Solutions to two functional equations arising in dynamic programming

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Abstract

This paper deals with the existence, uniqueness and iterative approximation of solutions for two functional equations arising in dynamic programming of multistage decision processes. The results presented in this paper extend, improve and unify the results due to Bellman, Bhakta and Choudhury, Bhakta and Mitra, Liu, Liu and Ume and others. © 2005 Elsevier B.V. All rights reserved.

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1. Introduction

The purpose of this work is to discuss the existence of solutions for the following two functional equations arising in dynamic programming of multistage decision processes:

\[ f(x) = \text{opt}_{y \in D} \{ p(x, y) + A(x, y, f(a(x, y))) \}, \quad x \in S \] (1.1)
and
\[ f(x) = \text{opt} \left\{ p(x, y) + q(x, y) f(a(x, y)) \right\} \]
\[ + \text{opt} \left\{ r(x, y), s(x, y) f(b(x, y)), t(x, y) f(c(x, y)) \right\}, \quad x \in S, \tag{1.2} \]
where \text{opt} denotes the sup or inf, \( x \) and \( y \) stand for the state and decision vectors, respectively, \( a, b \) and \( c \) represent the transformations of the processes, and \( f(x) \) denotes the optimal return function with initial state \( x \).

Several existence and uniqueness results for some special cases of the functional equations (1.1) and (1.2) have been established in [1–7, 9–12]. Bellman [2, 3] was the first to investigate the existence and uniqueness of solutions for the functional equations below
\[ f(x) = \text{inf} \left\{ r(x, y) \right\} \max \{ r(x, y), s(x, y) f(b(x, y)) \}, \quad x \in S \tag{1.3} \]
and
\[ f(x) = \text{inf} \left\{ r(x, y) \right\} \max \{ r(x, y), f(b(x, y)) \}, \quad x \in S \tag{1.4} \]
in a complete metric space \( BB(S) \). Bellman and Roosta [5] studied the iterative approximation of solutions for the functional equation
\[ f(x) = \max \left\{ p(x, y) \right\} \sup \{ A(x, y), f(a(x, y)) \}, \quad x \in S. \tag{1.5} \]
Bellman and Lee [4] pointed out that the basic form of the functional equations of dynamic programming is as follows:
\[ f(x) = \sup \left\{ A(x, y), f(a(x, y)) \right\}, \quad x \in S. \tag{1.6} \]
Bhakta and Mitra [7] obtained the existence and uniqueness of solutions for the functional equations
\[ f(x) = \sup \left\{ p(x, y) + A(x, y), f(a(x, y)) \right\}, \quad x \in S \tag{1.7} \]
in a Banach space \( B(S) \) and
\[ f(x) = \sup \left\{ p(x, y) + f(a(x, y)) \right\}, \quad x \in S \tag{1.8} \]
in \( BB(S) \), respectively. Bhakta and Choudhury [6] established the existence of solutions for the functional equations (1.3) and (1.4) in \( BB(S) \). Recently, Liu [11] and Liu and Ume [12] investigated properties of solutions for the functional equations (1.4) and
\[ f(x) = \text{opt} \left\{ x[p(x, y) + f(a(x, y))] + (1 - x)\text{opt}[r(x, y), f(a(x, y))] \right\}, \quad x \in S, \tag{1.9} \]
where \( x \) is a constant in \([0, 1]\), in \( BB(S) \).

This paper is divided into four sections. In Section 2, we recall some basic concepts, notations and lemmas, and establish a lemma that will be needed further on. In Section 3, we utilize the fixed-point theorem due to Boyd and Wong [8] to establish the existence, uniqueness and iterative approximation
of solution for the functional equation (1.1) in Banach spaces $BC(S)$ and $B(S)$, respectively. In Section 4, we use the fixed-point theorem due to Bhakta and Choudhury [6] to obtain the existence, uniqueness and iterative approximation of solution for the functional equation (1.2), and discuss, under suitable conditions, properties of solutions for the functional equations (1.2) in $BB(S)$. The results presented here generalize substantially the results due to Bellman [2], Bhakta and Choudhury [6], Bhakta and Mitra [7], Liu [11], Liu and Ume [12] and others.

2. Preliminaries

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{R}^- = (-\infty, 0]$. For any $t \in \mathbb{R}$, $[t]$ denotes the largest integer not exceeding $t$. Let $(X, \| \cdot \|)$ and $(Y, \| \cdot \|')$ be real Banach spaces, $S \subseteq X$ be the state space and $D \subseteq Y$ be the decision space. Define

$$
\Phi_1 = \{ \varphi : \varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ is right continuous at } t = 0 \},
$$

$$
\Phi_2 = \{ \varphi : \varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ is nondecreasing} \},
$$

$$
\Phi_3 = \{ \varphi : \varphi \in \Phi_1 \text{ and } \varphi(0) = 0 \},
$$

$$
\Phi_4 = \{ \varphi : \varphi \in \Phi_1 \cap \Phi_2 \text{ and } \varphi(t) < t \text{ for } t > 0 \},
$$

$$
\Phi_5 = \{ \varphi : \varphi \in \Phi_2 \text{ and } \varphi(t) < t \text{ for } t > 0 \},
$$

$$
\Phi_6 = \left\{ \varphi : \varphi \in \Phi_2 \text{ and } \sum_{n=0}^{\infty} \varphi^n(t) < t \text{ for } t > 0 \right\},
$$

$$
\Phi_7 = \left\{ (\varphi, \psi) : \varphi, \psi \in \Phi_2, \psi(t) > 0 \text{ and } \sum_{n=0}^{\infty} \psi(\varphi^n(t)) < \infty \text{ for } t > 0 \right\},
$$

$$
B(S) = \{ f : f : S \rightarrow \mathbb{R} \text{ is bounded} \},
$$

$$
BC(S) = \{ f : f \in B(S) \text{ is continuous} \},
$$

$$
BB(S) = \{ f : f : S \rightarrow \mathbb{R} \text{ is bounded on bounded subsets of } S \}.
$$

Clearly $(B(S), \| \cdot \|_1)$ and $(BC(S), \| \cdot \|_1)$ are Banach spaces with norm $\| f \|_1 = \sup_{x \in S} |f(x)|$. For any positive integer $k$ and $f, g \in BB(S)$, let

$$
d_k(f, g) = \sup\{ |f(x) - g(x)| : x \in \overline{B}(0, k) \},
$$

$$
d(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(f, g)}{1 + d_k(f, g)},
$$

where $\overline{B}(0, k) = \{ x : x \in S \text{ and } \| x \| \leq k \}$. Then $\{ d_k \}_{k \geq 1}$ is a countable family of pseudometrics on $BB(S)$. A sequence $\{x_n\}_{n \geq 1}$ in $BB(S)$ is said to converge to a point $x \in BB(S)$ if for any $k \geq 1$, $d_k(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$, and to be a Cauchy sequence if for any $k \geq 1$, $d_k(x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$. It is clear that $(BB(S), d)$ is a complete metric space. A metric space $(M, \rho)$ is said to be metrically convex if for
each \(x, y \in M\), there is a \(z \neq x, y\) for which \(\rho(x, y) = \rho(x, z) + \rho(z, y)\). Clearly any Banach space is metrically convex.

**Lemma 2.1** *(Boyd and Wong [8])*. Suppose that \((M, \rho)\) is a completely metrically convex metric space and that \(f : M \to M\) satisfies

\[
\rho(f(x), f(y)) \leq \varphi(\rho(x, y)) \quad \text{for } x, y \in M,
\]

where \(\varphi : \mathbb{T} \to \mathbb{R}^+\) satisfies \(\varphi(t) < t\) for \(t \in \mathbb{T} - \{0\}\), where \(P = \{\rho(x, y) : x, y \in M\}\) and \(\mathbb{T}\) denotes the closure of \(P\). Then \(f\) has a unique fixed point \(u \in M\) and \(\lim_{n \to \infty} f^n(x) = u\) for each \(x \in M\).

**Lemma 2.2** *(Liu and Ume [12])*.

Let \(a, b, c\) and \(d\) be in \(\mathbb{R}\). Then

\[
|\text{opt}\{a, b\} - \text{opt}\{c, d\}| \leq \max\{|a - c|, |b - d|\}.
\]

**Lemma 2.3**.

Let \(b, c\) be in \(\mathbb{R}\) and \(s, t\) be in \(\mathbb{R}^+\). Then

(i) \(\max\{sb, tc\} \geq \max\{s, t\} \min\{b, c\}\),
(ii) \(\min\{sb, tc\} \leq \min\{s, t\} \max\{b, c\}\).

**Proof**. Suppose that \(\min\{b, c\} \geq 0\). It follows from \(s, t \in \mathbb{R}^+\) that

\[
\max\{sb, tc\} \geq \max\{s \min\{b, c\}, t \min\{b, c\}\} = \max\{s, t\} \min\{b, c\}.
\]

Suppose that \(\min\{b, c\} < 0\). Then

\[
\max\{sb, tc\} \geq \max\{s \min\{b, c\}, t \min\{b, c\}\} = \min\{s, t\} \min\{b, c\} \geq \max\{s, t\} \min\{b, c\}.
\]

That is, (i) holds. Similarly, we can conclude that (ii) also hold. This completes the proof. \(\square\)

### 3. Existence and uniqueness of solutions in \(BC(S)\) and \(B(S)\)

Now we establish those conditions which guarantee the existence, uniqueness and iterative approximation of solution for the functional equation (1.1) in \(BC(S)\) and \(B(S)\), respectively.

**Theorem 3.1**. Let \(a : S \times D \to S\), \(p : S \times D \to \mathbb{R}\) and \(A : S \times D \times \mathbb{R} \to \mathbb{R}\) be mappings and let \(\varphi\) be in \(\Phi_3\) and \(\psi\) be in \(\Phi_4\) such that

(C1) \(p\) and \(A\) are bounded,
(C2) for each \(x_0 \in S\), \(p(x, y) \to p(x_0, y)\) and \(a(x, y) \to a(x_0, y)\) as \(x \to x_0\) uniformly for \(y \in D\),
(C3) \(|A(x, y, z) - A(x_0, y, z)| \leq \varphi(\|x - x_0\|)\) for \(x, x_0 \in S\), \(y \in D\), \(z \in \mathbb{R}\),
(C4) \(|A(x, y, z) - A(x, y, z_0)| \leq \psi(|z - z_0|)\) for \(x \in S\), \(y \in D\), \(z, z_0 \in \mathbb{R}\).
Then the functional equation (1.1) possesses a unique solution \( w \in BC(S) \) and \( \{H^n h\}_{n \geq 1} \) converges to \( w \) for each \( h \in BC(S) \), where \( H \) is defined by

\[
H h(x) = \operatorname{opt} \{ p(x, y) + A(x, y, h(a(x, y))) \}, \quad x \in S.
\]  

(3.1)

**Proof.** Let \( x_0 \in S \) and \( h \in BC(S) \). It is clear that (C1) means that \( H h \) is bounded. In view of (C2), \( \varphi \in \Phi_3 \) and \( \psi \in \Phi_4 \), we know that for given \( \varepsilon > 0 \), there exist \( \delta_1 > 0 \), \( \delta_2 > 0 \) and \( \delta_3 > 0 \) satisfying

\[
\varphi(\|x - x_0\|) < \frac{1}{4} \varepsilon \quad \text{for} \quad x \in S \quad \text{with} \quad \|x - x_0\| < \delta_1,
\]

(3.2)

\[
\psi(\delta_1) < \frac{1}{4} \varepsilon,
\]

(3.3)

\[
|p(x, y) - p(x_0, y)| < \frac{1}{4} \varepsilon \quad \text{for} \quad (x, y) \in S \times D \quad \text{with} \quad \|x - x_0\| < \delta_1,
\]

(3.4)

\[
|h(x) - h(x_0)| < \delta_1 \quad \text{for} \quad x \in S \quad \text{with} \quad \|x - x_0\| < \delta_2,
\]

(3.5)

\[
\|a(x, y) - a(x_0, y)\| < \delta_2 \quad \text{for} \quad (x, y) \in S \times D \quad \text{with} \quad \|x - x_0\| < \delta_3.
\]

(3.6)

On account of (3.3), (3.5) and (3.6), we conclude that

\[
\psi \left( \sup_{y \in D} |h(a(x, y)) - h(a(x_0, y))| \right) \leq \psi(\delta_1) < \frac{1}{4} \varepsilon \quad \text{for} \quad (x, y) \in S \times D \quad \text{with} \quad \|x - x_0\| < \delta_3.
\]

(3.7)

Set \( \delta = \min\{\delta_1, \delta_3\} \). In terms of (C3), (C4), (3.1), (3.2), (3.4) and (3.7), we deduce that for \( x \in S \) with \( \|x - x_0\| < \delta \)

\[
|H h(x) - H h(x_0)|
\]

\[
= \operatorname{opt} \{ p(x, y) + A(x, y, h(a(x, y))) \} - \operatorname{opt} \{ p(x_0, y) + A(x_0, y, h(a(x_0, y))) \}
\]

\[
\leq \sup_{y \in D} \{ |p(x, y) - p(x_0, y)| + |A(x, y, h(a(x, y))) - A(x_0, y, h(a(x_0, y)))| \}
\]

\[
\leq \sup_{y \in D} \{ |p(x, y) - p(x_0, y)| + \sup_{y \in D} |A(x, y, h(a(x, y))) - A(x_0, y, h(a(x, y)))| \}
\]

\[
+ \sup_{y \in D} |A(x_0, y, h(a(x, y))) - A(x_0, y, h(a(x_0, y)))| \leq \frac{1}{4} \varepsilon + \varphi(\|x - x_0\|) + \sup_{y \in D} \psi(|h(a(x, y)) - h(a(x_0, y))|)
\]

\[
< \varepsilon,
\]

which implies that \( H h \) is continuous at \( x_0 \). Thus \( H \) is a self mapping on \( BC(S) \). Given \( \varepsilon > 0 \), \( x \in S \) and \( h, g \in BC(S) \). Suppose that \( \operatorname{opt}_{y \in D} = \sup_{y \in D} \). Then there exist \( y, z \in D \) such that

\[
H h(x) < p(x, y) + A(x, y, h(a(x, y))) + \varepsilon,
\]

\[
H g(x) < p(x, z) + A(x, z, g(a(x, z))) + \varepsilon,
\]

\[
H h(x) \geq p(x, z) + A(x, z, h(a(x, z))),
\]

\[
H g(x) \geq p(x, y) + A(x, y, g(a(x, y))).
\]

(3.8)
Using (3.8) and (C4), we arrive at
\[ |Hh(x) - Hg(x)| < \max\{ |A(x, y, h(a(x, y))) - A(x, y, g(a(x, y)))|, \\
|A(x, z, h(a(x, z))) - A(x, z, g(a(x, z)))| + \varepsilon \}
\leq \psi(\max\{ |h(a(x, y)) - g(a(x, y))|, |h(a(x, z)) - g(a(x, z))| \}) + \varepsilon
\leq \psi(\|h - g\|_1) + \varepsilon,
\]
which gives that
\[ \|Hh - Hg\|_1 \leq \psi(\|h - g\|_1) + \varepsilon. \quad (3.9) \]

Similarly we conclude that (3.9) holds for \( \text{opt}_{y \in D} = \inf_{y \in D} \). As \( \varepsilon \to 0^+ \) in (3.9), we have
\[ \|Hh - Hg\|_1 \leq \psi(\|h - g\|_1). \]

Lemma 2.1 ensures that \( H \) has a unique fixed point \( w \in BC(S) \) and \( \{H^n h\}_{n \geq 1} \) converges to \( w \) for any \( h \in BC(S) \). It is evident that \( w \) is also a unique solution of the functional equation (1.1) in \( BC(S) \). This completes the proof. \( \Box \)

It follows from the proof of Theorem 3.1 that the following result holds.

**Theorem 3.2.** Let \( a : S \times D \to S \), \( p : S \times D \to \mathbb{R} \) and \( A : S \times D \times \mathbb{R} \to \mathbb{R} \) be mappings and let \( \psi \) be in \( \Phi_5 \) satisfying (C1) and (C4). Then the functional equation (1.1) possesses a unique solution \( w \in B(S) \) and \( \{H^n h\}_{n \geq 1} \) converges to \( w \) for each \( h \in B(S) \), where \( H \) is defined by (3.1).

**Remark 3.1.** Theorem 2.1 of Bhakta and Mitra [7] is a particular case of Theorem 3.2. The following example reveals that Theorem 3.2 is indeed an extension of Theorem 2.1 in [7].

**Example 3.1.** Let \( X = Y = \mathbb{R} \), \( S = [1, +\infty) \) and \( D = \mathbb{R}^- \). It follows from Theorem 3.2 that the functional equation below
\[ f(x) = \text{opt}_{y \in D} \left\{ \frac{x}{x + y^2} + \frac{1}{x^2 + |y|} + \frac{f(3 + \sin(2x - 5y))}{2 + 2[f(3 + \sin(2x - 5y))]} \right\}, \quad x \in S \quad (3.10) \]
possesses a unique solution in \( B(S) \). But Theorem 2.1 in [7] is not valid for the functional equation (3.10).

**Remark 3.2.** We point out that the functional equation (3.10) possesses also a unique solution in \( BC(S) \). In fact, taking
\[ p(x, y) = \frac{x}{x + y^2}, \quad a(x, y) = 3 + \sin(2x - 5y), \]
\[ A(x, y, z) = \frac{1}{x^2 + |y|} + \frac{z}{2 + 2z^2}, \quad \varphi(t) = 2t, \quad \psi(t) = \frac{1}{2} t \]
for \( x \in S, y \in D, z \in \mathbb{R} \) and \( t \in \mathbb{R}^+ \), we check easily that the assumptions of Theorem 3.1 are fulfilled. Thus Theorem 3.1 ensures that the functional equation (3.10) possesses a unique solution in \( BC(S) \).
4. Properties of solutions in $BB(S)$

Theorem 4.1. Let $a, b, c : S \times D \rightarrow S$ and $p, q, r, s, t : S \times D \rightarrow \mathbb{R}$ satisfy the following conditions:

\begin{enumerate}[(D1)]
    
    
    \text{(D1)} \ p \text{ and } r \text{ are bounded on } B(0, k) \times D \text{ for } k \geq 1,
    
    \text{(D2)} \ \max\{\|a(x, y)\|, \|b(x, y)\|, \|c(x, y)\|\} \leq \|x\| \text{ for } (x, y) \in S \times D,
    
    \text{(D3)} \ \text{there exists a constant } \alpha \text{ such that}
    
    \begin{equation*}
    \sup_{(x, y) \in S \times D} \{|q(x, y)| + \max\{|s(x, y)|, |t(x, y)|\}| \leq \alpha < 1.
    \end{equation*}

Then the functional equation (1.2) possesses a unique solution $w \in BB(S)$ and \{H_n h\}_{n \geq 1} converges to $w$ for each $h \in BB(S)$, where $H$ is defined by

\begin{equation*}
H h(x) = \operatorname{opt}_{y \in D} \{p(x, y) + q(x, y)h(a(x, y)) + \operatorname{opt}\{r(x, y), s(x, y)h(b(x, y)), t(x, y)h(c(x, y))\}\}, \quad x \in S. \quad (4.1)
\end{equation*}

Proof. It follows from (D1) and (D2) that for each $k \geq 1$ and $h \in BB(S)$, there exist $\beta(k) > 0$ and $\eta(k, h) > 0$ such that

\begin{align*}
&\sup_{(x, y) \in B(0, k) \times D} \{|p(x, y)|, |r(x, y)|\} \leq \beta(k), \\
&\sup_{(x, y) \in B(0, k) \times D} \{|h(a(x, y))|, |h(b(x, y))|, |h(c(x, y))|\} \leq \eta(k, h). \quad (4.2)
\end{align*}

By virtue of (D3), (4.1) and (4.2), we know that

\begin{align*}
&|H h(x)| \leq \sup_{y \in D} \{|p(x, y)| + |q(x, y)||h(a(x, y))| + |r(x, y)| + \max\{|s(x, y)||h(b(x, y))|, |t(x, y)||h(c(x, y))|\}\} \\
&\leq \sup_{y \in D} \{|p(x, y)| + |r(x, y)| + \max\{|h(a(x, y))|, |h(b(x, y))|, |h(c(x, y))|\}\} \\
&\leq 2\beta(k) + \eta(k, h)
\end{align*}

for $x \in B(0, k)$. This means that $H$ is a self mapping on $BB(S)$. Let $\varepsilon > 0, k \geq 1, x \in B(0, k)$ and $g, h \in BB(S)$. Suppose that $\operatorname{opt}_{y \in D} = \sup_{y \in D}$. Then there exist $u, v \in D$ satisfying

\begin{align*}
H g(x) &< p(x, u) + q(x, u)g(a(x, u)) + \operatorname{opt}\{r(x, u), s(x, u)g(b(x, u)), t(x, u)g(c(x, u))\} + \varepsilon, \\
H h(x) &< p(x, v) + q(x, v)h(a(x, v)) + \operatorname{opt}\{r(x, v), s(x, v)h(b(x, v)), t(x, v)h(c(x, v))\} + \varepsilon, \\
H g(x) &\geq p(x, v) + q(x, v)g(a(x, v)) + \operatorname{opt}\{r(x, v), s(x, v)g(b(x, v)), t(x, v)g(c(x, v))\}, \\
H h(x) &\geq p(x, u) + q(x, u)h(a(x, u)) + \operatorname{opt}\{r(x, u), s(x, u)h(b(x, u)), t(x, u)h(c(x, u))\}. \quad (4.3)
\end{align*}
In terms of (4.3), (D3) and Lemma 2.2, we derive that

\[ Hg(x) - Hh(x) \]

\[ \leq q(x, u)(g(a(x, u)) - h(a(x, u))) + \text{opt}\{r(x, u), s(x, u)g(b(x, u)), t(x, u)g(c(x, u))\} \]

\[ - \text{opt}\{r(x, u), s(x, u)h(b(x, u)), t(x, u)h(c(x, u))\} + \varepsilon \]

\[ \leq |q(x, u)||g(a(x, u)) - h(a(x, u))| + \max\{|s(x, u)||g(b(x, u)) - h(b(x, u))|, \]

\[ |t(x, u)||g(c(x, u)) - h(c(x, u))|\] + \varepsilon

\[ \leq |q(x, u)| + \max\{|s(x, u)|, |t(x, u)|\} \max\{|g(a(x, u)) - h(a(x, u))|, \]

\[ |g(b(x, u)) - h(b(x, u))|, |g(c(x, u)) - h(c(x, u))|\] + \varepsilon

\[ \leq x \max\{|g(a(x, u)) - h(a(x, u))|, |g(b(x, u)) - h(b(x, u))|, |g(c(x, u)) - h(c(x, u))|\} + \varepsilon \]

and

\[ Hg(x) - Hh(x) \]

\[ \geq q(x, v)(g(a(x, v)) - h(a(x, v))) + \text{opt}\{r(x, v), s(x, v)g(b(x, v)), t(x, v)g(c(x, v))\} \]

\[ - \text{opt}\{r(x, v), s(x, v)h(b(x, v)), t(x, v)h(c(x, v))\} - \varepsilon \]

\[ \geq - |q(x, v)||g(a(x, v)) - h(a(x, v))| + \max\{|s(x, v)||g(b(x, v)) - h(b(x, v))|, \]

\[ |t(x, v)||g(c(x, v)) - h(c(x, v))|\] - \varepsilon

\[ \geq - x \max\{|g(a(x, v)) - h(a(x, v))|, |g(b(x, v)) - h(b(x, v))|, |g(c(x, v)) - h(c(x, v))|\} - \varepsilon, \]

which yield that

\[ |Hg(x) - Hh(x)| \]

\[ \leq x \max\{|g(a(x, u)) - h(a(x, u))|, |g(a(x, v)) - h(a(x, v))|, |g(b(x, u)) - h(b(x, u))|, \]

\[ |g(b(x, v)) - h(b(x, v))|, |g(c(x, u)) - h(c(x, u))|, |g(c(x, v)) - h(c(x, v))|\} + \varepsilon

\[ \leq x d_k(g, h) + \varepsilon. \]  \hspace{1cm} (4.4)

In a similar way, we can show that (4.4) holds for \( \text{opt}_{y \in D} = \inf_{y \in D} \). It follows that

\[ d_k(Hg, Hh) \leq \varphi(d_k(g, h)) + \varepsilon, \]  \hspace{1cm} (4.5)

where \( \varphi(\lambda) = \alpha \lambda \) for \( \lambda \in \mathbb{R}^+ \). As \( \varepsilon \to 0^+ \) in (4.5), we get that

\[ d_k(Hg, Hh) \leq \varphi(d_k(g, h)). \]

It follows from Theorem 2.2 in [6] that \( H \) has a unique fixed point \( w \in BB(S) \) and \( \{H^n h\}_{n \geq 1} \) converges to \( w \) for each \( h \in BB(S) \). Obviously, \( w \) is also a unique solution of the functional equation (1.2). This completes the proof. \( \Box \)

**Remark 4.1.** In case \( \text{opt}_{y \in D} = \inf_{y \in D} \), \( \text{opt} = \max \) and \( p(x, y) = q(x, y) = t(x, y) = 0 \) for \( (x, y) \in S \times D \), then Theorem 4.1 reduces to Theorem 3.4 in [6] and a result in [2, p. 149]. The example below shows that Theorem 4.1 extends substantially the results in [2,6].
Example 4.1. Let $X = Y = \mathbb{R}$ and $S = D = \mathbb{R}^+$. Consider the following functional equation:

$$f(x) = \text{opt}_{y \in D} \left\{ \frac{5x^3y}{1 + xy} \sin(x^2 + 2y \ln(1 + xy^2)) + \frac{x^2 + y^2}{1 + 3(x^2 + y^2)} f \left( \frac{x^2}{1 + x + y} \right) \right. $$

$$+ \text{opt} \left\{ x^2 \cos(x - 3y^2 + 1), \frac{\sin(x - 2y + 1)}{4 + x^2 + x(1 + y) + x^3y} f \left( \frac{x^4y^2}{1 + x^6 + y^4} \right) \right. $$

$$\left. \frac{1 + \ln(1 + x^2y^3)}{3 + 3x^2y^3} f \left( \frac{x^3y \sin(1 - 2xy)}{1 + x^2y} \right) \right\}, \quad x \in S. \quad (4.6)$$

Note that

$$\sup_{(x,y) \in B(0,k) \times D} \left\{ \frac{5x^3y}{1 + xy} | \sin(x^2 + 2y \ln(1 + xy^2))|, x^2 | \cos(x - 3y^2 + 1)| \right\} \leq 5k^2, \quad k \geq 1,$$

$$\max \left\{ \frac{x^2}{1 + x + y}, \frac{x^4y^2}{1 + x^6 + y^4}, \frac{x^3y | \sin(1 - 2xy)|}{1 + x^2y} \right\} \leq |x|, \quad (x, y) \in S \times D$$

and

$$\sup_{(x,y) \in S \times D} \left\{ \frac{x^2 + y^2}{1 + 3(x^2 + y^2)} + \max \left\{ \frac{| \sin(x - 2y + 1)|}{4 + x^2 + x(1 + y) + x^3y}, \frac{1 + \ln(1 + x^2y^3)}{3 + 3x^2y^3} \right\} \right\} \leq \frac{2}{3}.$$

It follows from Theorem 4.1 that the functional equation (4.6) possesses a unique solution in $BB(S)$. However, the results in [2,6] are not applicable for the functional equation (4.6) because

$$\frac{1 + \ln(1 + x^2y^3)}{3 + 3x^2y^3} > 0, \quad (x, y) \in S \times D.$$

Theorem 4.2. Let $a, b, c : S \times D \to S$ and $p, q, r, s, t : S \times D \to \mathbb{R}$ be mappings and let $(\varphi, \psi)$ be in $\Phi_7$ satisfying the following conditions:

(D4) $|p(x,y)| + |r(x,y)| \leq \psi(\|x\|)$ for $(x,y) \in S \times D$,

(D5) $\max \{|a(x, y)|, \|b(x, y)\|, \|c(x, y)\| \} \leq \varphi(\|x\|)$ for $(x,y) \in S \times D$,

(D6) $\sup_{(x,y) \in S \times D} \{|q(x, y)| + \max \{|s(x, y)|, |t(x, y)|\} \} \leq 1.$

Then the functional equation (1.2) possesses a solution $w \in BB(S)$ that satisfies the following conditions:

(D7) the sequence $\{w_n\}_{n \geq 0}$ defined by

$$w_0(x) = \text{opt}_{y \in D} \{ p(x, y) + r(x, y) \},$$

$$w_n(x) = \text{opt}_{y \in D} \{ p(x, y) + q(x, y)w_{n-1}(a(x, y)) \$$

$$+ \text{opt}(r(x, y), s(x, y)w_{n-1}(b(x, y)), t(x, y)w_{n-1}(c(x, y))) \}, \quad x \in S, \quad n \geq 1$$

converges to $w$,.
\( \lim_{n \to \infty} w(x_n) = 0 \) for any \( x_0 \in S \), \( \{y_n\}_{n \geq 1} \subset D \) and \( x_n \in \{a(x_{n-1}, y_n), b(x_{n-1}, y_n), c(x_{n-1}, y_n)\}, \ n \geq 1 \).

\( w \) is unique with respect to condition (D8).

**Proof.** Since \((\varphi, \psi)\) is in \( \Phi_\gamma \), it is easy to verify that

\[ \varphi(t) < t \quad \text{for } t > 0. \quad (4.7) \]

First of all we claim that the mapping \( H \) defined by (4.1) is nonexpansive on \( BB(S) \). Let \( k \) be a positive integer and \( h \) be in \( BB(S) \). On account of (4.7) and (D5), we obtain that

\[ \max\{\|a(x, y)\|, \|b(x, y)\|, \|c(x, y)\|\} \leq \varphi(\|x\|) < k \quad \text{for } (x, y) \in \overline{B}(0, k) \times D, \]

which implies that there exists a constant \( \theta(k, h) > 0 \) with

\[ \max\{|h(a(x, y)|, |h(b(x, y)|, |h(c(x, y))|)\} \leq \theta(k, h) \quad \text{for } (x, y) \in \overline{B}(0, k) \times D. \quad (4.8) \]

By virtue of (D4), (D6), (4.1) and (4.8), we deduce that

\[
|H h(x)| = \left| \operatorname{opt}_{y \in D} \{p(x, y) + q(x, y)h(a(x, y)) + \operatorname{opt}_{y \in D} \{r(x, y), s(x, y)h(b(x, y)), t(x, y)h(c(x, y))\} \right| \\
\leq \sup_{y \in D} \{|p(x, y)| + |q(x, y)||h(a(x, y))| \\
+ \max\{|r(x, y)|, |s(x, y)||h(b(x, y))|, |t(x, y)||h(c(x, y))|\} \}
\leq \sup_{y \in D} \{|p(x, y)| + |r(x, y)| + |q(x, y)| + \max\{|s(x, y)|, |t(x, y)|\} \}
\times \max\{|h(a(x, y)|), |h(b(x, y)|, |h(c(x, y))|\} \\
\leq \psi(k) + \theta(k, h)
\]

for \( x \in \overline{B}(0, k) \). Therefore \( H \) is a self mapping on \( BB(S) \). As in the proof of Theorem 4.1, by (D6) we immediately conclude that for \( g, h \in BB(S) \) and \( k \geq 1 \),

\[ d_k(Hg, Hh) \leq d_k(g, h), \]

which yields that

\[ d(Hg, Hh) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(Hg, Hh)}{1 + d_k(Hg, Hh)} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(g, h)}{1 + d_k(g, h)} = d(g, h) \]

for \( g, h \in BB(S) \). That is, \( H \) is nonexpansive.

Now we assert that for each \( n \geq 0 \),

\[ |w_n(x)| \leq \sum_{j=0}^{n} \psi(\varphi^j(\|x\|)), \quad x \in S. \quad (4.9) \]
In fact, by (D4) we see that
\[
|w_0(x)| \leq \sup_{y \in D} \{ |p(x, y)| + |r(x, y)| \} \leq \psi(\|x\|), \quad x \in S.
\]
That is, (4.9) holds for \( n = 0 \). Suppose that (4.9) holds for some \( n \geq 0 \). From (D4)–(D6) we know that
\[
|w_{n+1}(x)| = \left| \sup_{y \in D} \{ |p(x, y)| + |q(x, y)| w_n(a(x, y)) + \psi(r(x, y), s(x, y)w_n(b(x, y)), t(x, y)w_n(c(x, y))) \} \right|
\]
\[
\leq \sup_{y \in D} \{ |p(x, y)| + |q(x, y)| w_n(a(x, y)) + \psi(r(x, y), s(x, y)w_n(b(x, y)), t(x, y)w_n(c(x, y))) \}
\]
\[
\leq \psi(\|x\|) + \sup_{y \in D} \max \left\{ \sum_{j=0}^{n} \psi(\phi^j(\|a(x, y)\|)), \sum_{j=0}^{n} \psi(\phi^j(\|b(x, y)\|)), \sum_{j=0}^{n} \psi(\phi^j(\|c(x, y)\|)) \right\}
\]
\[
\leq \psi(\|x\|) + \sum_{j=0}^{n} \psi(\phi^{j+1}(\|x\|))
\]
\[
= \sum_{j=0}^{n+1} \psi(\phi^j(\|x\|)).
\]
Hence (4.9) holds for \( n \geq 0 \).

Next we claim that \( \{w_n\}_{n \geq 0} \) is a Cauchy sequence in \( BB(S) \). Given \( k \geq 1 \) and \( x_0 \in \overline{B}(0, k) \). Let \( \varepsilon > 0 \), \( n \geq 1 \) and \( m \geq 1 \). Suppose that \( \text{opt}_{y \in D} = \sup_{y \in D} \). Then we can select \( y, z \in D \) such that
\[
w_n(x_0) < p(x_0, y) + q(x_0, y)w_{n-1}(a(x_0, y))
\]
\[
+ \psi(r(x_0, y), s(x_0, y)w_{n-1}(b(x_0, y)), t(x_0, y)w_{n-1}(c(x_0, y))) + 2^{-1}\varepsilon,
\]
\[
w_{n+m}(x_0) < p(x_0, z) + q(x_0, z)w_{n+m-1}(a(x_0, z))
\]
\[
+ \psi(r(x_0, z), s(x_0, z)w_{n+m-1}(b(x_0, z)), t(x_0, z)w_{n+m-1}(c(x_0, z))) + 2^{-1}\varepsilon,
\]
\[
w_n(x_0) > p(x_0, z) + q(x_0, z)w_{n-1}(a(x_0, z))
\]
\[
+ \psi(r(x_0, z), s(x_0, z)w_{n-1}(b(x_0, z)), t(x_0, z)w_{n-1}(c(x_0, z)))
\]
\[
w_{n+m}(x_0) > p(x_0, y) + q(x_0, y)w_{n+m-1}(a(x_0, y))
\]
\[
+ \psi(r(x_0, y), s(x_0, y)w_{n+m-1}(b(x_0, y)), t(x_0, y)w_{n+m-1}(c(x_0, y)))
\]
\[
(4.10)
\]
In view of (4.10), (D6) and Lemma 2.2, we infer that

\[
\begin{align*}
& w_{n+m}(x_0) - w_n(x_0) \\
& \quad < q(x_0, z)[w_{n+m-1}(a(x_0, z)) - w_{n-1}(a(x_0, z))] \\
& \quad + \text{opt}\{r(x_0, z), s(x_0, z)w_{n+m-1}(b(x_0, z)), t(x_0, z)w_{n+m-1}(c(x_0, z))\} \\
& \quad - \text{opt}\{r(x_0, z), s(x_0, z)w_{n-1}(b(x_0, z)), t(x_0, z)w_{n-1}(c(x_0, z))\} + 2^{-1} \varepsilon \\
& \leq |q(x_0, z)||w_{n+m-1}(a(x_0, z)) - w_{n-1}(a(x_0, z))| \\
& \quad + \max\{|s(x_0, z)||w_{n+m-1}(b(x_0, z)) - w_{n-1}(b(x_0, z))|, \\
& \quad |t(x_0, z)||w_{n+m-1}(c(x_0, z)) - w_{n-1}(c(x_0, z))|\} + 2^{-1} \varepsilon \\
& \leq \max\{|w_{n+m-1}(a(x_0, z)) - w_{n-1}(a(x_0, z))|, |w_{n+m-1}(b(x_0, z)) - w_{n-1}(b(x_0, z))|, \\
& \quad |w_{n+m-1}(c(x_0, z)) - w_{n-1}(c(x_0, z))|\} + 2^{-1} \varepsilon
\end{align*}
\]

and

\[
\begin{align*}
& w_{n+m}(x_0) - w_n(x_0) \\
& \quad > - |q(x_0, z)||w_{n+m-1}(a(x_0, y)) - w_{n-1}(a(x_0, y))| \\
& \quad - \max\{|s(x_0, y)||w_{n+m-1}(b(x_0, y)) - w_{n-1}(b(x_0, y))|, \\
& \quad |t(x_0, y)||w_{n+m-1}(c(x_0, y)) - w_{n-1}(c(x_0, y))|\} - 2^{-1} \varepsilon \\
& \quad \geq - \max\{|w_{n+m-1}(a(x_0, y)) - w_{n-1}(a(x_0, y))|, |w_{n+m-1}(b(x_0, y)) - w_{n-1}(b(x_0, y))|, \\
& \quad |w_{n+m-1}(c(x_0, y)) - w_{n-1}(c(x_0, y))|\} - 2^{-1} \varepsilon
\end{align*}
\]

which means that

\[
|w_{n+m}(x_0) - w_n(x_0)|< \begin{align*}
& \quad \leq \max\{|w_{n+m-1}(a(x_0, z)) - w_{n-1}(a(x_0, z))|, \\
& \quad |w_{n+m-1}(b(x_0, z)) - w_{n-1}(b(x_0, z))|, |w_{n+m-1}(c(x_0, z)) - w_{n-1}(c(x_0, z))|\}, \\
& \quad \max\{|w_{n+m-1}(a(x_0, y)) - w_{n-1}(a(x_0, y))|, |w_{n+m-1}(b(x_0, y)) - w_{n-1}(b(x_0, y))|, \\
& \quad |w_{n+m-1}(c(x_0, y)) - w_{n-1}(c(x_0, y))|\}\}
\end{align*} + 2^{-1} \varepsilon
\]

\[
= |w_{n+m-1}(x_1) - w_{n-1}(x_1)| + 2^{-1} \varepsilon \tag{4.11}
\]

for some \(y_1 \in [y, z]\) and \(x_1 \in \{a(x_0, y_1), b(x_0, y_1), c(x_0, y_1)\}\). Similarly, we can conclude that (4.11) holds for \(\text{opt}_{y \in D} = \inf_{y \in D}\). Proceeding in this way, we select \(y_j \in D\) and \(x_j \in \{a(x_{j-1}, y_j), b(x_{j-1}, y_j), c(x_{j-1}, y_j)\}\) for \(j \in \{2, 3, \ldots, n\}\) such that

\[
\begin{align*}
& |w_{n+m-1}(x_1) - w_{n-1}(x_1)| < |w_{n+m-2}(x_2) - w_{n-2}(x_2)| + 2^{-2} \varepsilon, \\
& |w_{n+m-2}(x_2) - w_{n-2}(x_2)| < |w_{n+m-3}(x_3) - w_{n-3}(x_3)| + 2^{-3} \varepsilon, \\
& \ldots \cdot \\
& |w_{m+1}(x_{n-1}) - w_1(x_{n-1})| < |w_{m}(x_n) - w_0(x_n)| + 2^{-n} \varepsilon. \tag{4.12}
\end{align*}
\]
It follows from (D5), (4.7), (4.9), (4.11) and (4.12) that

\[
|w_{n+m}(x_0) - w_n(x_0)| < |w_m(x_n) - w_0(x_n)| + \sum_{i=1}^{n} 2^{-i} \varepsilon \\
< |w_m(x_n)| + |w_0(x_n)| + \varepsilon \\
\leq \sum_{i=0}^{m} \psi(\varphi^i(\|x_n\|)) + \psi(\|x_n\|) + \varepsilon \\
\leq \sum_{i=0}^{m} \psi(\varphi^{i+n}(\|x_0\|)) + \psi(\varphi^n(\|x_0\|)) + \varepsilon \\
\leq \sum_{j=n-1}^{\infty} \psi(\varphi^j(k)) + \varepsilon,
\]

which implies that

\[
d_k(w_{n+m}, w_n) \leq \sum_{j=n-1}^{\infty} \psi(\varphi^j(k)) + \varepsilon.
\]

As \( \varepsilon \to 0^+ \) in the above inequality, we deduce that

\[
d_k(w_{n+m}, w_n) \leq \sum_{j=n-1}^{\infty} \psi(\varphi^j(k)),
\]

which yields that \( \{w_n\}_{n \geq 0} \) is a Cauchy sequence in \( (BB(S), d) \) since \( \sum_{i=0}^{\infty} \psi(\varphi^n(t)) < \infty \) for each \( t > 0 \). Suppose that \( \{w_n\}_{n \geq 0} \) converges to some \( w \in BB(S) \). Notice that \( H \) is nonexpansive. It follows that

\[
d(w, Hw) \leq d(w, Hw_n) + d(Hw_n, Hw) \\
\leq d(w, w_{n+1}) + d(w_n, w) \to 0 \quad \text{as} \quad n \to \infty.
\]

That is, \( Hw = w \). Thus the functional equation (1.2) possesses a solution \( w \).

Now we show that (D8) holds. Let \( \varepsilon > 0, x_0 \in S, \{y_n\}_{n \geq 1} \subset D \) and \( x_n \in \{a(x_n-1, y_n), b(x_n-1, y_n), c(x_n-1, y_n)\} \) for \( n \geq 1 \). Put \( k = [\|x_0\|] + 1 \). Then there exists a positive integer \( m \) satisfying

\[
d_k(w, w_n) + \sum_{j=n}^{\infty} \psi(\varphi^j(k)) < \varepsilon \quad \text{for} \quad n > m. \tag{4.13}
\]
By (4.9), (D5) and (4.13), we infer that for \( n > m \),
\[
|w(x_n)| \leq |w(x_n) - w_n(x_n)| + |w_n(x_n)| \\
\leq d_k(w, w_n) + \sum_{j=0}^{\infty} \psi(\varphi^j(\|x_n\|)) \\
\leq d_k(w, w_n) + \sum_{j=n}^{\infty} \psi(\varphi^j(k)) \\
\leq \varepsilon,
\]
which means that \( \lim_{n \to \infty} w(x_n) = 0 \).

Finally, we show that (D9) holds. Suppose that the functional equation (1.2) possesses another solution \( h \in \mathcal{BB}(S) \), which satisfies condition (D8). Let \( \varepsilon > 0 \) and \( x_0 \in S \). If \( \text{opt}_{y \in D} = \sup_{y \in D} \), then there exist \( y, z \in S \) such that
\[
w(x_0) < p(x_0, y) + q(x_0, y)w(a(x_0, y)) + \text{opt}\{r(x_0, y), s(x_0, y)w(b(x_0, y)), \}
\]
\[
t(x_0, y)w(c(x_0, y))\} + 2^{-1}\varepsilon,
\]
\[
h(x_0) < p(x_0, z) + q(x_0, z)h(a(x_0, z)) + \text{opt}\{r(x_0, z), s(x_0, z)h(b(x_0, z)), \}
\]
\[
t(x_0, z)h(c(x_0, z))\} + 2^{-1}\varepsilon,
\]
\[
w(x_0) \geq p(x_0, z) + q(x_0, z)w(a(x_0, z)) + \text{opt}\{r(x_0, z), s(x_0, z)w(b(x_0, z)), \}
\]
\[
t(x_0, z)w(c(x_0, z))\}, \]
\[
h(x_0) \geq p(x_0, y) + q(x_0, y)h(a(x_0, y)) + \text{opt}\{r(x_0, y), s(x_0, y)h(b(x_0, y)), \}
\]
\[
t(x_0, y)h(c(x_0, y))\}. \tag{4.14}
\]

Using Lemma 2.2, (D6) and (4.14), we derive that
\[
w(x_0) - h(x_0)
\leq q(x_0, y)[w(a(x_0, y)) - h(a(x_0, y))]
\]
\[
+ \text{opt}\{r(x_0, y), s(x_0, y)w(b(x_0, y)), t(x_0, y)w(c(x_0, y))\}
\]
\[
- \text{opt}\{r(x_0, y), s(x_0, y)h(b(x_0, y)), t(x_0, y)h(c(x_0, y))\} + 2^{-1}\varepsilon
\]
\[
\leq |q(x_0, y)||w(a(x_0, y)) - h(a(x_0, y))| + \max\{|s(x_0, y)||w(b(x_0, y)) - h(b(x_0, y))|,
\]
\[
|t(x_0, y)||w(c(x_0, y)) - h(c(x_0, y))|\} + 2^{-1}\varepsilon
\]
\[
\leq \max\{|q(x_0, y)|, |s(x_0, y)|, |t(x_0, y)|\}]
\]
\[
\times \max\{|w(a(x_0, y)) - h(a(x_0, y))|, |w(b(x_0, y)) - h(b(x_0, y))|,
\]
\[
|w(c(x_0, y)) - h(c(x_0, y))|\} + 2^{-1}\varepsilon
\]
\[
\leq \max\{|w(a(x_0, y)) - h(a(x_0, y))|, |w(b(x_0, y)) - h(b(x_0, y))|,
\]
\[
|w(c(x_0, y)) - h(c(x_0, y))|\} + 2^{-1}\varepsilon
\]
and
\[ w(x_0) - h(x_0) > - |q(x_0, z)||w(a(x_0, z)) - h(a(x_0, z))| \]
\[ - \max\{|s(x_0, z)||w(b(x_0, z)) - h(b(x_0, z))|, |r(x_0, z)||w(c(x_0, z)) - h(c(x_0, z))|\} - 2^{-1}e \]
\[ \geq - \max\{|w(a(x_0, z)) - h(a(x_0, z))|, |w(b(x_0, z)) - h(b(x_0, z))|, |w(c(x_0, z)) - h(c(x_0, z))|\} - 2^{-1}e, \]
which imply that
\[ |w(x_0) - h(x_0)| < \max\{|w(a(x_0, y)) - h(a(x_0, y))|, |w(b(x_0, y)) - h(b(x_0, y))|, |w(c(x_0, y)) - h(c(x_0, y))|\}, \max\{|w(a(x_0, z)) - h(a(x_0, z))|, |w(b(x_0, z)) - h(b(x_0, z))|, |w(c(x_0, z)) - h(c(x_0, z))|\} + 2^{-1}e, \]
\[ = |w(x_1) - h(x_1)| + 2^{-1}e \quad (4.15) \]
for some \( y_1 \in \{y, z\} \) and \( x_1 \in \{a(x_0, y_1), b(x_0, y_1), c(x_0, y_1)\} \). Similarly, we can conclude that (4.15) holds for \( \text{opt}_{y \in D} = \inf_{y \in D} \). Proceeding in this way, we select \( y_j \in D \) and \( x_j \in \{a(x_{j-1}, y_j), b(x_{j-1}, y_j), c(x_{j-1}, y_j)\} \) for \( j = 2, 3, \ldots, n \) satisfying
\[ |w(x_1) - h(x_1)| < |w(x_2) - h(x_2)| + 2^{-2}e, \]
\[ |w(x_2) - h(x_2)| < |w(x_3) - h(x_3)| + 2^{-3}e, \]
\[ \ldots \]
\[ |w(x_{n-1}) - h(x_{n-1})| < |w(x_n) - h(x_n)| + 2^{-n}e. \quad (4.16) \]
Combining (4.15) and (4.16), we obtain that
\[ |w(x_0) - h(x_0)| < |w(x_n) - h(x_n)| + \sum_{j=1}^{n} 2^{-j}e < |w(x_n) - h(x_n)| + e. \]
Letting \( n \to \infty \) in the above inequalities, by (D8) we get that
\[ |w(x_0) - h(x_0)| \leq e. \]
As \( e \to 0^+ \) in the above inequality, we know that \( w(x_0) = h(x_0) \). This completes the proof. \( \square \)

**Remark 4.2.** In case \( a(x, y) = b(x, y) = c(x, y) = t(x, y) = 1 - \lambda, r(x, y) = (1 - \lambda)r_1(x, y), p(x, y) = \lambda p_1(x, y), q(x, y) = \lambda, \psi(t) = Mt, \varphi \in \Phi_6 \) and \( \max\{|r_1(x, y)|, |p_1(x, y)| \} \leq \psi(||x||) \) for \( x \in S, y \in D, t \in \mathbb{R}^+ \), where \( \lambda \) is a constant in \([0, 1]\), then Theorem 4.2 reduces to Theorem 3.1 in [12], which, in turn, generalizes Theorem 3.5 in [6], Theorem 2.4 in [7] and a result in [2, p. 149]. The example below reveals that Theorem 4.2 extends properly the results in [2,6,7,12].
Example 4.2. Let \( X = Y = \mathbb{R} \), \( S = D = [1, \infty) \), \( \psi(t) = 3t^3 \) and \( \varphi(t) = 2^{-1}t \) for \( t \in \mathbb{R}^+ \). It follows from Theorem 4.2 that the following functional equation:

\[
f(x) = \underset{y \in D}{\text{opt}} \left\{ x^3 \left( 1 + \frac{1}{x^2 + 2y} \right) + \frac{x^2 y}{x^4 + y^2} f \left( \frac{x^3}{2x^2 + \ln(x^4 + xy^2 + 2y^3)} \right) \right. \\
+ \left. \underset{y \in D}{\text{opt}} \left\{ \frac{x^10}{x^7 + \cos(x^2 - xy + y^3)} \right\} \right. \\
+ \left. \frac{1}{2} \sin[x^y - \ln(x + y^3)] f \left( \frac{\sin(xy)}{2y} \right) \right\}, \quad x \in S
\]

possesses a solution \( w \in BB(S) \). However, the results in [2,6,7,14] are not applicable for the above functional equation since

\[
\left| x^3 \left( 1 + \frac{1}{x^2 + 2y} \right) \right| \leq M|x|
\]

does not hold for \( (x, y) = (1 + M, 1) \in S \times D \), where \( M \) is a positive constant.

Theorem 4.3. Let \( a, b, c : S \times D \to S \) and \( p, q, r, s, t : S \times D \to \mathbb{R} \) be mappings and let \((\varphi, \psi)\) be in \( \Phi_7 \) satisfying (D4)–(D6). Then the functional equation

\[
f(x) = \underset{y \in D}{\text{opt}} \left\{ p(x, y) + q(x, y) f(a(x, y)) \right. \\
+ \left. \max\{r(x, y), s(x, y) f(b(x, y)), t(x, y) f(c(x, y))\} \right\}, \quad x \in S,
\]

(4.17)

possesses a solution \( w \in BB(S) \) that satisfies (D8), (D9) and

(D10) the sequence \( \{w_n\}_{n \geq 0} \) defined by

\[
w_0(x) = \underset{y \in D}{\text{opt}} \left\{ p(x, y) + r(x, y) \right\}, \\
w_n(x) = \underset{y \in D}{\text{opt}} \left\{ p(x, y) + q(x, y) w_{n-1}(a(x, y)) \right. \\
+ \left. \max\{r(x, y), s(x, y) w_{n-1}(b(x, y)), t(x, y) w_{n-1}(c(x, y))\} \right\}, \quad x \in S, \quad n \geq 1
\]

converges to \( w \),

(D11) if \( q, s \) and \( t \) are nonnegative and

\[
q(x, y) + \max\{s(x, y), t(x, y)\} \equiv \beta, \quad (x, y) \in S \times D,
\]

(4.18)

where \( \beta \) is a constant, then for given \( \varepsilon > 0 \) and \( x_0 \in S \), there exist \( \{y_n\}_{n \geq 1} \subset D \) and \( x_n \in \{a(x_{n-1}, y_n), b(x_{n-1}, y_n), c(x_{n-1}, y_n)\} \) for \( n \geq 1 \), such that

\[
w(x_0) - \sum_{n=1}^{\infty} \beta^{n-1} p(x_{n-1}, y_n) \geq - \varepsilon.
\]

Moreover, if \( p \) is nonnegative, then \( w(x) \geq 0 \) for \( x \in S \).
**Proof.** It follows from Theorem 4.2 that the functional equation (4.17) possesses a solution \( w \in BB(S) \) that satisfies (D8)–(D10).

Now we show that (D11) holds. Let \( \varepsilon > 0 \) and \( x_0 \in S \). In terms of Lemma 2.3 and (4.18), we infer that there exist \( y_1 \in D \) and \( x_1 \in \{ a(x_0, y_1), b(x_0, y_1), c(x_0, y_1) \} \) satisfying

\[
w(x_0) > p(x_0, y_1) + q(x_0, y_1)w(a(x_0, y_1)) \\
+ \max\{r(x_0, y_1), s(x_0, y_1)w(b(x_0, y_1)), t(x_0, y_1)w(c(x_0, y_1))\} - 2^{-1}\varepsilon \\
\geq p(x_0, y_1) + q(x_0, y_1)w(a(x_0, y_1)) \\
+ \max\{s(x_0, y_1), t(x_0, y_1)\} \min\{w(b(x_0, y_1)), w(c(x_0, y_1))\} - 2^{-1}\varepsilon \\
\geq p(x_0, y_1) + \beta w(x_1) - 2^{-1}\varepsilon.
\]

(4.19)

Similarly we select \( y_j \in D \) and \( x_j \in \{ a(x_j-1, y_j), b(x_j-1, y_j), c(x_j-1, y_j) \} \) such that

\[
w(x_j-1) > p(x_j-1, y_j) + \beta w(x_j) - 2^{-j} \beta^{-j+1}\varepsilon, \quad j \in \{2, 3, \ldots, n\}.
\]

(4.20)

Using (4.19) and (4.20), we deduce that

\[
w(x_0) - \sum_{j=1}^{n} \beta^{j-1} p(x_j-1, y_j) > \beta^n w(x_n) - \sum_{j=1}^{n} 2^{-j}\varepsilon > \beta^n w(x_n) - \varepsilon.
\]

(4.21)

By virtue of (D4)–(D6), (4.18) and \((\varphi, \psi) \in \Phi_7\), we see that

\[
\sum_{n=1}^{\infty} \beta^{n-1} |p(x_{n-1}, y_n)| \leq \sum_{n=1}^{\infty} \beta^{n-1} \psi(\|x_n\|) \leq \sum_{n=1}^{\infty} \beta^{n-1} \psi(\varphi^n(\|x_0\|)) < \infty.
\]

(4.22)

Letting \( n \to \infty \) in (4.21), by (D8) and (4.22) we get that

\[
w(x_0) - \sum_{n=1}^{\infty} \beta^{n-1} p(x_{n-1}, y_n) \geq - \varepsilon.
\]

(4.23)

Suppose that \( p \) is nonnegative. In view of (4.23) we obtain that

\[
w(x_0) \geq - \varepsilon.
\]

As \( \varepsilon \to 0^+ \) in the above inequality, we derive that \( w(x_0) \geq 0 \) for \( x_0 \in S \). This completes the proof. \( \Box \)

A proof similar to that of Theorem 4.3 gives the following result and is thus omitted.
Theorem 4.4. Let \( a, b, c : S \times D \to S \) and \( p, q, r, s, t : S \times D \to \mathbb{R} \) be mappings and let \((\varphi, \psi)\) be in \( \Phi_7 \) satisfying (D4)–(D6). Then the functional equation
\[
f(x) = \operatorname{opt}_{y \in D} \{ p(x, y) + q(x, y)f(a(x, y)) + \min\{ r(x, y), s(x, y)f(b(x, y)), t(x, y)f(c(x, y)) \} \}, \quad x \in S,
\]
possesses a solution \( w \in BB(S) \) that satisfies (D8), (D9) and the sequence \( \{w_n\}_{n \geq 0} \) defined by
\[
w_0(x) = \operatorname{opt}_{y \in D} \{ p(x, y) + r(x, y) \},
\]
\[
w_n(x) = \operatorname{opt}_{y \in D} \{ p(x, y) + q(x, y)w_{n-1}(a(x, y)) + \min\{ r(x, y), s(x, y)w_{n-1}(b(x, y)), t(x, y)w_{n-1}(c(x, y)) \} \}, \quad x \in S, \quad n \geq 1
\]
converges to \( w \),

(D12) if \( q, s \) and \( t \) are nonnegative and
\[
q(x, y) + \min\{ s(x, y), t(x, y) \} \equiv \sigma, \quad (x, y) \in S \times D,
\]
where \( \sigma \) is a constant, then for given \( \varepsilon > 0 \) and \( x_0 \in S \), there exist \( \{y_n\}_{n \geq 1} \subset D \) and \( x_n \in \{a(x_{n-1}, y_n), b(x_{n-1}, y_n), c(x_{n-1}, y_n)\} \) for \( n \geq 1 \), such that
\[
w(x_0) - \sum_{n=1}^{\infty} \sigma^{n-1} p(x_{n-1}, y_n) \leq \varepsilon.
\]
Moreover, if \( p \) is nonpositive, then \( w(x) \leq 0 \) for \( x \in S \).

Remark 4.3. Theorems 4.3 and 4.4 extend, improve and unify Theorem 3.5 in [6], Theorems 2.3 and 2.4 in [7], Theorem 3.5 in [11], Theorems 3.2 and 3.3 in [12] and a result in [2, p. 149]. The following example shows that Theorems 4.3 and 4.4 are indeed extension of the results in [2,6,7,11,12].

Example 4.3. Let \( X = Y = S = \mathbb{R}, D = [1, \infty), \psi(t) = t^2 \) and \( \varphi(t) = \frac{1}{3} t \) for \( t \in \mathbb{R}^+ \). Then Theorems 4.3 and 4.4 ensure that the following functional equations
\[
f(x) = \operatorname{opt}_{y \in D} \left\{ \frac{x^2 y}{|\sin(xy + y - 1)| + 2y + x^2 + y^2 + \ln(1 + |x|y)} f\left( \frac{x \cos(x - y)}{3 + |x|y} \right) \right. \\
+ \max \left\{ \frac{x^2 \sin(x - y)}{2 + |\cos(xy - x + y)|}, \frac{y^2 + \ln(1 + |x|y)}{x^2 + y^2 + \ln(1 + |x|y)} f\left( \frac{\sin x}{3 + \cos^2 xy} \right), \right. \\
\left. \frac{\ln(1 + |x|y)}{x^2 + y^2 + \ln(1 + |x|y)} f\left( \frac{x}{3 + \sin^2 xy} \right) \right\}, \quad x \in S
\]
and

\[ f(x) = \operatorname{opt}_{y \in D} \left\{ \frac{-x^2 \ln y}{1 + |\sin x| + 2 \ln y} + \frac{x^2 y + \max \{\sin^2(x^2 + y), \cos^2(x^2 + y)\}}{3 + 3x^2 y} \right. \]
\[ \times f \left( \frac{x \cos(x + y^2)}{3 + |\cos(x^2 + xy + 1)|} \right) + \min \left\{ \frac{x^3}{1 + 2|x|}, \frac{\sin^2(x^2 + y)}{3 + 3x^2 y} f \left( \frac{\sin x}{3 + \ln(1 + x^2 y)} \right), \right. \]
\[ \left. \frac{\cos^2(x^2 + y)}{3 + 3x^2 y} f \left( \frac{x}{3 + \sin^2 xy} \right) \right\} \}

possess, respectively, a nonnegative and nonpositive solutions in \(BB(S)\). But the results in \([2,6,7,11,12]\) are not valid for the above functional equations.

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References