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Imprimitive flag-transitive symmetric designs

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Abstract

A recent paper of O'Reilly Regueiro obtained an explicit upper bound on the number of points of a flag-transitive, point-imprimitive, symmetric design in terms of the number of blocks containing two points. We improve that upper bound and give a complete list of feasible parameter sequences for such designs for which two points lie in at most ten blocks. Classifications are available for some of these parameter sequences.

Keywords: Permutation group; Flag-transitive; Symmetric design

1. Introduction

This paper was inspired by a recent paper of O'Reilly Regueiro [22] on flag-transitive symmetric block designs. In 1987 Davies [7] proved that, for a given value of the parameter λ, the block size k of a point-imprimitive, flag-transitive $2-(v,k,\lambda)$ design (not necessarily symmetric) was bounded. This implies that, for a given λ, the number of such designs is bounded (see [8]). However Davies did not give an explicit upper bound for k in terms of λ.

O'Reilly Regueiro (as she prefers to be called) obtained in [22, Theorem 1] an explicit upper bound as the first main result of her paper, and in the rest of her paper studied point-primitive,

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flag-transitive designs, see also [23]. Our aim is to refine O'Reilly Regueiro's bound on the size of point-imprimitive, flag-transitive, symmetric designs, and to study some of the extreme cases.

A $t-(v,k,\lambda)$ design $\mathcal{D}=(\Omega,\mathcal{B})$ consists of a set Ω of v points, and a set $\mathcal{B}$ of k-element subsets of Ω, called blocks, such that every t-element subset of points lies in exactly λ blocks. The design is *nontrivial* if $t<k<v-t$, and is *symmetric* if $|\mathcal{B}|=v$, that is if there is an equal number of points and blocks. By [4, Theorem 1.1] (or see [12, Theorem 1.27]), if $\mathcal{D}$ is symmetric and nontrivial, then $t\leqslant 2$. Thus we study nontrivial symmetric $2-(v,k,\lambda)$ designs. Our main result is the following improvement of O'Reilly Regueiro's result [22, Theorem 1].

Theorem 1.1. *Let $\mathcal{D}=(\Omega,\mathcal{B})$ be a nontrivial symmetric $2-(v,k,\lambda)$ design admitting a flag-transitive, point-imprimitive subgroup of automorphisms G that leaves invariant a nontrivial partition $\mathcal{C}$ of Ω with d classes of size c. Then there is a constant ℓ such that, for each $B\in\mathcal{B}$ and $\Delta\in\mathcal{C}$, the size $|B\cap\Delta|$ is either 0 or ℓ, and one of the following holds*:

(a) $k\leqslant\lambda(\lambda-3)/2$;
(b) $(v,k,\lambda)=(\lambda^2(\lambda+2),\lambda(\lambda+1),\lambda)$ *with* $(c,d,\ell)=(\lambda^2,\lambda+2,\lambda)$ *or* $(\lambda+2,\lambda^2,2)$;
(c) $(v,k,\lambda,c,d,\ell)=((\frac{\lambda+2}{2})(\frac{\lambda^2-2\lambda+2}{2}),\frac{\lambda^2}{2},\lambda,\frac{\lambda+2}{2},\frac{\lambda^2-2\lambda+2}{2},2)$, *and either* $\lambda\equiv 0\pmod 4$, *or* $\lambda=2u^2$ *where u is odd, $u\geqslant 3$, and $2(u^2-1)$ is a square*;
(d) $(v,k,\lambda,c,d,\ell)=((\lambda+6)(\frac{\lambda^2+4\lambda-1}{4}),\frac{\lambda(\lambda+5)}{2},\lambda,\lambda+6,\frac{\lambda^2+4\lambda-1}{4},3)$, *where* $\lambda\equiv 1$ *or* $3\pmod 6$.

Remark 1.2.

(1) Cases (a) to (d) of Theorem 1.1 are pairwise disjoint except for the parameter sequence $(v,k,\lambda,c,d,\ell)=(45,12,3,9,5,3)$ which arises in both case (b) and case (d).
(2) By a well-known result due to Schützenberger, Chowla & Ryser, and Shrikhande, if the number v of points is even then $k-\lambda$ is a square (see Lemma 2.2). We take account of this result in the proof and we note that it holds in those cases of Theorem 1.1(b)–(d) where v is even, namely in case (b) with λ even, and in case (c) with $\lambda=2u^2\equiv 2\pmod 4$ and $\lambda-2$ a nonzero square.

Symmetric designs with λ small are of interest. For example, those with $\lambda=1$ are the projective planes, while those with $\lambda=2$ are called biplanes. We evaluated the parameter sequences from Theorem 1.1 with small λ until we found sequences arising from each of the cases, that did not correspond to Hadamard designs or their complements. To do this we needed to consider values of λ up to 10. We obtained the following corollary. This extends [22, Corollary 1] that dealt with values of λ up to 4.

Corollary 1.3. *Using the notation of Theorem 1.1, let $\lambda\leqslant 10$. Then one of the lines of Table 1 holds, where the column headed 'Case' records the case of Theorem 1.1. Moreover, in each of these lines, the permutation groups $G^{\mathcal{C}}$ and G_Δ^Δ (where $\Delta\in\mathcal{C}$) are both primitive.*

Faced with such a list, questions arise about existence and classification of designs with these properties and parameters. We summarise what is known about such examples in Remark 1.4. The cases occurring in O'Reilly Regueiro [22] are those with $\lambda\leqslant 4$, that is, Lines 1–6 of Table 1. The information she gave about examples in these cases is included in Remark 1.4.

Table 1
Parameter sequences for Corollary 1.3

Line	v	k	λ	c	d	ℓ	Case	Examples	Reference
1	16	6	2	4	4	2	(b)	2	[13,22]
2	45	12	3	5	9	2	(b)	None	[21]
3	45	12	3	9	5	3	(b) and (d)	1	[21]
4	15	8	4	3	5	2	(c)	1	Proposition 1.5
5	96	20	4	16	6	4	(b)	$\geqslant 4$	[16,22]
6	96	20	4	6	16	2	(b)	$\geqslant 3$	[16]
7	175	30	5	25	7	5	(b)		
8	175	30	5	7	25	2	(b)		
9	288	42	6	36	8	6	(b)		
10	288	42	6	8	36	2	(b)		
11	441	56	7	49	9	7	(b)		
12	441	56	7	9	49	2	(b)		
13	247	42	7	13	19	3	(d)	None	[22] and Proposition 1.6
14	125	32	8	5	25	2	(c)		
15	640	72	8	64	10	8	(b)		
16	640	72	8	10	64	2	(b)		
17	891	90	9	81	11	9	(b)		
18	891	90	9	11	81	2	(b)		
19	435	63	9	15	29	3	(d)	None	Proposition 1.6
20	120	35	10	15	8	5	(a)		
21	1200	110	10	100	12	10	(b)		
22	1200	110	10	12	100	2	(b)		

Remark 1.4.

(1) (Line 1) In 1945 Hussain [13] proved that there are exactly three $2-(16,6,2)$ symmetric designs (biplanes). O'Reilly Regueiro [22, p. 139] described the three designs and showed that exactly two of them are flag-transitive, and, moreover, that both of the flag-transitive examples admit a point-imprimitive, flag-transitive subgroup of automorphisms. Hussain's examples were obtained also by Nandi [18] by an independent construction in 1946.

(2) (Lines 2–3) Mathon and Spence [17] constructed 3752 pairwise nonisomorphic $2-(45,12,3)$ symmetric designs, and showed that at least 1136 of them have a trivial automorphism group. The existence of a flag-transitive, point-imprimitive example was not resolved in [22]. However in [21] it has been shown that there are exactly two flag-transitive $2-(45,12,3)$ symmetric designs, exactly one of which admits a point-imprimitive, flag-transitive group. This example satisfies Line 3, but not Line 2.

(3) (Line 4) In 1946 Nandi [19] proved, with an enumeration by hand, that there are exactly five pairwise nonisomorphic symmetric $2-(15,7,3)$ designs, and hence that there are precisely five pairwise-nonisomorphic $2-(15,8,4)$ designs, namely, the complements of the $2-(15,7,3)$ designs. (The complement of a design $\mathcal{D}=(\Omega,\mathcal{B})$ is the design whose blocks are the complements in Ω of the blocks in $\mathcal{B}$.) According to Nandi, one of these five designs had appeared earlier in the 1938 tables of Fisher and Yates [10] and also (the same design) in the 1939 paper of Bose [2]. However, Nandi did not investigate the automorphism groups of these designs. O'Reilly Regueiro [22, p. 139] gave a construction of one of them, proving that it was flag-transitive and point-imprimitive. We prove, in Proposition 1.5, that there is a unique flag-transitive, point-imprimitive, example with the parameters of Line 4, namely the design of points and hyperplane complements of the projective geometry $\mathrm{PG}(3,2)$.

An accessible reference for the five $2-(15,7,3)$ designs is the CRC Handbook [5, p. 11, Table 1.23] which lists the block sets. The first computer-aided classification of these five designs was done by P. Gibbons in 1976, and confirmed by Ivanov, first in a preprint (in Russian) in 1981, and then in [14,15]. Ivanov mentions that they had been constructed by Nandi. We are grateful to Mikhail Klin for drawing our attention to the work of Nandi and Ivanov.

(4) (Lines 5–6) O'Reilly Regueiro [22, pp. 139–140] gave a construction of a flag-transitive, point-imprimitive symmetric $2-(96,20,4)$ design associated with a generalised quadrangle with parameters $(3,5)$. We are grateful to Maska Law, both for explaining the construction, and also for computing the automorphism group for us. In Remark 4.1 more details about this design are given. It admits a flag-transitive, point-imprimitive subgroup of automorphisms satisfying Line 5 of Table 1, but no subgroup satisfying Line 6.
Designs with these parameters arose also in the study in [3] of certain strongly regular graphs, and it turns out that there are some designs from [3] corresponding to Lines 5 and 6 of Table 1. We are grateful to Sven Reichard for computing this information about the automorphism groups. Designs satisfying Line 5 or 6 are the subject of further study in [16].

(5) (Other lines) Symmetric designs with the values of v, k, λ in Lines 7–8, 11–12, and 15–18, and admitting a point-transitive (and hence block-transitive) group of automorphisms, were constructed by McFarland in 1973 (see [1, VI.7.1]). However, except in the case of Lines 13 and 19, it is not known whether there exist examples that admit a point-imprimitive, flag-transitive group. The nonexistence of such a design in Line 13 was shown at the end of the proof of [22, Theorem 1, p. 142], although not required to prove the assertions of that theorem. The proof is simple, but unfortunately rather hidden in [22], so we give the details in Proposition 1.6 and extend the argument to eliminate Line 19 also. At the time of publication of [1] the existence of symmetric designs with v, k, λ as in Lines 9–10, 13, 14, 19–22 was unresolved.

Proposition 1.5. *The design of points and hyperplane complements of the projective geometry* $\mathrm{PG}(3,2)$ *is the unique design admitting a flag-transitive, point-imprimitive subgroup of automorphisms satisfying Line* 4 *of Table* 1.

Proposition 1.6. *There are no designs admitting a flag-transitive, point-imprimitive subgroup of automorphisms satisfying Line* 13 *or Line* 19 *of Table* 1.

We prove Theorem 1.1 in Section 2, Corollary 1.3 in Section 3, and Propositions 1.5 and 1.6 in Section 4.

2. Proof of Theorem 1.1

Let $\mathcal{D}, G, \mathcal{C}$ be as in Theorem 1.1. First we give some combinatorial information about $\mathcal{D}$ and G. Choose a point α and a block $B \in \mathcal{B}$ containing α. Let $\Delta \in \mathcal{C}$ be the class containing α. By hypothesis $|\Delta| = c$ and

$$v = cd. \tag{1}$$

Lemma 2.1. *G is transitive on $F := \{(\Delta', B') \mid \Delta' \in \mathcal{C},\ B' \in \mathcal{B},\ \Delta' \cap B' \neq \emptyset\}$, and hence $\ell := |\Delta' \cap B'|$ is independent of the pair (Δ', B') in F. Moreover,*

$$k = \ell s, \tag{2}$$

where s is the number of classes of $\mathcal{C}$ that intersect a block B' nontrivially, and $\ell > 1, s > 1$.

Proof. Let $(\Delta', B') \in F$ and let $\alpha' \in \Delta' \cap B'$. Since G is flag-transitive there exists $g \in G$ such that $(\alpha', B')^g = (\alpha, B)$. Since $(\alpha')^g = \alpha$ we have $(\Delta')^g = \Delta$, and hence $(\Delta', B')^g = (\Delta, B)$. Thus G is transitive on F and the rest of the lemma follows. Note that $\ell > 1, s > 1$ follow since each pair of points lies in a block. □

By the definition of a symmetric 2-design, counting (α, α', B) with $\alpha, \alpha' \in B \in \mathcal{B}$ gives

$$\lambda(v-1) = k(k-1). \tag{3}$$

Similarly, counting (α, α', B) with $\alpha, \alpha' \in B \in \mathcal{B}$ and α, α' in the same class of $\mathcal{C}$, gives

$$\lambda(c-1) = k(\ell-1). \tag{4}$$

Arguing as in [22, p. 141], $x := k-1-d(\ell-1)$ and $\lambda - x(\ell-1)$ are positive integers and

$$k = \frac{\lambda(x+\ell)}{\lambda - x(\ell-1)}, \tag{5}$$

$$\lambda(d-1) = kx. \tag{6}$$

We make a few trivial observations about these parameters:

$$\lambda < k, \qquad \ell < k, \qquad k+1 < v. \tag{7}$$

The first inequality follows from (3) since $v > k$; the second follows from (3) and (4) since $v > c$. For the third inequality, observe that, if $v = k+1$ then (3) implies that $\lambda = k-1$, and then (4) implies further that $k-1$ divides $\ell - 1$, contradicting $\ell < k$.

We will make use of the following lemma, part of which is the result of Schützenberger, Chowla and Ryser, and Shrikhande mentioned in Remark 1.2. Proofs of the lemma can be found in [1, II.3.11 and II.3.12] for part (a) and [1, II.3.9] for part (b).

Lemma 2.2. *Let $\mathcal{D}$ be a symmetric $2-(v,k,\lambda)$ design and set $n = k - \lambda$. Then the following hold*:

(a) $4n-1 \leqslant v \leqslant n^2+n+1$. *Moreover, $v = 4n-1$ if and only if $\mathcal{D}$ is an Hadamard design* (*with* $k = 2n-1$, $\lambda = n-1$) *or its complement*; *and $v = n^2+n+1$ if and only if $\mathcal{D}$ is a projective plane* (*with* $\lambda = 1$) *or its complement.*
(b) *If v is even and $v > k$, then $k - \lambda$ is a square.*

The following result about orbits of automorphism groups on points and blocks will be used frequently, often without explicit reference. Proofs may be found in [12, Theorem 1.46] and [1, III.4.2].

Lemma 2.3. *Let $\mathcal{D} = (\Omega, \mathcal{B})$ be a symmetric $2-(v,k,\lambda)$ design and let $G \leqslant \mathrm{Aut}(\mathcal{D})$. Then the numbers of G-orbits in Ω and in $\mathcal{B}$ are equal.*

Before we begin the detailed analysis, refining the arguments given in [22], we consider the parameters in case (c) of Theorem 1.1.

Lemma 2.4. *Suppose that $(v,k,\lambda) = ((\frac{\lambda+2}{2})(\frac{\lambda^2-2\lambda+2}{2}), \frac{\lambda^2}{2}, \lambda)$, and $\mathcal{D}$ is nontrivial. Then either $\lambda \equiv 0 \pmod 4$, or $\lambda = 2u^2$ where u is odd, $u \geqslant 3$, and $2(u^2-1)$ is a square.*

Proof. For these parameters λ must be even, and for $\mathcal{D}$ to have blocks of size greater than 2, and hence be nontrivial, we must have $\lambda > 2$. Moreover, v is even if and only if $\lambda \equiv 2 \pmod 4$, so suppose this is the case. Then by Lemma 2.2, $k-\lambda = \lambda(\frac{\lambda-2}{2})$ is a square. Since $\gcd(\lambda, \lambda-2) = 2$, this implies that $\lambda = 2u^2$ where u is odd, and $2(u^2-1)$ is a square. $\square$

We treat the cases $x(\ell-1) < x+\ell$ and $x(\ell-1) \geqslant x+\ell$ separately. First we use the information above to prove that Theorem 1.1(a)–(d) holds except for a short list of parameter values. Then we consider these exceptional parameters and show that they do not correspond to point-imprimitive, flag-transitive symmetric designs.

Lemma 2.5. *If $x(\ell-1) < x+\ell$ then either Theorem* 1.1 *holds or we have* $(k, \lambda, c, d, \ell) = (15, 6, 6, 6, 3)$.

Proof. The inequality $x(\ell-1) < x+\ell$ holds if either $x = 1$ or $\ell = 2$ (recall that $\ell \geqslant 2$). On the other hand, if $x > 1$ and $\ell > 2$ then $2 \leqslant x < \frac{\ell}{\ell-2} = 1 + \frac{2}{\ell-2} \leqslant 3$, and so $(x, \ell) = (2, 3)$. Thus one of (i) $x = 1$, (ii) $\ell = 2$, or (iii) $(x, \ell) = (2, 3)$. Also, as noted before (5), $x(\ell-1) < \lambda$.

Case 1. $x(\ell-1) < x+\ell < \lambda$. Here $\lambda - x(\ell-1) \geqslant \lambda - (x+\ell) + 1 \geqslant 2$ and hence by (5), $k \leqslant \lambda(x+\ell)/2$. If $x+\ell \leqslant \lambda - 3$ then Theorem 1.1(a) holds, so we may assume that this is not the case.

Suppose next that $x+\ell = \lambda-2$. Then, by (5), $k = \frac{\lambda(\lambda-2)}{\lambda - x(\ell-1)}$ and $\lambda - x(\ell-1) \geqslant 3$. If $\lambda - x(\ell-1) \geqslant 4$ then $\lambda \geqslant 5$, and hence $k \leqslant \frac{\lambda(\lambda-2)}{4} \leqslant \frac{\lambda(\lambda-3)}{2}$ and again Theorem 1.1(a) holds. Thus we may assume that $\lambda - x(\ell-1) = 3$. Then we have $x+\ell = \lambda-2 = x(\ell-1)+1$ and in particular $x \neq 1$ and $\ell \neq 2$. Thus, by the observation in the first paragraph, $(x, \ell) = (2, 3)$ and hence $\lambda = 7$ and $k = 35/3$ which is a contradiction.

The remaining case to be considered is $x+\ell = \lambda-1$. By (5), $k = \frac{\lambda(\lambda-1)}{\lambda-x(\ell-1)}$ and $\lambda - x(\ell-1) \geqslant 2$. If $\lambda - x(\ell-1) \geqslant 4$ then $\lambda \geqslant 5$ and hence $k \leqslant \frac{\lambda(\lambda-1)}{4} \leqslant \frac{\lambda(\lambda-3)}{2}$, and again Theorem 1.1(a) holds. Thus we may suppose that $\lambda - x(\ell-1) = 2$ or 3. Consider first $\lambda - x(\ell-1) = 3$ so that $x+\ell = \lambda-1 = x(\ell-1)+2$ and in particular $(x, \ell) \neq (2, 3)$. In this case $k = \frac{\lambda(\lambda-1)}{3}$. If $\lambda \geqslant 7$ then this expression for k is at most $\frac{\lambda(\lambda-3)}{2}$ and Theorem 1.1(a) holds. Thus we may assume that $\lambda \leqslant 6$. If $x = 1$ then $\lambda = \ell+2$ so $k = \frac{(\ell+1)(\ell+2)}{3}$, and as ℓ divides k it follows that $\ell = 2$ and $\lambda = 4 = k$, contradicting (7). Thus $x \neq 1$ and so by the observation from the first paragraph, $\ell = 2$. Hence $5 \leqslant 3+x = \lambda \leqslant 6$. Since $k = \frac{\lambda(\lambda-1)}{3}$ it follows that $\lambda = 6$, $k = 10$, but then $c-1 = 10/6$ which is a contradiction. Thus $\lambda - x(\ell-1) = 2$. Then $x+\ell = \lambda-1 = x(\ell-1)+1$, and hence $x = \frac{\ell-1}{\ell-2}$ which implies $(\ell, x, \lambda) = (3, 2, 6)$, and by (5), $k = 15$. Thus by (3) and (4), $v = 36$ and $c = d = 6$.

Case 2 *(The remaining case).* $x(\ell-1) < \lambda \leqslant x+\ell$. We work through the cases given in the first paragraph of the proof. Suppose that $x = 1$. Then $\ell - 1 < \lambda \leqslant \ell+1$. If $\lambda = \ell+1$ then $k = (\ell+1)^2/2$ contradicting (2). Hence $\lambda = \ell$ and so using the equations above, $k = \lambda(\lambda+1)$, $c = \lambda^2$, $d = \lambda+2$ and Theorem 1.1(b) holds. Next suppose that $\ell = 2$ so that $x < \lambda \leqslant x+2$. If $\lambda = x+1$ then the equations give $k = \lambda(\lambda+1)$, $v = \lambda^2(\lambda+2)$, $c = \lambda+2$, $d = \lambda^2$, and again Theorem 1.1(b) holds. If $\lambda = x+2$ then the equations give $k = \lambda^2/2$, $c = \lambda/2+1$, $d = \frac{\lambda}{2}(\lambda-2)+1$, so that, using Lemma 2.4, Theorem 1.1(c) holds. Finally suppose that $(x, \ell) = (2, 3)$ so that $4 < \lambda \leqslant 5$. Then $\lambda = 5$ so that by (5), $k = 25$. However this contradicts the fact that ℓ divides k. $\square$

Table 2
Parameter sequences for Lemma 2.6

k	λ	c	d	ℓ
24	8	7	10	3
24	8	10	7	4
104	13	25	33	4
210	21	41	51	5
210	21	51	41	6
300	25	37	97	4

Lemma 2.6. *If $x(\ell-1) \geqslant x+\ell$, then either Theorem* 1.1 *holds or k, λ, c, d, ℓ are as in one of the lines of Table* 2.

Proof. Here we have $\lambda > x(\ell-1) \geqslant x+\ell$, and this implies in particular that $x \geqslant 2$ and $\ell \geqslant 3$.

Case 1. $\lambda \geqslant x(\ell-1)+2$. If in addition $\lambda \geqslant x+\ell+3$ then by (5), $k \leqslant \lambda(x+\ell)/2 \leqslant \lambda(\lambda-3)/2$ so Theorem 1.1(a) holds. Thus we may assume that $\lambda < x+\ell+3$ and then we have $x+\ell+2 \leqslant x(\ell-1)+2 \leqslant \lambda < x+\ell+3$. It follows that $\lambda = x+\ell+2 = x(\ell-1)+2$ and hence $x = \frac{\ell}{\ell-2}$ which implies that $(x,\ell) = (3,3)$ or $(2,4)$. Solving the equations for λ, k, c, d we find the values in Lines 1 and 2 of Table 2 for these two cases.

Case 2 *(The remaining case).* $\lambda = x(\ell-1)+1$. Here, by (5),

$$k = \lambda(x+\ell) = \lambda\left(\frac{\lambda-1}{\ell-1}+\ell\right).$$

Note that, since ℓ divides $k = \lambda(x+\ell)$ it follows that ℓ divides $\lambda x = x^2(\ell-1)+x$, and hence that ℓ divides $x(x-1)$. Then, since $\ell \geqslant 3$ we must have $x \geqslant 3$ also. Suppose now that Theorem 1.1(a) does not hold, that is, $k > \lambda(\lambda-3)/2$. It follows from the displayed equation above that

$$\frac{\lambda-1}{\ell-1}+\ell > \frac{\lambda-3}{2}. \tag{8}$$

Multiplying out this gives $\ell^2 - (\frac{\lambda-1}{2})\ell + \frac{3\lambda-5}{2} > 0$, or equivalently

$$\left(\ell - \frac{\lambda-1}{4}\right)^2 > \frac{\lambda^2-2\lambda+1}{16} - \frac{3\lambda-5}{2} = \frac{\lambda^2-26\lambda+41}{16}.$$

Suppose first that $\lambda \geqslant 25$. Then $0 < (\frac{\lambda}{4} - \frac{21}{4})^2 = \frac{\lambda^2}{16} - \frac{42}{16}\lambda + \frac{441}{16} \leqslant \frac{\lambda^2-26\lambda+41}{16}$, and hence the inequality above implies that either (i) $\ell - \frac{\lambda-1}{4} > \frac{\lambda}{4} - \frac{21}{4}$, or (ii) $\frac{\lambda-1}{4} - \ell > \frac{\lambda}{4} - \frac{21}{4}$. Consider first case (i) so that $\ell > \frac{\lambda}{2} - \frac{22}{4}$. Here, if $x \geqslant 4$, then $\lambda = x(\ell-1)+1 > 4(\frac{\lambda}{2} - \frac{26}{4})+1 = 2\lambda-25$ which is impossible since $\lambda \geqslant 25$. Thus, since $x \geqslant 3$, we must have $x = 3$, and since $\ell \geqslant 3$ and ℓ divides $x(x-1)$, it follows that $\ell = 3$ or 6. Also, we have that $\lambda = 3(\ell-1)+1 = 7$ or 16 as $\ell = 3$ or 6, respectively, contradicting $\lambda \geqslant 25$. Thus we may assume that case (ii) holds and therefore $\ell = 3$ or 4. If $\ell = 4$ then, since $\frac{\lambda-1}{\ell-1}+\ell > \frac{\lambda-3}{2}$, we find that $\lambda < 31$. Moreover, since in this case $\lambda = 3x+1$ we must have $\lambda = 25$ or 28. For $\lambda = 25$, solving for k, c, d we find that Line 6 of Table 2 holds; for $\lambda = 28$ we find $k = \lambda(x+\ell) = 13\lambda$ so $k - \lambda = 12\lambda$ is not a square, while $c = 40$ so v is even, contradicting Lemma 2.2. Suppose now that $\ell = 3$. Then $\lambda = 2x+1$, so λ is odd and $x = \frac{\lambda-1}{2}$, and then we find that $k = \lambda(x+3) = \lambda(\frac{\lambda+5}{2})$, $c = \lambda+6$ and $d = \frac{\lambda^2+4\lambda-1}{4}$. Since $\ell = 3$ divides k it follows that $\lambda \equiv 1$ or 3 (mod 6). Thus Theorem 1.1(d) holds.

It remains to deal with the case $\lambda \leqslant 24$. We consider the various possibilities for ℓ. Suppose first that $\ell = 3$. Then as in the previous paragraph, $\lambda = 2x + 1$ and we find that Theorem 1.1(d) holds. Thus we may assume that $\ell \geqslant 4$. Suppose next that $\ell = 4$. Then $\lambda = 3x + 1 \leqslant 24$ so $3 \leqslant x \leqslant 7$ and as $\ell = 4$ divides $x(x-1)$ we have $x = 4$ or 5, and hence $\lambda = 13$ or 16, respectively. For $\lambda = 13$, solving for k, c, d gives the values in Line 3 of Table 2; for $\lambda = 16$, we find $k = \lambda(x+\ell) = 9\lambda$ so $k - \lambda = 8\lambda$ is not a square, while $c = 28$ so v is even, contradicting Lemma 2.2. Now consider $\ell = 5$. Here $\lambda = 4x+1 \leqslant 24$ and $\ell = 5$ divides $x(x-1)$, so we have $x = 5$. Solving for λ, k, c, d gives the values in Line 4 of Table 2. Similarly if $\ell = 6$ then $\lambda = 5x + 1 \leqslant 24$, and so $x = 3$ or 4, and $\lambda = 16$ or 21, respectively. For $\lambda = 21$, solving for k, c, d gives the values in Line 5 of Table 2; for $\lambda = 16$, we find $k = \lambda(x + \ell) = 9\lambda$ so $k - \lambda = 8\lambda$ is not a square, while $c = 46$ so v is even, contradicting Lemma 2.2. Thus we may assume that $\ell \geqslant 7$. Then $6x + 1 \leqslant \lambda \leqslant 24$ and hence $x = 3$. However in this case ℓ cannot divide $x(x-1)$. □

To complete the proof of Theorem 1.1 we must deal with the seven parameter sequences occurring in Lemmas 2.5 and 2.6. To do this we employ ad hoc group theoretic methods. Note that, since G is flag-transitive, a point stabiliser G_α is transitive on the set of k blocks containing α, and a block stabiliser G_B is transitive on the k points of B.

Lemma 2.7. *There is no flag-transitive, point-imprimitive, symmetric design with* (k, λ, c, d, ℓ) *either equal to* $(15, 6, 6, 6, 3)$*, or as in one of the lines of Table* 2*.*

Proof. Suppose there is such a design admitting a flag-transitive group G. For each of the possibilities for the parameter sequence, the pair (k, λ) does not satisfy any of (a)–(d) of Theorem 1.1. It follows that (k, λ) arises only with (c, d) equal to $(6, 6)$, or as in one of the lines of Table 2, and in all cases the unordered pair $\{c, d\}$ is uniquely determined, given (k, λ). This implies that the only nontrivial point partitions preserved by G consist of d blocks of size c (or possibly c blocks of size d). In particular $G^{\mathcal{C}}$, and G_Δ^Δ are primitive subgroups of degree d and c, respectively.

We deal with each of the parameter sequences in turn. Let $B \in \mathcal{B}$ so that the setwise stabiliser G_B is transitive on the k/ℓ classes of $\mathcal{C}$ that intersect B in ℓ points. Let K be the kernel of the action of G on $\mathcal{C}$, and let $S = \mathrm{Soc}(K)$. If $K \neq 1$ then, for $\Delta \in \mathcal{C}$, both K^Δ and S^Δ are nontrivial normal subgroups of the primitive group G_Δ^Δ, and hence both are transitive. Thus in this case, K and S have d orbits of length c in Ω and, by Lemma 2.3 and since they are normal subgroups of G (transitive on $\mathcal{B}$), they also have d orbits of length c in $\mathcal{B}$.

Case. $(k, \lambda, c, d, \ell) = (15, 6, 6, 6, 3)$. Here G_B is transitive on the $k/\ell = 5$ classes of $\mathcal{C}$ that intersect B in 3 points, and hence G_B must fix setwise the unique class $\Delta \in \mathcal{C}$ disjoint from B. In particular $G^{\mathcal{C}}$ is 2-transitive. Since $|G : G_B| = 36$, G_B contains a Sylow 5-subgroup P of G. Let $\alpha \in B$. Then $15 = |G_B : G_{B,\alpha}|$ and hence P_α has index 5 in P. Moreover, P_α fixes Δ and also the class Δ' containing α. It follows that P_α fixes all classes in $\mathcal{C}$, and so P_α is a Sylow 5-subgroup of K. Now P_α fixes the 3 points of $B \cap \Delta'$ setwise and hence P_α fixes Δ' pointwise. Now $G = N_G(P_\alpha)K$ by the 'Frattini argument,' and so $N_G(P_\alpha)$ is transitive on $\mathcal{C}$. Hence P_α fixes every class pointwise, so $P_\alpha = 1$ and $|K|$ is not divisible by 5.

Now $G^{\mathcal{C}}$ and G_Δ^Δ are both primitive of degree 6, and hence they have socles isomorphic to A_6 or $\mathrm{PSL}(2, 5)$ (see [9, p. 324]). Now K^Δ is a normal subgroup of G_Δ^Δ, and since $|K|$ is not divisible by 5 it follows that K^Δ is trivial, whence $K = 1$. Thus G is isomorphic to a subgroup of S_6, and in particular $|G|$ is not divisible by 3^3, so that $|G_B|$ is not divisible by 3. Thus G_B is not transitive on the fifteen points of B, contradicting flag-transitivity.

Now we consider the lines of Table 2.

Line 1: Here $(k,\lambda,c,d,\ell)=(24,8,7,10,3)$ and a block stabiliser G_B is transitive on the 8 classes of $\mathcal{C}$ that meet B in 3 points. Hence G_B fixes the two remaining classes, say Δ_1, Δ_2, setwise. Moreover, since G_B is transitive on B it follows that a Sylow 3-subgroup P of G_B is nontrivial, and since $|G:G_B|=70$, P is a Sylow 3-subgroup of G. Suppose that $P^{\mathcal{C}}\neq 1$. Then the primitive subgroup $G^{\mathcal{C}}$ of S_{10} contains an element of order 3 fixing at least two classes of $\mathcal{C}$. It follows from a consideration of the primitive subgroups of S_{10}, see [9, p. 324] that $G^{\mathcal{C}}$ contains A_{10} and hence that $|G:G_{\{\Delta_1,\Delta_2\}}|=45$. This is a contradiction since $|G:G_B|=70$ and $G_B\leqslant G_{\{\Delta_1,\Delta_2\}}$. Thus $P^{\mathcal{C}}=1$, and hence $|G^{\mathcal{C}}|$ is not divisible by 3. This is a contradiction (see [9, p. 324]).

Line 2: Here $(k,\lambda,c,d,\ell)=(24,8,10,7,4)$ and this time G_B is transitive on the 6 classes of $\mathcal{C}$ that meet B in 4 points, and fixes setwise the unique class Δ disjoint from B. In particular $G^{\mathcal{C}}$ is 2-transitive, and $G_B<G_\Delta$. If $K=1$ then G_Δ acts faithfully on $\mathcal{C}$ and also induces a primitive group on Δ. The only possibility is $G_\Delta\cong A_6$ or S_6 and $G\cong A_7$ or S_7, respectively. However a computation in GAP [11] shows that, for the group $G=S_7$ acting transitively of degree 70, a point stabiliser has orbit lengths $1,6,9,18,36$. Thus for neither $G=A_7$ nor $G=S_7$ does the subgroup G_B of index 70 have B as an orbit of length 24.

Hence $K\neq 1$. Then also $S=\mathrm{Soc}(K)\neq 1$, and it follows from the second paragraph of the proof that S has 7 orbits of length 10 in $\mathcal{B}$, so $|S:S_B|=10$. Now K^Δ is a nontrivial normal subgroup of G_Δ^Δ which (see [9, p. 324]) is almost simple with socle $T=A_5,A_6$ or A_{10}. Thus K^Δ contains T, and hence $S=\mathrm{Soc}(K)\cong T^s$ for some $s\geqslant 1$. We claim that $s=1$. Suppose to the contrary that $s\geqslant 2$, and let S_i be the pointwise stabiliser in S of the ith class Δ_i of $\mathcal{C}$, for $1\leqslant i\leqslant 7$. Then each S_i is nontrivial and G permutes the S_i primitively by conjugation. It follows that the S_i are all distinct, and that S_i is transitive on Δ_j for each $j\neq i$. Since $|S:S_B|=10$, and in all cases there is a unique G-conjugacy class of subgroups of S of index 10, it follows that S_B contains $T^{s-1}=S_i$ for some i, and hence S_B is transitive on at least 6 of the Δ_j. On the other hand, S_B fixes $\Delta_j\cap B$ setwise for each j, and $|\Delta_j\cap B|=4$ for 6 of the classes Δ_j. This contradiction proves the claim. Thus $S\cong T$.

As mentioned above, S_B has index 10 in S, and in all cases S has a unique conjugacy class of subgroups of index 10. Hence S_B is the stabiliser in S of a point in each of the classes Δ_i and is either transitive on the remaining 9 points of Δ_i (if $T=A_6$ or A_{10}) or has orbits of lengths 3 and 6 on these points (if $T=A_5$). However if $B\cap\Delta_i\neq\emptyset$, then G_{B,Δ_i} is transitive on the four points of $B\cap\Delta_i$, and contains S_B as a normal subgroup. This is a contradiction since the normal subgroup S_B should have equal length orbits in $B\cap\Delta_i$.

Line 3: Here $(k,\lambda,c,d,\ell)=(104,13,25,33,4)$. By [9, p. 324], the only primitive groups of degree 33 have socle A_{33} or $\mathrm{PSL}(2,32)$, and hence $G^{\mathcal{C}}$ contains one of these groups. Consider the group $G_B^{\mathcal{C}}$ induced by G_B on $\mathcal{C}$. The subset of $k/\ell=26$ classes that intersect B in 4 points forms an orbit for $G_B^{\mathcal{C}}$. In particular, 13 divides $|G^{\mathcal{C}}|$ and hence $A_{33}\leqslant G^{\mathcal{C}}$. Then it follows from [9, Theorem 5.2A], and the fact that $|G:G_B|=v=25\cdot 33<\binom{33}{3}$, that $G_B^{\mathcal{C}}$ contains A_{31}. This group has no orbit of length 26 in $\mathcal{C}$ so again we have a contradiction.

Line 4: Here $(k,\lambda,c,d,\ell)=(210,21,41,51,5)$. By [9, p. 324], the only primitive groups of degree 51 are A_{51} and S_{51}, and hence $G^{\mathcal{C}}$ contains A_{51}. Consider the group $G_B^{\mathcal{C}}$ induced by

G_B on $\mathcal{C}$. Since $|G : G_B| = v = 41 \cdot 51 < \binom{51}{3}$, it follows from [9, Theorem 5.2A] that $G_B^{\mathcal{C}}$ contains A_{49}. This contradicts the fact that G_B fixes setwise a subset of $k/\ell = 42$ classes.

Line 5: Here $(k, \lambda, c, d, \ell) = (210, 21, 51, 41, 6)$. This time we have, from [9, p. 324], that either $G^{\mathcal{C}}$ contains A_{41} or $G^{\mathcal{C}} \leqslant \mathrm{AGL}(1, 41)$. Now the block stabiliser G_B has index $v = 41 \cdot 51$ in G, and the subset of $k/\ell = 35$ classes that intersect B in 6 points forms an orbit for $G_B^{\mathcal{C}}$. Thus 7 divides $|G^{\mathcal{C}}|$ and hence $G^{\mathcal{C}}$ contains A_{41}. Then arguing as for Line 4 we find that $G_B^{\mathcal{C}}$ contains A_{39}. This contradicts the fact that G_B has an orbit of length 35 in $\mathcal{C}$.

Line 6: Here $(k, \lambda, c, d, \ell) = (300, 25, 37, 97, 4)$. The argument is similar to previous ones. We have, from [9, p. 324], that either $G^{\mathcal{C}}$ contains A_{97} or $G^{\mathcal{C}} \leqslant \mathrm{AGL}(1, 97)$. The stabiliser G_B has index $v = 37 \cdot 97$ in G and has an orbit of length $k/\ell = 75$ in $\mathcal{C}$, namely the set of classes that intersect B in 4 points. Thus 25 divides $|G^{\mathcal{C}}|$ and hence $G^{\mathcal{C}}$ contains A_{97}. Then arguing as for Line 4 we find that $G_B^{\mathcal{C}}$ contains A_{96} which is a contradiction. □

We note that the proof of Theorem 1.1 follows immediately from the results of this section.

3. Proof of Corollary 1.3

We use Theorem 1.1 and the information at the beginning of Section 2 to identify a list of feasible parameter sequences for point-imprimitive, flag-transitive symmetric designs for $\lambda \leqslant 10$, thereby proving Corollary 1.3.

Proof. Let $\mathcal{D}$ be a nontrivial, flag-transitive, point-imprimitive symmetric design with parameters $v, k, \lambda, c, d, \ell$ as in Theorem 1.1, where $\lambda \leqslant 10$. Suppose first that one of parts (b)–(d) of Theorem 1.1 holds. A straightforward computation gives the values in all lines of Table 1 except Line 20. Since each pair (v, k) occurs with a unique unordered pair $\{c, d\}$, it follows that the only possible sizes for nontrivial blocks of imprimitivity for G on points are c and d. Thus the groups $G^{\mathcal{C}}$ and G_Δ^Δ (where $\Delta \in \mathcal{C}$) are primitive.

By Theorem 1.1, we may suppose therefore that $k \leqslant \lambda(\lambda - 3)/2$. Then $3 \leqslant k \leqslant \lambda(\lambda - 3)/2 \leqslant 35$, and hence $5 \leqslant \lambda \leqslant 10$. Moreover, by (7) it follows that $\lambda < k \leqslant v - 2$, and in particular, $\lambda \neq 5$. Note that $2 < k < v$ since $\mathcal{D}$ is nontrivial, and $v = cd$ with $c > 1, d > 1$, so v is not a prime. Also all of (3)–(7) hold. We deal with these values of λ, one by one. As usual, $B \in \mathcal{B}$, and K is the kernel of G on $\mathcal{C}$.

If $\lambda = 6$, then $7 \leqslant k \leqslant 9$, and (3) implies that $(k, v) = (7, 8)$ or $(9, 13)$; but since $v \geqslant k + 2$ and v is not prime, neither of these pairs is possible.

If $\lambda = 7$, then $8 \leqslant k \leqslant 14$, and (3) implies that $(k, v) = (8, 9)$ or $(14, 27)$; but since $v \geqslant k + 2$, only the second pair is allowed (and $\mathcal{D}$ is the complement of an Hadamard design). If $c = 9$ then (4) implies that $\ell = 5$, which does not divide k. Hence $(c, d, x, \ell) = (3, 9, 4, 2)$. Now G_B is transitive on the set of 7 classes that meet B in 2 points. Hence 7 divides $|G^{\mathcal{C}}|$, and so by [9, p. 324], $G^{\mathcal{C}}$ has socle $\mathrm{PSL}(2, 8)$ or A_9. In particular, $G^{\mathcal{C}}$ is 2-transitive, and hence the stabiliser in G of any unordered pair of classes has index 36. This is a contradiction since G_B fixes setwise the two classes disjoint from B, and $|G : G_B| = 27$.

If $\lambda = 8$, then $9 \leqslant k \leqslant 20$, and (3) implies that $(k, v) = (9, 10)$, $(16, 31)$ or $(17, 35)$; but since $v \geqslant k + 2$ and v is not prime, only the last pair is allowed (and $\mathcal{D}$ is an Hadamard design). However (4) implies that 17 divides $c - 1$, which is impossible for any proper divisor of 35.

Table 3

Column	1	2	3	4	5	6	7	8	9	10
k	11	15	16	20	21	25	26	30	31	35
v	12	22	25	39	43	61	66	88	94	120

If $\lambda = 9$, then $10 \leqslant k \leqslant 27$, and (3) implies that $(k, v) = (10, 11), (18, 35), (19, 39)$, or $(27, 79)$; but since $v \geqslant k + 2$ and v is not prime, only the second and third pairs are allowed (and $\mathcal{D}$ is an Hadamard design or a complement of an Hadamard design). If $k = 19$, then (4) implies that $v = c$ and we have a contradiction. Hence $(k, v) = (18, 35)$. If $c = 7$ then (4) implies that $\ell = 4$, which does not divide k. Hence $(c, d, x, \ell) = (5, 7, 3, 3)$. Now G_B is transitive on the set of 6 classes that meet B in 3 points, and hence G_B fixes the unique class, say Δ, disjoint from B. Thus $G^{\mathcal{C}}$ is 2-transitive. If $K = 1$ then $G \cong G^{\mathcal{C}} \leqslant S_7$, and as G_B^B is transitive, 5 divides $|G|$, so $G = A_7$ or S_7 (see [9, p. 324]). However neither of these groups has a subgroup $G_B \cong G_B^{\mathcal{C}}$ of index 35 with orbits in $\mathcal{C}$ of lengths 1, 6. Hence $K \neq 1$. The argument just given also shows that $|G^{\mathcal{C}} : G_B^{\mathcal{C}}| = 7$ (not 35), and hence $G_\Delta = KG_B$ and $|K : K_B| = 5$. Now K_B fixes setwise the 3-element set $B \cap \Delta'$, for each $\Delta' \neq \Delta$. Then, since K_B has index 5 in K, it follows that $K^{\Delta'}$ is not 2-transitive. Therefore $G_{\Delta'}^{\Delta'} \leqslant \mathrm{AGL}(1, 5)$. However, the setwise stabiliser in G_B of Δ' is transitive on the 3 points of $B \cap \Delta'$, and hence $G_{\Delta'}^{\Delta'}$ has order divisible by 3. This is a contradiction.

If $\lambda = 10$, then $11 \leqslant k \leqslant 35$, and (3) implies that (k, v) is as in one of the columns in Table 3. Since c is a proper divisor of v, (4) implies that k, v are not as in Columns 1, 2, 3, 5, 6, 7, 9. This leaves $(k, v) = (20, 39), (30, 88)$ or $(35, 120)$. Consider the first pair. If $c = 13$ then $\ell = 7$ which does not divide k. Hence $(c, d, x, \ell) = (3, 13, 6, 2)$ (here $\mathcal{D}$ is a complement of an Hadamard design). Since G_B is transitive on the set of 10 classes that intersect B in 2 points, 5 divides $|G^{\mathcal{C}}|$ and it follows from [9, p. 324] that $G^{\mathcal{C}}$ contains A_{13}. This however implies that 11 divides $|G_B^{\mathcal{C}}|$, so G_B cannot fix a set of 10 classes setwise. Next consider the pair $(k, v) = (30, 88)$. In this case (4) and (6) imply that 3 divides both $c - 1$ and $d - 1$, and hence $\{c, d\} = \{4, 22\}$. However if $c = 22$ then $\ell = 8$ which does not divide k. So $(c, d, x, \ell) = (4, 22, 7, 2)$. By [9, p. 324] it follows that $G^{\mathcal{C}}$ contains A_{22} or M_{22}, and hence G_B has index 4 in the stabiliser of a class, say Δ (since the class-stabilisers are the only intransitive subgroups of $G^{\mathcal{C}}$ of index dividing 88). Since $G_\Delta^{\mathcal{C}}$ has socle A_{21} or $\mathrm{PSL}(3, 4)$ (with no proper subgroups of index dividing 4), it follows that $G_B^{\mathcal{C}}$ contains this socle and so is transitive on $\mathcal{C} \setminus \{\Delta\}$, contradicting the fact that G_B fixes setwise the set of 15 classes that intersect B in 2 points. Finally consider the pair $(k, v) = (35, 120)$. In this case (4) and (6) imply that 7 divides both $c - 1$ and $d - 1$, and hence $\{c, d\} = \{8, 15\}$. However if $c = 8$ then $\ell = 3$ which does not divide k. So $(c, d, x, \ell) = (15, 8, 2, 5)$, as in Line 20 of Table 1. This completes the proof. □

4. Final commentary on examples

In this section we discuss in Remark 4.1 the example for Line 5 of Table 1 given in [22]. We are grateful to Maska Law for giving us the information contained in this remark. Also we prove Proposition 1.5 establishing the uniqueness of the example for Line 4 mentioned in Remark 1.4, and Proposition 1.6 proving nonexistence of examples for Lines 13 and 19.

Remark 4.1. The example of a flag-transitive, point-imprimitive symmetric $2 - (96, 20, 4)$ design in [22] arises from a certain configuration of lines and points of a generalised quadrangle of order $(3,5)$. We explain this construction for generalised quadrangles of order $(q-1,q+1)$ since the same construction gives one of the examples for Line 1.

These designs are obtained by taking the point set Ω as the set of lines of a generalised quadrangle (GQ) with parameters $(q-1,q+1)$ associated with a 2-transitive hyperoval H in a projective plane $\mathrm{PG}(2,q)$ contained in projective space $\mathrm{PG}(3,q)$, where $q=2$ or 4 for Line 1 and Line 5, respectively. The lines of the GQ are precisely the $q^2(q+2)$ lines of $\mathrm{PG}(3,q)$ that meet the hyperoval H and that do not lie in the plane containing H. The block set $\mathcal{B}$ of the design is also labelled by the lines of the GQ: for a block $B\in\mathcal{B}$, the set of lines of the GQ (points of Ω) it contains is the set of $q(q+1)$ lines of the GQ that meet the line labelling B in a point not on the hyperoval. Since 2-transitive hyperovals exist only for $q=2,4$, and since 2-transitivity seems to be necessary for flag-transitivity of the design, this construction may not give a larger family of flag-transitive examples.

It was suggested in [22] that the flag-transitive $2-(96,20,4)$ design obtained in this way has automorphism group $2^4 3.S_6$. However computation, using GAP [11], verified that the automorphism group G is $2^6.3S_6$, the stabiliser of a line of the GQ is $2^2(3:S_5)$, and this subgroup (which is the stabiliser of both a point and a block of the design) is transitive on the 20 lines forming the block corresponding to this line. Moreover, there is a unique G-invariant partition consisting of 6 classes of size 16. Each of these classes is a spread in the GQ (that is, a set of lines of the GQ such that each point of the GQ is on exactly one of these lines). Thus this design is flag-transitive and point-imprimitive with the parameters of Line 5. For more details about the geometrical nature of this construction, a reader may wish to consult [20].

4.1. Proof of Proposition 1.5

In this subsection we prove the uniqueness of the flag-transitive, point-imprimitive $2-(15,8,4)$ design. The five pairwise nonisomorphic $2-(15,7,3)$ designs are listed in [5, Table 1.23, p. 11]. One way to establish uniqueness would be to examine all five of these designs, compute their automorphism groups, and prove that only one has a point-imprimitive subgroup that acts flag-transitively on the complementary design. However, to be consistent with the spirit of the paper, we decided to give a theoretical proof. First we identify the example.

Lemma 4.2. *Let $\mathcal{D}=(\Omega,\mathcal{B})$ be the design of points and complements of hyperplanes of* $\mathrm{PG}(3,2)$. *Then* $\mathrm{Aut}(\mathcal{D})$ *has subgroups $G_1\cong S_5$ and $G_2=G_1.3$ acting flag-transitively and point-imprimitively, satisfying Line* 4 *of Table* 1.

Proof. The automorphism group $\mathrm{Aut}(\mathcal{D})\cong A_8$ has a subgroup $X=A_7$ that is 2-transitive on points and on hyperplane complements. Let $\alpha\in\Omega$ and $B\in\mathcal{B}$. Since X_B has orbits of lengths 1, 14 in $\mathcal{B}$, it follows from Lemma 2.3 that X_B has two orbits in Ω, and these must be the set B of size 8, and its complement.

Let $G=S_5$ be the stabiliser of an unordered pair in the natural action of X on 7 points. Then $X_\alpha\cong\mathrm{PSL}(2,7)$ and $X_\alpha G=X$, so G is transitive on the point set Ω, and hence also on the block set $\mathcal{B}$. Now $G_B=X_B\cap G\cong D_8$ is a Sylow 2-subgroup of X_B, and since X_B is transitive on B (of size 8), it follows that its Sylow 2-subgroup G_B must be transitive on B also. Thus G is flag-transitive. Finally $G_\alpha=X_\alpha\cap G$ is also isomorphic to D_8, and there is exactly one proper subgroup of G properly containing G_α, namely $D_8<S_4<S_5$. Thus G is point-imprimitive, preserving a point-partition with 5 classes of size 3, and Line 4 of Table 1 holds.

Thus the lemma is proved for the subgroup $G_1 = G$. The normaliser of G_1 in A_8 is $G_2 = G_1.3 \cong (A_5 \times Z_3).2$. Since G_2 contains G_1, it acts flag-transitively on $\mathcal{D}$. The point stabiliser $(G_2)_\alpha$ is $D_8.3 \cong S_4$, and since $(G_2)_\alpha < (A_4 \times Z_3).2 < G_2$, it follows that G_2 also preserves a point-partition with 5 classes of size 3 (the same partition that is preserved by G_1). Thus Line 4 of Table 1 also holds for G_2. $\square$

Now we begin the proof of Proposition 1.5. Let $\mathcal{D} = (\Omega, \mathcal{B})$ be a symmetric $2-(15, 8, 4)$ design admitting a flag-transitive, point-imprimitive subgroup of automorphisms G. Then G satisfies Line 4 of Table 1, so G leaves invariant a partition $\mathcal{C}$ of Ω with 5 classes of size 3, and each block meets $k/\ell = 4$ of the classes in $\ell = 2$ points. Let $B \in \mathcal{B}$. Then B is disjoint from exactly one class of $\mathcal{C}$, say Δ. Thus $G_B < G_\Delta < G$, and $|G_\Delta : G_B| = 3$, $|G : G_\Delta| = 5$. Moreover, G_B is transitive on the 4 classes of $\mathcal{C}$ that meet B in 2 points. Hence $G^{\mathcal{C}}$ is 2-transitive of degree 5, and so $G^{\mathcal{C}}$ is one of F_{20}, A_5 or S_5 (see [24, p. 178]). First we handle the case where G acts faithfully on $\mathcal{C}$.

Lemma 4.3. *If G is faithful on $\mathcal{C}$, then $\mathcal{D}, G$ are as in Lemma 4.2 with $G = G_1$.*

Proof. Suppose that $G \cong G^{\mathcal{C}}$. Since G is flag-transitive, its order is divisible by 120, and so $G \cong S_5$. This implies that, for $\alpha \in \Omega$, $G_\alpha \cong D_8$ (a Sylow 2-subgroup of G), and so the action of G on Ω is permutationally isomorphic to the action of the group $G_1 \cong S_5$ of Lemma 4.2 on the points of $\mathrm{PG}(3, 2)$. Thus we may identify Ω with the set of points of $\mathrm{PG}(3, 2)$. Similarly $G_B \cong D_8$, and the action of G on $\mathcal{B}$ is permutationally isomorphic to the action of G_1 on the (complements of) hyperplanes of $\mathrm{PG}(3, 2)$. Since B is a G_B-orbit in Ω of size 8, and since G_B has only one orbit of this size, it follows that B is the complement of a hyperplane of $\mathrm{PG}(3, 2)$, and the lemma is proved. $\square$

Now we deal with the case where G is not faithful on $\mathcal{B}$.

Lemma 4.4. *If G is not faithful on $\mathcal{C}$, then $\mathcal{D}, G$ are as in Lemma 4.2 with $G = G_2$.*

Proof. Suppose that the kernel K of the action of G on $\mathcal{C}$ is nontrivial, and let $\mathcal{C} = \{\Delta_1, \dots, \Delta_5\}$. Then K^{Δ_i} is a normal subgroup of the primitive group $G_{\Delta_i}^{\Delta_i}$ and hence K is transitive on Δ_i, for each i. This means that K is isomorphic to a subgroup of S_3^5. By Lemma 2.3 it follows that K has exactly 5 orbits in $\mathcal{B}$, say $\mathcal{B}_1, \dots, \mathcal{B}_5$, and since K is normal in G, each $\mathcal{B}_i$ has length 3. Thus, for each $B \in \mathcal{B}$, $|K : K_B| = 3$ and hence $|G_B K| = 3|G_B|$. Therefore $|G^{\mathcal{C}} : G_B^{\mathcal{C}}| = |G : G_B| = 5$. For each of the possibilities for $G^{\mathcal{C}}$ (namely F_{20}, A_5 and S_5), the group $G^{\mathcal{C}}$ has a unique conjugacy class of subgroups of index 5, and hence $G_B K = G_\Delta$ for some $\Delta \in \mathcal{C}$. We may therefore label the $\mathcal{B}_i$ in such a way that, for $B \in \mathcal{B}_i$, $G_B K = G_{\Delta_i}$, and $G_B^{\mathcal{C}} = G_{\Delta_i}^{\mathcal{C}}$. We observed above that G_B fixes the unique class of $\mathcal{C}$ from which it is disjoint. Since $G_B^{\mathcal{C}}$ is transitive on $\mathcal{C} \setminus \{\Delta_i\}$, it follows that, for $B \in \mathcal{B}_i$, $B \cap \Delta_i = \emptyset$.

Let $S = O_3(K) = Z_3^s$, the largest normal 3-subgroup of K. Then S is transitive on each of the Δ_i and each of the $\mathcal{B}_i$. Let $B \in \mathcal{B}_1$. Then $G_B^{\mathcal{C}} = G_{\Delta_1}^{\mathcal{C}}$, $B \cap \Delta_1 = \emptyset$, and $|S : S_B| = 3$. Now S_B fixes setwise the two points of $B \cap \Delta_i$ for each $i \neq 1$. Since S_B is a 3-group this implies that S_B fixes Δ_i pointwise for each $i \neq 1$. Suppose that $S_B \neq 1$. Then S_B induces a cyclic group of order 3 on Δ_1, fixing all other points of Ω. For each i there exists $g_i \in G$ such that $\Delta_1^{g_i} = \Delta_i$, and the conjugate $S_B^{g_i}$ permutes the three points of Δ_i and no others. This implies that S is the direct product of the five subgroups $S_B^{g_i}$, for $1 \leqslant i \leqslant 5$, and hence $|S| = 3^5$. However this means that $|S_B| = 3^4$, which is a contradiction. Hence $|S| = 3$.

Now $S \leqslant K \leqslant S_3^5$ and $|K : S|$ is a power of 2. This means that S is self-centralising in K, and hence either $K = S$ or $K = S.2 \cong S_3$, and in both cases K acts faithfully on each of the Δ_i. Let $C = C_G(K)$, so G/C is isomorphic to a subgroup of $\mathrm{Aut}(K)$. If $C^{\mathcal{C}} = 1$ then $C \leqslant K$ and $|G/C|$ is divisible by 5, which is a contradiction. Hence $C^{\mathcal{C}} \neq 1$, and so $C^{\mathcal{C}}$ is transitive. Now C is normal in G and so either C has 3 orbits of length 5 in Ω, or C is transitive on Ω. Now the C-orbits in Ω form a G-invariant partition, and by Corollary 1.3, G has no invariant partition consisting of 3 classes of size 5. Hence C is transitive on Ω and, by Lemma 2.3, C is also transitive on $\mathcal{B}$.

We claim that G has a normal subgroup $C_0 \times K$, where $C_0 \cong A_5$, $C_0 \cap K = 1$, and $|G : (C_0 \times K)| \leqslant 2$.

Case 1. $K \cong S_3$. Here $\mathrm{Aut}(K) = \mathrm{Inn}(K) \cong K$, and so $G = CK$ and $C \cap K = 1$. Hence $G = C \times K$, and $C \cong C^{\mathcal{C}} = G^{\mathcal{C}} = F_{20}$, A_5 or S_5. Since C is transitive on the 15 points of Ω it follows that $C \cong A_5$ or S_5. Define C_0 to be the derived subgroup of C. Then C_0 has all the required properties with $|G : (C_0 \times K)| = |G^{\mathcal{C}} : C_0^{\mathcal{C}}| \leqslant 2$.

Case 2. $K = S$. Here $|G : C| \leqslant |\mathrm{Aut}(K)| = 2$, $K < C$, and $K_B = 1$. Thus $G_B^{\mathcal{C}} \cong G_B$ and, since G_B is transitive on B, 8 divides $|G_B|$ which in turn divides $|G^{\mathcal{C}}|$. It follows that $G^{\mathcal{C}} \cong S_5$. Then, since $|G : C| \leqslant 2$, $C^{\mathcal{C}}$ contains A_5. Let C_0 be the derived subgroup of C. Then $C_0^{\mathcal{C}} = A_5$ and, since the Schur multiplier of A_5 is only Z_2 (see [6]), it follows that $C_0 \cap K = 1$. Thus again we have $C_0 \cong A_5$ and $C_0 \times K$ normal in G of index at most 2.

Thus the claim is proved. By Corollary 1.3 and Lemma 2.3, it follows as before that C_0 is transitive on both Ω and $\mathcal{B}$. Now C_0 has a unique conjugacy class of subgroups of index 15, namely the Sylow 2-subgroups. Hence each Sylow 2-subgroup H of C_0 is the stabiliser in C_0 of both a point, say α, and a block, say B. Using a small computation in GAP we find that the orbit lengths of H in Ω (and also in $\mathcal{B}$) are 1, 1, 1, 4, 4, 4. Hence B is the union of two of the three H-orbits of length 4 in Ω.

Since H fixes exactly 3 points of Ω, it follows that the centraliser of C_0 in $\mathrm{Sym}(\Omega)$ has order 3. In particular, C_0 does not centralise a subgroup S_3, and hence we are in Case 2 with $K = S = Z_3$. In fact we have that $G = N_{\mathrm{Sym}(\Omega)}(C_0) = (A_5 \times Z_3).2$ (a subgroup of index 2 in $S_5 \times S_3$). In particular G contains a subgroup $H_1 \cong S_5$ acting transitively on Ω, G is isomorphic to the group G_2 of Lemma 4.2, and the action of G on Ω is permutationally isomorphic to the action of G_2 on the points of $\mathrm{PG}(3, 2)$. Moreover, $(H_1)_\alpha$ has a unique orbit of length 8 in Ω; this must be the block B, and it follows that $\mathcal{D}$ is the design of Lemma 4.2. □

Now the proof of Proposition 1.5 is complete. Existence of the design with the required symmetry properties follows from Lemma 4.2, and uniqueness from Lemmas 4.3 and 4.4.

4.2. Proof of Proposition 1.6

Finally we give a proof of Proposition 1.6, similar to the proof in [22, p. 142]. Suppose (for a contradiction) that $\mathcal{D} = (\Omega, \mathcal{B})$ is a symmetric $2 - (247, 42, 7)$ or $2 - (435, 63, 9)$ design admitting a flag-transitive, point-imprimitive subgroup of automorphisms G. Then G satisfies Line 13 or 19 of Table 1, so G leaves invariant a partition $\mathcal{C}$ of Ω with 19 classes of size 13 or 29 classes of size 15, and each block meets $k/\ell = 14$ or 21 of the classes in $\ell = 3$ points, respectively. Let $B \in \mathcal{B}$, $\Delta \in \mathcal{C}$.

Then G_B is transitive on the k/ℓ classes of $\mathcal{C}$ that meet B in 3 points. Hence $G^{\mathcal{C}}$ is primitive of degree 19 or 29, and has order divisible by 14 or 21, respectively. It follows from [9, p. 324] that $G^{\mathcal{C}}$ is S_d or A_d. This means that $G^{\mathcal{C}}$ is transitive on the (k/ℓ)-element subsets of $\mathcal{C}$, and hence, as g ranges over G, each (k/ℓ)-element subset of $\mathcal{C}$ occurs as the set of classes intersecting some block B^g. Thus $247 = |\mathcal{B}| \geqslant \binom{19}{14}$ or $435 = |\mathcal{B}| \geqslant \binom{29}{21}$, respectively, which is a contradiction, thereby proving Proposition 1.6. □

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