressed as

$$\tilde{\mathcal{B}}(\mathbf{x},\mathbf{x}') = \tilde{\mathcal{B}}_L(\mathbf{x},\mathbf{x}') + \tilde{\mathcal{B}}_R(\mathbf{x},\mathbf{x}'), \tag{2.7}$$

where

$$\tilde{\mathcal{B}}_L(\mathbf{x},\mathbf{x}') = \frac{\kappa}{8\pi\gamma^2}\left[\gamma_L \tilde{I} + \frac{1}{\gamma_L}\nabla\nabla + \nabla \times \tilde{I}\right] g_L(u) \tag{2.8}$$

and

$$\tilde{\mathcal{B}}_R(\mathbf{x},\mathbf{x}') = \frac{\kappa}{8\pi\gamma^2}\left[\gamma_R \tilde{I} + \frac{1}{\gamma_R}\nabla\nabla - \nabla \times \tilde{I}\right] g_R(u), \tag{2.9}$$

with

$$\gamma_L = \frac{\kappa}{1-\kappa\beta}, \qquad \gamma_R = \frac{\kappa}{1+\kappa\beta}, \qquad \mathbf{u} = \mathbf{x} - \mathbf{x}', \qquad u = |\mathbf{u}|,$$
$$g_A(u) = \frac{e^{i\gamma_A u}}{u}, \qquad A = L, R,$$

and

$$\tilde{I} = \hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}}$$

is the identity dyadic [8,9].

3. THE EXPANSION THEOREM

In order to simplify the statement of the theorem, we give the following definition as in [2,3] and [6].

DEFINITION 1. *Let V be the exterior domain $\mathbb{R}^3 \setminus \bar{\Omega}$ described in Section 2. The vector field $\mathbf{U}^s : V \longrightarrow \mathbb{C}^3$ is called an* **E/H** *chiral radiation function if it has the following properties:*

(i) *$\mathbf{U}^s \in C^2(V)$ and satisfies (2.2), and*

(ii) *$\mathbf{U}^s$ satisfies (2.5) uniformly over all directions.*

LEMMA 1. *Let $\tilde{\mathcal{B}}_A(\mathbf{x},\mathbf{x}')$ be the $A = L, R$-part of the function $\tilde{\mathcal{B}}(\mathbf{x},\mathbf{x}')$ as in Section 2. Then we have*

$$\tilde{\mathcal{B}}_A(\mathbf{x},\mathbf{x}') = \frac{\kappa}{8\pi\gamma^2}\left\{\gamma_A \tilde{K}_A(\hat{\mathbf{u}}) + \frac{i}{u}\left[\tilde{K}_A(\hat{\mathbf{u}}) - 2\hat{\mathbf{u}}\hat{\mathbf{u}}\right] - \frac{1}{\gamma_A u^2}\left[\tilde{I} - 3\hat{\mathbf{u}}\hat{\mathbf{u}}\right]\right\} g_A(u) \tag{3.1}$$

for $A = L, R$, where

$$\tilde{K}_L(\hat{\mathbf{u}}) = \tilde{I} - \hat{\mathbf{u}}\hat{\mathbf{u}} + i\hat{\mathbf{u}} \times \tilde{I} \quad \text{and} \quad \tilde{K}_R(\hat{\mathbf{u}}) = \tilde{I} - \hat{\mathbf{u}}\hat{\mathbf{u}} - i\hat{\mathbf{u}} \times \tilde{I}. \tag{3.2}$$

PROOF. After some calculations, we have

$$\nabla g_A(u) = \left(i\gamma_A - \frac{1}{u}\right) g_A(u)\hat{\mathbf{u}}, \tag{3.3}$$

$$\nabla_x \nabla_x g_A(u) = \left\{\left(\frac{i\gamma_A}{u} - \frac{1}{u^2}\right)\left(\tilde{I} - 3\hat{\mathbf{u}}\hat{\mathbf{u}}\right) - \gamma_A^2 \hat{\mathbf{u}}\hat{\mathbf{u}}\right\} g_A(u), \tag{3.4}$$

$$\nabla_x \times \tilde{I} g_A(u) = \left(i\gamma_A - \frac{1}{u}\right) g_A(u)\hat{\mathbf{u}} \times \tilde{I}. \tag{3.5}$$

The result now is clear from (2.8) and (2.9).

LEMMA 2. *Let $\mathbf{U}^s : V \longrightarrow \mathbb{C}^3$ be an $\mathbf{E}/\mathbf{H}$ chiral radiation function and let 2α be the maximal diameter of the scatterer $\bar{\Omega}$. Then we have*

$$\mathbf{U}^s(\mathbf{x}) = \frac{\kappa\alpha^2}{8\pi\gamma^2}\sum_{n=1}^{3}\int_{|\mathbf{x}'|=1} u^{1-n}\left[g_L(u)\boldsymbol{\Psi}_n^L(\hat{\mathbf{x}}') - g_R(u)\boldsymbol{\Psi}_n^R(\hat{\mathbf{x}}')\right] ds(\hat{\mathbf{x}}'), \tag{3.6}$$

where

$$\boldsymbol{\Psi}_1^L(\hat{\mathbf{x}}') = \gamma_L \tilde{K}_L(\hat{\mathbf{u}}) \cdot \mathbf{T}_L(\hat{\mathbf{x}}'), \tag{3.7}$$

$$\boldsymbol{\Psi}_2^L(\hat{\mathbf{x}}') = i\left(\tilde{K}_L(\hat{\mathbf{u}}) - 2\hat{\mathbf{u}}\hat{\mathbf{u}}\right) \cdot \mathbf{T}_L(\hat{\mathbf{x}}'), \tag{3.8}$$

$$\boldsymbol{\Psi}_3^L(\hat{\mathbf{x}}') = -\frac{1}{\gamma_L}\left(\tilde{I} - 3\hat{\mathbf{u}}\hat{\mathbf{u}}\right) \cdot \mathbf{T}_L(\hat{\mathbf{x}}'), \tag{3.9}$$

with

$$\mathbf{T}_L(\hat{\mathbf{x}}') = \gamma_R \hat{\mathbf{n}} \times \mathbf{U}^t(\hat{\mathbf{x}}') + \hat{\mathbf{n}} \times \left(\nabla \times \mathbf{U}^t(\hat{\mathbf{x}}')\right), \tag{3.10}$$

and

$$\boldsymbol{\Psi}_1^R(\hat{\mathbf{x}}') = \gamma_R \tilde{K}_R(\hat{\mathbf{u}}) \cdot \mathbf{T}_R(\hat{\mathbf{x}}'), \tag{3.11}$$

$$\boldsymbol{\Psi}_2^R(\hat{\mathbf{x}}') = i\left(\tilde{K}_R(\hat{\mathbf{u}}) - 2\hat{\mathbf{u}}\hat{\mathbf{u}}\right) \cdot \mathbf{T}_R(\hat{\mathbf{x}}'), \tag{3.12}$$

$$\boldsymbol{\Psi}_3^R(\hat{\mathbf{x}}') = -\frac{1}{\gamma_R}\left(\tilde{I} - 3\hat{\mathbf{u}}\hat{\mathbf{u}}\right) \cdot \mathbf{T}_R(\hat{\mathbf{x}}'), \tag{3.13}$$

with

$$\mathbf{T}_R(\hat{\mathbf{x}}') = \gamma_L \hat{\mathbf{n}} \times \mathbf{U}^t(\hat{\mathbf{x}}') - \hat{\mathbf{n}} \times \left(\nabla \times \mathbf{U}^t(\hat{\mathbf{x}}')\right). \tag{3.14}$$

PROOF. We apply the divergence theorem to the first integral in (2.6) and Green's second theorem to the second integral, in the space between the surfaces S and $|\mathbf{x}'| = \alpha$. The volume integrals vanish since $\mathbf{U}^t$ is a solution of (2.2) in this region. Then we have

$$\begin{aligned}\mathbf{U}^s(\mathbf{x}) = {} & -2\gamma^2\alpha^2\beta\int_{|\mathbf{x}'|=1} \tilde{\mathcal{B}}(\mathbf{x},\mathbf{x}') \cdot \left[\hat{\mathbf{n}} \times \mathbf{U}^t(\mathbf{x}')\right] ds(\mathbf{x}') \\ & + \alpha^2\int_{|\mathbf{x}'|=1}\left\{\tilde{\mathcal{B}}(\mathbf{x},\mathbf{x}') \cdot \left(\hat{\mathbf{n}} \times \left[\nabla \times \mathbf{U}^t(\mathbf{x}')\right]\right) + \left[\nabla_x \times \tilde{\mathcal{B}}(\mathbf{x},\mathbf{x}')\right] \cdot \left(\hat{\mathbf{n}} \times \mathbf{U}^t(\mathbf{x}')\right)\right\} ds(\mathbf{x}').\end{aligned} \tag{3.15}$$

Using (2.7), Lemma 1, and the relation

$$\nabla \times \tilde{\mathcal{B}}(\mathbf{x},\mathbf{x}') = \gamma_L \tilde{\mathcal{B}}_L(\mathbf{x},\mathbf{x}') - \gamma_R \tilde{\mathcal{B}}_R(\mathbf{x},\mathbf{x}'), \tag{3.16}$$

we deduce expansion (3.6).

LEMMA 3. *For every $r \geq r_0 > \alpha$, the functions $u^{-n}\exp\{i\kappa(u-r)\}$, $n = 1, 2, 3, \ldots$, where $u = |\mathbf{r} - \mathbf{r}'|$ and $r' = \alpha$, are analytic in the variable $\rho = \alpha/r$. Their Laurent series expansions converge absolutely and uniformly for $r \geq r_0 > \alpha$ and all directions $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$. Moreover, their expansions can be differentiated term-by-term with respect to r, θ, φ any number of times and the resulting series are absolutely and uniformly convergent.*

PROOF. (See [6].) From the above lemmas one may infer the following theorem.

THEOREM 1. EXPANSION THEOREM. *Let $\mathbf{U}^s$ be an $\mathbf{E}/\mathbf{H}$ chiral radiation function for the domain $r > \alpha$, where (r, θ, φ) are the spherical coordinates of the observation point $\mathbf{r}$. Then*

$$\mathbf{U}^s(\mathbf{r}) = \frac{e^{i\gamma_L r}}{r}\sum_{n=0}^{\infty}\frac{\mathbf{F}_n^L(\theta,\varphi)}{r^n} + \frac{e^{i\gamma_R r}}{r}\sum_{n=0}^{\infty}\frac{\mathbf{F}_n^R(\theta,\varphi)}{r^n}, \tag{3.17}$$

which converges for $r > \alpha$.

The series in (3.17), as well as those obtained by term-by-term differentiation of any order, converge absolutely and uniformly in the closed domain $r \geq r_0 > \alpha$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$.

A consequence of the above expansion theorem is the following proposition by which we recurrently obtain all the coefficients in series (3.17) by the leading coefficient.

PROPOSITION 1. RECURSION RELATIONS. *The coefficients $\mathbf{F}_n^A(\theta,\varphi)$ of series (3.17) can be determined from the "radiation pattern" $\mathbf{F}_0^A(\theta,\varphi)$ by the recurrence relations*

$$\left(\gamma^2 - \gamma_A^2\right)\mathbf{F}_1^A + 2i\gamma^2\beta\gamma_A\hat{\mathbf{r}} \times \mathbf{F}_1^A = 2\gamma^2\beta\,(\hat{\mathbf{r}} - D) \times \mathbf{F}_0^A, \tag{3.18}$$

and, for $n \geq 1$,

$$\begin{aligned}\left(\gamma^2 - \gamma_A^2\right)\mathbf{F}_{n+1}^A + 2i\gamma^2\beta\gamma_A\hat{\mathbf{r}} \times \mathbf{F}_{n+1}^A = 2ni\gamma_A\mathbf{F}_n^A + 2\gamma^2\beta\left((n+1)\,\hat{\mathbf{r}} - D\right) \times \mathbf{F}_n^A \\ - [B + n(n-1)]\mathbf{F}_{n-1}^A,\end{aligned} \tag{3.19}$$

where $A = L, R$ and

$$D = \hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\hat{\varphi}\frac{\partial}{\partial\varphi}, \tag{3.20}$$

with $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\varphi}$ being the unit vectors of the spherical coordinate system and B the Beltrami operator

$$B = D \cdot D = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{3.21}$$

PROOF. Since $\mathbf{U}^s(\mathbf{r})$ is divergence-free, (2.2) becomes

$$\Delta\mathbf{U}^s + \gamma^2\mathbf{U}^s + 2\gamma^2\beta\nabla \times \mathbf{U}^s = \mathbf{0}. \tag{3.22}$$

From the expansion theorem and equation (3.17) we have

$$\mathbf{U}^s = \mathbf{U}_L^s + \mathbf{U}_R^s, \tag{3.23}$$

with

$$\mathbf{U}_A^s = \sum_{n=0}^{\infty} g_A^n(r)\mathbf{F}_n^A(\theta,\varphi), \qquad A = L, R, \tag{3.24}$$

where

$$g_A^n(r) = \frac{e^{i\gamma_A r}}{r^{n+1}}. \tag{3.25}$$

Since the set

$$\left\{\frac{e^{i\gamma_L r}}{r^{n+1}}, \frac{e^{i\gamma_R r}}{r^{n+1}}\right\}_{n\geq 0} \tag{3.26}$$

is linearly independent in the domain $r > \alpha$ [6], we deduce that $\mathbf{U}_A^s$ for $A = L, R$ satisfies equation (3.22). Moreover, we have

$$\Delta\left(g_A^n \mathbf{F}_n^A\right) = -\gamma_A^2 g_A^n \mathbf{F}_n^A - \frac{2ni\gamma_A}{r} g_A^n \mathbf{F}_n^A + \frac{1}{r^2}[B + n(n+1)] g_A^n \mathbf{F}_n^A, \tag{3.27}$$

$$\nabla \times \left(g_A^n \mathbf{F}_n^A\right) = i\gamma_A g_A^n \hat{\mathbf{r}} \times \mathbf{F}_n^A + \frac{1}{r} g_A^n \left[D - (n+1)\hat{\mathbf{r}}\right] \times \mathbf{F}_n^A. \tag{3.28}$$

Substituting (3.24), (3.27), and (3.28) into equation (3.22), we obtain desired relations (3.18) and (3.19), as well as the $n = 0$ terms of the series which give

$$\left(\gamma^2 - \gamma_A^2\right) \mathbf{F}_0^A + 2i\beta\gamma^2\gamma_A \hat{\mathbf{r}} \times \mathbf{F}_0^A = \mathbf{0}, \qquad A = L, R. \tag{3.29}$$

NOTE. Since $\gamma_L - \gamma_R = 2\gamma^2\beta$ and $\gamma^2 = \gamma_L\gamma_R$, relations (3.18), (3.19), and (3.29) can be written as follows:

$$\begin{aligned} \gamma_L \left[i\hat{\mathbf{r}} \times \mathbf{F}_1^L - \mathbf{F}_1^L\right] &= [\hat{\mathbf{r}} - D] \times \mathbf{F}_0^L, \\ \gamma_R \left[i\hat{\mathbf{r}} \times \mathbf{F}_1^R + \mathbf{F}_1^R\right] &= [\hat{\mathbf{r}} - D] \times \mathbf{F}_0^R, \end{aligned} \tag{3.18$'$}$$

$$\begin{aligned} 2\gamma^2\beta\gamma_L \left[i\hat{\mathbf{r}} \times \mathbf{F}_{n+1}^L - \mathbf{F}_{n+1}^L\right] &= 2in\gamma_L \mathbf{F}_n^L + 2\gamma^2\beta \left[(n+1)\hat{\mathbf{r}} - D\right] \times \mathbf{F}_n^L \\ &\quad - \left[B + n(n-1)\right] \mathbf{F}_{n-1}^L, \\ 2\gamma^2\beta\gamma_R \left[i\hat{\mathbf{r}} \times \mathbf{F}_{n+1}^R + \mathbf{F}_{n+1}^R\right] &= 2in\gamma_R \mathbf{F}_n^R + 2\gamma^2\beta \left[(n+1)\hat{\mathbf{r}} - D\right] \times \mathbf{F}_n^R \\ &\quad - \left[B + n(n-1)\right] \mathbf{F}_{n-1}^R, \end{aligned} \tag{3.19$'$}$$

and

$$\mathbf{F}_0^L = i\hat{\mathbf{r}} \times \mathbf{F}_0^L, \qquad \mathbf{F}_0^R = -i\hat{\mathbf{r}} \times \mathbf{F}_0^R. \tag{3.29$'$}$$

As in the classical electromagnetic and acoustic scattering (see [5, pp. 75,117]) we have the following corollary.

COROLLARY 1. *Let $\mathbf{U}^s$ be an $\mathbf{E}/\mathbf{H}$ chiral radiation function for which the radiation pattern $\mathbf{F}_0^A$, $A = L, R$ vanishes identically. Then $\mathbf{U}^s = \mathbf{0}$ in V.*

PROOF. This is a straightforward consequence of the recursion relations (3.18) and (3.19).

REMARK. For the achiral electromagnetic scattering, Wilcox in [3] has given the expansion

$$\mathbf{E}^s(\mathbf{r}) = \frac{e^{i\kappa r}}{r} \sum_{n=0}^{\infty} \mathbf{F}_n(\theta, \varphi) r^{-n}, \tag{3.30}$$

where the coefficients $\mathbf{F}_n$ are given by the following recurrence relations:

$$2i\kappa n \left(\hat{\mathbf{r}} \cdot \mathbf{F}_{n+1}\right) = \left[B + n(n-1)\right] \left(\hat{\mathbf{r}} \cdot \mathbf{F}_n\right), \tag{3.31}$$

$$2i\kappa n \left(\hat{\boldsymbol{\theta}} \cdot \mathbf{F}_n\right) = \left[B + n(n-1)\right] \left(\hat{\boldsymbol{\theta}} \cdot \mathbf{F}_{n-1}\right) + D_\theta \mathbf{F}_{n-1}, \tag{3.32}$$

$$2i\kappa n \left(\hat{\varphi} \cdot \mathbf{F}_n\right) = \left[B + n(n-1)\right] \left(\hat{\varphi} \cdot \mathbf{F}_{n-1}\right) + D_\varphi \mathbf{F}_{n-1}, \tag{3.33}$$

for $n = 1, 2, \dots$, where the differential operators D_θ, D_φ used by Wilcox [3] are given by

$$D_\theta \mathbf{F} = 2\frac{\partial\left(\hat{\mathbf{r}} \cdot \mathbf{F}\right)}{\partial\theta} - \frac{1}{\sin^2\theta}\left(\hat{\boldsymbol{\theta}} \cdot \mathbf{F}\right) - 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial(\hat{\varphi} \cdot \mathbf{F})}{\partial\varphi} \tag{3.34}$$

and

$$D_\varphi \mathbf{F} = \frac{2}{\sin\theta}\frac{\partial\left(\hat{\mathbf{r}} \cdot \mathbf{F}\right)}{\partial\varphi} + 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial\left(\hat{\boldsymbol{\theta}} \cdot \mathbf{F}\right)}{\partial\varphi} - \frac{1}{\sin^2\theta}\left(\hat{\varphi} \cdot \mathbf{F}\right). \tag{3.35}$$

We now check the results of our work against Wilcox's results. In fact, if $\beta = 0$, then $\gamma_L = \gamma_R = \gamma = \kappa$, where κ is the wave number. Then, equation (2.2) becomes $\nabla \times \nabla \times \mathbf{U} - \kappa^2 \mathbf{U} = 0$ and the expansion relation (3.17) takes the form (3.30) with

$$\mathbf{F}_n(\theta, \varphi) = \mathbf{F}_n^L(\theta, \varphi) + \mathbf{F}_n^R(\theta, \varphi), \qquad n = 0, 1, 2, \dots. \tag{3.36}$$

From (3.29′), (3.18′), (3.19′), and the relation $D \times \hat{\mathbf{r}} = \mathbf{0}$, we obtain

$$i\kappa \hat{\mathbf{r}} \cdot \mathbf{F}_1 = -D \cdot \mathbf{F}_0, \tag{3.37}$$

and

$$2in\kappa \mathbf{F}_n = [B + n(n-1)]\mathbf{F}_{n-1}. \tag{3.38}$$

Since $\nabla \cdot \mathbf{U}^s = 0$, we have

$$i\kappa \hat{\mathbf{r}} \mathbf{F}_n = n\hat{\mathbf{r}} \cdot \mathbf{F}_{n-1} - D \cdot \mathbf{F}_{n-1}. \tag{3.39}$$

We use the spherical decomposition of $B\mathbf{F}$,

$$B\mathbf{F} = [B(\hat{\mathbf{r}} \cdot \mathbf{F}) + 2(\hat{\mathbf{r}} \cdot \mathbf{F}) - 2D \cdot \mathbf{F}]\,\hat{\mathbf{r}} + \left[B\left(\hat{\boldsymbol{\theta}} \cdot \mathbf{F}\right) + D_\theta \mathbf{F}\right]\hat{\boldsymbol{\theta}} + [B(\hat{\varphi} \cdot \mathbf{F}) + D_\varphi \mathbf{F}]\,\hat{\varphi}. \tag{3.40}$$

Hence, equations (3.31)–(3.33) can easily be obtained from (3.37)–(3.39) and are exactly the equations for the corresponding (achiral) electromagnetic problem given by Wilcox [3].

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