Higher linear syzygies of inner projections

Youngook Choi\textsuperscript{a}, Pyung-Lyun Kang\textsuperscript{b}, Sijong Kwak\textsuperscript{c,*}

\textsuperscript{a} Department of Mathematics, Seoul National University, Seoul 151-742, Republic of Korea
\textsuperscript{b} Department of Mathematics, Chungnam National University, 305-764 Daejeon, Republic of Korea
\textsuperscript{c} Department of Mathematics, Korea Advanced Institute of Science and Technology, Daejeon and Korea Institute for Advanced Study, Seoul, Republic of Korea

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Abstract

Let $X \subset \mathbb{P}^r$ be a smooth projective variety embedded by the complete linear system associated to a very ample line bundle $L$ on $X$. In this paper, we prove that if $X$ satisfies property $N_p$ for $p \geq 1$, then every isomorphic inner projection $\tilde{X} \subset \mathbb{P}^{r-1}$ of $X$ satisfies $N_{p-1}$. We also give some applications of our results to various examples.

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Contents

1. Introduction ............................................... 860
2. Notations and preliminaries ................................ 861
3. Syzygies of inner projections of curves ................... 864
4. Inner projections of higher dimensional projective varieties ........................................... 867
5. Applications to some examples and questions ..................... 873

References ................................................ 875

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Corresponding author.

E-mail addresses: ychoi@math.kaist.ac.kr (Y. Choi), plkang@cnu.ac.kr (P.-L. Kang), skwak@kaist.ac.kr, sjkwak@kias.re.kr (S. Kwak).

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1. Introduction

For a smooth projective variety $X \subset \mathbb{P}(H^0(L))$ embedded by the complete linear system of a very ample line bundle $L$ on $X$, one can ask more detailed information about defining equations of $X$, i.e., the syzygies of $X$. As M. Green defined [11], we can say that $L$ satisfies property $N_0$ if it gives the projectively normal embedding and $L$ satisfies property $N_1$ if property $N_0$ holds and $X$ is cut out by quadrics. In general, $L$ satisfies property $N_p$, $p \geq 1$, if $X$ is projectively normal and the projective coordinate ring $S(X)$ of $X$ has the following minimal free resolution of the simplest type up to $p$th steps as a graded $S$-module:

$$
\cdots \rightarrow E_p \rightarrow E_{p-1} \rightarrow \cdots \rightarrow E_1 \rightarrow S \rightarrow S(X) \rightarrow 0,
$$

where $S = \text{Sym}(H^0(L))$ and $E_i = \bigoplus_{\beta_i} S(-i - 1)$ for all $1 \leq i \leq p$. In other words, property $N_p$ means the minimal free resolution of the homogeneous coordinate ring $S(X)$ of the projectively normal embedding of $X$ is linear until the $p$th step.

In general, we can consider two kinds of subsystems of $H^0(L)$ with respect to projections. One gives an isomorphic outer projection of $X \subset \mathbb{P}(H^0(L))$ with the center outside Sec($X$) which is not linearly normal and the other is a subsystem with base points that is the very ample complete linear system of the blow-up of $X$ and gives an inner projection of $X$. Recently, we have been interested in the geometric and syzygetic effects of property $N_p$ of $X \subset \mathbb{P}(H^0(L))$ to the isomorphic projection of $X$ in $\mathbb{P}(W)$ by a very ample subsystem $W \subset H^0(L)$.

More precisely, a generalization of property $N_p$ to a smooth nonlinearly normal variety $X \subset \mathbb{P}(W)$ can be made as follows [13]: $X$ satisfies property $N^S_p$ if for $R = \bigoplus_{\ell \in \mathbb{Z}} H^0(X, L^\ell)$, it has the following minimal free resolution of the simplest type as a graded $S_W$-module:

$$
\cdots \rightarrow E_p \rightarrow E_{p-1} \rightarrow \cdots \rightarrow E_1 \rightarrow S_W \oplus S_W(-1)^t \rightarrow R \rightarrow 0,
$$

where $S_W = \text{Sym}(W)$, $t = \text{codim}(W, H^0(L))$ and $E_i = S_W(-i - 1)^{\bigoplus \beta_i}$ for all $1 \leq i \leq p$. In other words, property $N^S_p$ means the minimal free resolution of $R$ as a graded $S_W$-module is linear until the $p$th step. Generally speaking, for a nonlinearly normal embedding of $X$, it is hard to find defining equations or to control the degree bound of defining equations of $X$ in $\mathbb{P}(W)$. However, it can be shown that if $X \subset \mathbb{P}(H^0(L))$ satisfies property $N_p$, then the isomorphic projection of $X$ in $\mathbb{P}(W)$ satisfies property $N^S_p$ for $t = \text{codim}(W, H^0(L)) \leq p$. Furthermore, if $X$ satisfies property $N^S_p$, then defining equations of $X$ in $\mathbb{P}(W)$ have degree at most $(t + 2)$, see [13] for details.

In the present paper, we are mainly interested in an inner projection of $X$ which is given by subsystem $W$ of $H^0(L)$ with given base points in $X$. Let $X$ be a smooth projective variety in $\mathbb{P}(H^0(L))$. For a closed point $q \in X$, the inner projection $\pi_q : X \rightarrow \mathbb{P}(W)$ defined by $\pi_q(p) = \overline{qP \cap \mathbb{P}(W)}$, $p \neq q$, is a rational map. We can understand this situation in the following diagram; for the blow-up $\text{Bl}_q(X)$ of $X$ at $q$, one has the regular morphism $\tilde{\pi}_q : \text{Bl}_q(X) \rightarrow \mathbb{P}(W)$ with the following commutative diagram:

$$
\begin{array}{ccc}
\text{Bl}_q(X) & \xrightarrow{\tilde{\pi}_q} & \mathbb{P}(W) \\
\sigma \downarrow & & \\
X & \xrightarrow{\pi_q} & X' = \pi_q(X \setminus \{q\}) \subset \mathbb{P}(W)
\end{array}
$$
It is said that \( X \) admits an inner projection at a point \( q \in X \) if the morphism \( \tilde{\pi}_q : \text{Bl}_q(X) \to \pi_q(X \setminus \{q\}) \) is an embedding, i.e., \( \sigma^*L - E \) is very ample. When one considers this situation in the projective embedding, we have a nice criterion for \( \sigma^*L - E \) to be very ample. Note that the inner projection \( \tilde{\pi}_q : \text{Bl}_q(X) \to \mathbb{P}(W) \) with center \( q \in X \) is a closed embedding if and only if \( q \notin \text{Trisec}(X) \) where \( \text{Trisec}(X) \) is the union of all trisecant lines to \( X \) and all lines contained in \( X \) [8, pp. 268–275]. In this case, \( \tilde{\pi}_q(\text{Bl}_q(X)) \) is equal to \( \pi_q(X \setminus \{q\}) \), the Zariski closure of \( \pi_q(X \setminus \{q\}) \).

With these in mind, we get the following theorem without any assumption on a very ample line bundle \( \mathcal{L} \).

**Theorem 1.1.** Let \( X \subset \mathbb{P}(H^0(\mathcal{L})) \) be a smooth irreducible projective variety. Suppose \( \mathcal{L} \) satisfies property \( N_p \) for \( p \geq 1 \). For any \( q \in X \setminus \text{Trisec}(X) \), \( \tilde{\pi}_q(\text{Bl}_q(X)) = \pi_q(X \setminus \{q\}) \) in \( \mathbb{P}(W) \) is smooth and satisfies property \( N_{p-1} \), i.e., property \( N_{p-1} \) holds for \( (\text{Bl}_q(X), \sigma^*\mathcal{L} - E) \).

Let \( \sigma : \tilde{X} \to X \) be the blowing-up at distinct \( s \) points \( q_1, q_2, \ldots, q_s \) in \( X \) with the exceptional divisors \( E_1, \ldots, E_s \). If \( \mathcal{L}' := \sigma^*\mathcal{L}(-E_1 - \cdots - E_s) \) on \( \tilde{X} \) is very ample and \( \mathcal{L} \) satisfies \( N_p \) for \( p \geq 1 \), then we can check from Theorem 1.1 that \( \mathcal{L}' \) satisfies \( N_{p-s} \) for \( p \geq s \).

As another simple corollary, we have the following:

**Corollary 1.2.** Let \( X \subset \mathbb{P}(W) \) be a smooth variety with property \( N_p^s \) and \( t = \text{codim}(W, H^0(\mathcal{L})) \). Then, for any \( q \in X \setminus \text{Trisec}(X) \) we have:

1. An inner projection \( \pi_q(X \setminus \{q\}) \subset \mathbb{P}^{N-1} \) satisfies property \( N_{p-1}^s \).
2. The defining equations of \( \pi_q(X \setminus \{q\}) \) in \( \mathbb{P}^{N-1} \) have degree at most \( p + 2 \) for \( p \geq 2 \).

By using Koszul cohomology technique due to M. Green and the vanishing of higher cohomology groups of related vector bundles, we can show inductively \( \text{Tor}^i_S(R', R_i) = 0 \), \( 0 \leq i \leq p - 1 \) and \( j \geq 2 \), for a finitely generated graded \( S \) module \( R' = \bigoplus_{\ell \in \mathbb{Z}} H^0((\sigma^*\mathcal{L} - \mathcal{E})^{\otimes \ell}) \). In Section 3, we prove the main Theorem 1.1 for curves and then, we deal with the higher dimensional case in Section 4.

As a typical example of our main Theorem 1.1, it is well known that a Del Pezzo surface \( v_3(\mathbb{P}^2) \) of degree 9 in \( \mathbb{P}^3 \) satisfies property \( N_6 \) (and fails to hold \( N_2 \)) and by taking successive inner projections at general points \( \{q_1, q_2, \ldots, q_i, \ i \leq 6\} \), we get smooth Del Pezzo surfaces of degree \( 9 - i \) in \( \mathbb{P}^3 \) with property \( N_{6-i} \). In general, it would be very interesting to know syzygies of the form \( \sigma^*\mathcal{L}(-m_1E_1 - m_2E_2 - \cdots - m_\alpha E_\alpha) \) if it is very ample where \( \sigma : \tilde{X} \to X \) is the blowing-up with exceptional divisors \( E_i, \ i = 1, 2, \ldots, \alpha \).

In Section 5, we give some applications of our results to various varieties and their embeddings, such as the inner projections of Veronese embeddings, Calabi–Yau manifolds, rational surfaces, and adjoint linear series of projective varieties. We also relate multisecant spaces of \( X \) to syzygies of \( X \).

## 2. Notations and preliminaries

### 2.1. Notations

Throughout this paper the following are assumed.

1. We work throughout over the complex numbers.
(2) For a finite dimensional vector space $V$, $\mathbb{P}(V)$ is the projective space of one-dimensional quotients of $V$.

(3) When a projective variety $X$ is embedded in a projective space $\mathbb{P}^r$, we always assume that it is nondegenerate, i.e., does not lie on any hyperplane in $\mathbb{P}^r$.

(4) For a smooth projective variety $X \subset \mathbb{P}^r$ embedded by the complete linear system associated to very ample line bundle $L$ on $X$ and a closed point $q \in X$, we use the following notations:

- $V = H^0(X, L) = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$.
- $S = \oplus_{\ell \in \mathbb{Z}} \text{Sym}^\ell(V) = \oplus_{\ell \in \mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(\ell))$: the homogeneous coordinate ring of $\mathbb{P}^r$.
- $R = \oplus_{\ell \in \mathbb{Z}} H^0(X, L^{\ell}) = \oplus_{\ell \in \mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{O}_X(\ell))$: the associated graded $S$-module of $\mathcal{O}_X$.
- $\tilde{X} = \text{Bl}_q(X)$: a blowing-up of $X$ at a point $q$ with a morphism $\sigma : \tilde{X} \to X$.
- $E$: the exceptional divisor of $\tilde{X}$.
- $W = H^0(\tilde{X}, \sigma^*L(-E)) = H^0(X, L(-q))$.
- $S_W = \oplus_{\ell \in \mathbb{Z}} \text{Sym}^\ell(W)$: the homogeneous coordinate ring of $\mathbb{P}(W) = \mathbb{P}^{r-1}$.
- $R' = \oplus_{\ell \in \mathbb{Z}} H^0(\tilde{X}, (\sigma^*L - E)^\ell)$: the associated graded $S_W$-module.

2.2. Criteria for $N_p$ property

Suppose that $X \subset \mathbb{P}(V)$ is defined by the complete linear system associated to very ample line bundle $L$ on $X$ and consider the natural exact sequence

$$0 \to \mathcal{M}_V \to H^0(X, L) \otimes \mathcal{O}_X \xrightarrow{\varphi} L \to 0,$$

where $\mathcal{M}_V$ is the kernel of the surjective map $\varphi$. Taking $(i+1)$st exterior powers and twisting by $L^{j-1}$ yield

$$0 \to \wedge^{i+1} \mathcal{M}_V \otimes L^{j-1} \to \wedge^{i+1} H^0(X, L) \otimes L^{j-1} \to \wedge^i \mathcal{M}_V \otimes L^j \to 0.$$

Let $R = \bigoplus_{\ell \in \mathbb{Z}} H^0(X, L^{\ell})$ and $V = H^0(X, L)$. Then $\text{Tor}_i^S(R, \mathbb{C})_{i+j}$ fits in the following exact sequence:

$$0 \to \text{Tor}_i^S(R, \mathbb{C})_{i+j} \to H^1(X, \wedge^{i+1} \mathcal{M}_V \otimes L^{j-1}) \to \wedge^{i+1} V \otimes H^1(X, L^{j-1})$$

$$\to H^1(X, \wedge^i \mathcal{M}_V \otimes L^j) \to \cdots.$$

Note that $\beta_{i,j} = \dim(\text{Tor}_i^S(R, \mathbb{C})_{i+j})$ where $\beta_{i,j}$ is the Betti number of the minimal free resolution of a graded $S$-module $R$.

**Lemma 2.1.** A smooth variety $X$ satisfies property $N_p$ if and only if the homomorphism

$$\wedge^{i+1} V \otimes H^0(X, L^{j-1}) \to H^0(X, \wedge^i \mathcal{M}_V \otimes L^j)$$

is surjective, equivalently the homomorphism

$$H^1(X, \wedge^{i+1} \mathcal{M}_V \otimes L^{j-1}) \to \wedge^{i+1} V \otimes H^1(X, L^{j-1})$$

is injective for $0 \leq i \leq p$ and $j \geq 2$.

**Proof.** See [12, Section 1].
2.3. Blow-up and regularity

Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety $X$, and let $\mathcal{L}$ be a very ample line bundle on $X$. A coherent sheaf $\mathcal{F}$ is said to be $m$-regular with respect to $\mathcal{L}$ in the sense of Castelnuovo–Mumford if $H^i(X, \mathcal{F} \otimes \mathcal{L}^{m-i}) = 0$ for $i \geq 1$. It was shown in [15, Lecture 14], [14] that if $\mathcal{F}$ is $m$-regular, then $\mathcal{F}$ is $(m+1)$-regular. So one can define the regularity of $\mathcal{F}$ with respect to $\mathcal{L}$ to be the least integer $m$ such that $\mathcal{F}$ is $m$-regular, i.e.,

$$\text{reg}(\mathcal{F}, \mathcal{L}) = \min\{m \in \mathbb{Z} | \mathcal{F} \text{ is } m\text{-regular with respect to } \mathcal{L}\}.$$ 

On the other hand, if a graded $S$-module $M$ has the following minimal free resolution

$$\cdots \rightarrow F_i \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0,$$

where $F_i = \bigoplus S(-i-j)\beta_{i,j}$, then regularity of $M$ is defined by $\text{reg}(M) = \max\{j | \beta_{i,j} \neq 0 \text{ for all } i \geq 0\}$. Let $\mathcal{F}$ be a coherent sheaf on the projective space $\mathbb{P}^r$, and let $F = \bigoplus_{t \in \mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{F}(t))$ be the corresponding graded $S$-module of twisted global sections. If $F$ is finitely generated as a graded $S$-module, it is proved in [5, Exercise 20.20] that $\text{reg}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^r}(1)) = \text{reg}(F)$.

We have the following well-known cohomological properties between a smooth variety $X$ and its blow-up $\tilde{X}$.

**Lemma 2.2.** Let $Z \subset X$ be a smooth codimension $e$ subvariety of a smooth variety $X$, let $\sigma : \tilde{X} = \text{Bl}_Z(X) \rightarrow X$ be the blowing-up of $X$ along $Z$, and let $E \subset \tilde{X}$ be the exceptional divisor.

1. If $0 \leq t \leq e - 1$, then

$$H^i(\tilde{X}, \sigma^* \mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(tE)) = H^i(X, \mathcal{F}) \quad \text{for } i \geq 0,$$

for any locally free sheaf $\mathcal{F}$ on $X$.

2. If $t > 0$, then

$$H^i(\tilde{X}, \sigma^* \mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(-tE)) = H^i(X, \mathcal{F} \otimes \mathcal{I}_Z^t) \quad \text{for } i \geq 0,$$

for any locally free sheaf $\mathcal{F}$ on $X$ where $\mathcal{I}_Z$ is the sheaf of ideals defining $Z$.

**Proof.** See [1, p. 592]. □

When $\sigma^* \mathcal{L} - E$ is very ample, $R' = \bigoplus_{t \in \mathbb{Z}} H^0(\tilde{X}, (\sigma^* \mathcal{L} - E)^{\otimes t})$ is a finitely generated $S_W$-module. In particular, we have the following fact about the regularity of $R'$.

**Proposition 2.3.** Assume that $\mathcal{L}$ and $(\sigma^* \mathcal{L} - E)$ are very ample line bundles on varieties $X$ and $\tilde{X} = \text{Bl}_q(X)$, respectively. Then $\text{reg}(\mathcal{O}_X, \mathcal{L}) = m$ if and only if $\text{reg}(\mathcal{O}_{\tilde{X}}, (\sigma^* \mathcal{L} - E)) = m$.

**Proof.** It is enough to show that $\mathcal{O}_X$ is $m$-regular with respect to $\mathcal{L}$ if and only if $\mathcal{O}_{\tilde{X}}$ is $m$-regular with respect to $\sigma^* \mathcal{L}(-E)$, i.e., for all $i \geq 1$,

$$H^i(X, \mathcal{L}^{m-i}) = 0 \quad \text{if and only if } \quad H^i(\tilde{X}, (\sigma^* \mathcal{L} - E)^{m-i}) = 0.$$
(1) $1 \leq i < m$. By the above Lemma 2.2, $H^i(\tilde{X}, (\sigma^* \mathcal{L} - E)^{m-i}) = H^i(X, \mathcal{L}^{m-i} \otimes m_q^{m-i})$ where $m_q$ is the sheaf of ideals corresponding to the point $q \in X$.

For $2 \leq i < m$, from the following exact sequence

$$0 \to \mathcal{L}^{m-i} \otimes m_q^{m-i} \to \mathcal{L}^{m-i} \to \mathcal{O}_X/m_q^{m-i} \to 0$$

and $\dim \text{Spec}(\mathcal{O}_X/m_q^{m-i}) = 0$, we get

$$H^i(X, \mathcal{L}^{m-i}) \simeq H^i(\mathcal{L}^{m-i} \otimes m_q^{m-i}) \simeq H^i(\tilde{X}, (\sigma^* \mathcal{L} - E)^{m-i}).$$

For $i = 1$, notice that the natural morphism $H^0(\mathcal{L}^{m-1}) \to H^0(\mathcal{O}_X/m_q^{m-1})$ is surjective. In fact, we have the following natural isomorphism as a $\mathbb{C} \simeq k(q) = \mathcal{O}_X/m_q$ vector spaces:

$$\mathcal{O}_X/m_q^{m-1} \simeq \mathcal{O}_X/m_q \oplus m_q/m_q^2 \oplus m_q^2/m_q^3 \oplus \cdots \oplus m_q^{m-2}/m_q^{m-1},$$

and $H^0(\mathcal{L}^{m-1}) \to O_X/m_q^{m-1} \simeq \bigoplus_{i=0}^{m-2} m_q^i/m_q^{i+1}$ is surjective since $\mathcal{L}$ is very ample and $m_q^i/m_q^{i+1} \simeq S^i(m_q/m_q^2)$ for a smooth point $q \in X$ where $S^i$ denotes the $i$th symmetric power. Thus,

$$H^1(X, \mathcal{L}^{m-1}) \simeq H^1(\tilde{X}, (\sigma^* \mathcal{L} - E)^{m-1}).$$

(2) $m \leq i \leq \dim(X)$. Since $m \geq 1$, we have $0 \leq i - m \leq \dim(X) - m \leq \dim(X) - 1 = e - 1$ where $e$ is the codimension of $q$ in $X$. By Lemma 2.2,

$$H^i(X, \mathcal{L}^{m-i}) \simeq H^i(\tilde{X}, (\sigma^* \mathcal{L} - E)^{m-i}), \quad i \geq m. \quad \square$$

We have the following easy corollary which we need in the subsequent sections.

**Corollary 2.4.** Under the same conditions as in Proposition 2.3, we have $H^i(\tilde{X}, (\sigma^* \mathcal{L} - E)^i) = H^i(\tilde{X}, \sigma^* \mathcal{L}^i(-kE))$ for $i \geq 1$ and $0 \leq k \leq j$.

**Proof.** In Proposition 2.3, we showed that $H^i(\tilde{X}, (\sigma^* \mathcal{L} - E)^i) \simeq H^i(X, \mathcal{L}^i), \ i \geq 1$. From the short exact sequence $0 \to \mathcal{L}^i \otimes m_q^k \to \mathcal{L}^i \to \mathcal{O}_X/m_q^k \to 0$ and the same argument as in Proposition 2.3, we are done. \(\square\)

### 3. Syzygies of inner projections of curves

In this section, we assume that $X \subset \mathbb{P}(H^0(\mathcal{L}))$ is a smooth projective curve embedded by complete linear system of a very ample line bundle $\mathcal{L}$ on $X$. If $X$ has no trisecant line, then the inner projection $\pi_q(X)$ for a point $q \in X$ is embedded in $\mathbb{P}(W)$ where $W = H^0(\mathcal{L}(-q))$. Let $S_W$ be the coordinate ring of $\mathbb{P}(W)$ and $R' = \bigoplus_{\ell \in \mathbb{Z}} H^0(X, \mathcal{L}(-q)^{\otimes \ell})$ a finite $S_W$-module. The following theorem is a generalization of Theorem 1 in [4].

**Theorem 3.1.** Let $X \subset \mathbb{P}(H^0(\mathcal{L}))$ be a smooth irreducible curve with property $N_p$, $p \geq 1$. For every point $q \in X$, the Zariski closure of the linear projection $\pi_q(X \setminus \{q\})$ has property $N_{p-1}$. In other words, $\mathcal{L}(-q)$ satisfies $N_{p-1}$ for every point $q \in X$. 
**Proof.** Note that $\mathcal{L}(-q)$ is very ample for any point $q \in X$ since $\mathcal{L}$ satisfies property $N_1$, i.e., $X$ has no trisecant line. Consider the following diagram:

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{M}_W & W \otimes \mathcal{O}_X & \mathcal{L}(-q) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{M}_V & V \otimes \mathcal{O}_X & \mathcal{L} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{O}_X(-q) & \mathcal{O}_X & k(q) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 
\end{array}
$$

The above commutative diagram induces the following diagram:

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \wedge^{i+1} \mathcal{M}_W & \wedge^{i+1} W \otimes \mathcal{O}_X & \wedge^i \mathcal{M}_W \otimes \mathcal{L}(-q) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \wedge^{i+1} \mathcal{M}_V & \wedge^{i+1} V \otimes \mathcal{O}_X & \wedge^i \mathcal{M}_V \otimes \mathcal{L} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \wedge^i \mathcal{M}_W \otimes \mathcal{O}_X(-q) & \wedge^i W \otimes \mathcal{O}_X & \text{coker } \alpha \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 
\end{array}
$$

(3.1)

Here, $\alpha$ is a composition map $\wedge^i \mathcal{M}_W \otimes \mathcal{L}(-q) \rightarrow \wedge^i \mathcal{M}_V \otimes \mathcal{L}(-q) \rightarrow \wedge^i \mathcal{M}_V \otimes \mathcal{L}$. Twisting through by $\mathcal{L} \otimes (j-1)$ in diagram (3.1) and taking cohomology, we have the following diagram:

$$
\begin{array}{cccc}
\wedge^{i+1} W \otimes H^0(\mathcal{L}^{j-1}) & \tilde{\rho}_{i,j} & H^0(\wedge^i \mathcal{M}_W \otimes \mathcal{L}^{j-1}) & \tilde{\delta}_{i,j} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\wedge^{i+1} V \otimes H^0(\mathcal{L}^{j-1}) & \tilde{\mu}_{i,j} & H^0(\wedge^i \mathcal{M}_V \otimes \mathcal{L}^{j-1}) & \tilde{\beta}_{i,j} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\wedge^i \mathcal{M}_W \otimes H^0(\mathcal{L}^{j-1}) & \gamma_{i,j} & H^0(\text{coker } \alpha \otimes \mathcal{L}^{j-1}) & \tilde{\xi}_{i,j} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & H^1(\wedge^i \mathcal{M}_W \otimes \mathcal{L}^{j-1}(-q)) & H^1(\wedge^i \mathcal{M}_W \otimes \mathcal{L}^{j-1}(-q)) & \delta_{i,j} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & H^1(\wedge^i \mathcal{M}_V \otimes \mathcal{L}^{j-1}) & \tilde{\beta}_{i-1,j+1} & \rho_{i,j} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
H^1(\wedge^i \mathcal{M}_V \otimes \mathcal{L}^{j}) & \tilde{\phi}_{i-1,j+1} & \wedge^i \mathcal{V} \otimes H^1(\mathcal{L}^{j}) 
\end{array}
$$

(3.2)

Observe in the above diagram that $\mu_{i,j}$ is surjective and $\rho_{i,j}$ is injective for all $i \geq 0$ and $j \geq 0$. 
Now suppose that \( N_p \) holds for \((X, \mathcal{L})\), i.e., \( \alpha_{i,j} \) is surjective for \( 0 \leq i \leq p \) and \( j \geq 2 \). Then we claim that \( \delta_{i,j} \) is injective for all \( 1 \leq i \leq p \) and \( j \geq 2 \). Since \( H^1(\wedge^i \mathcal{M}_W \otimes \mathcal{L}(-q)) = 0 \) for all \( j \gg 0 \), \( \delta_{i,j+1} \) is injective for all sufficiently large \( j \). Then from the previous diagram, we have, for \( 1 \leq i \leq p \),

\[
\delta_{i,j+1} \text{ is injective} \iff \omega_{i,j} \text{ is injective} \iff \nu_{i,j} \text{ is surjective} \iff \gamma_{i,j} \text{ is surjective} \iff \delta_{i,j} \text{ is injective}
\]

until \( j \geq 2 \). Therefore, \( \delta_{i,j} \) is injective for \( 1 \leq i \leq p \) and \( j \geq 2 \).

To show that \((X, \mathcal{L}(-q))\) has property \( N_{p-1} \), it suffices to show that, for all \( 0 \leq i \leq p - 1 \) and \( j \geq 2 \),

\[
\wedge^{i+1} W \otimes H^0(\mathcal{L}(-q)^{\otimes (j-1)}) \to H^0(\wedge^i \mathcal{M}_W \otimes \mathcal{L}(-q)^{\otimes j})
\]

is surjective, or

\[
H^1(\wedge^{i+1} \mathcal{M}_W \otimes \mathcal{L}(-q)^{\otimes j-1}) \xrightarrow{\tilde{\beta}_{i,j}} \wedge^{i+1} W \otimes H^1(\mathcal{L}(-q)^{\otimes (j-1)})
\]

is injective.

Now, consider the following diagram for \( 0 \leq i \leq p - 1 \) and \( j \geq 2 \):

\[
\begin{array}{c}
H^1(\wedge^{i+1} \mathcal{M}_W \otimes \mathcal{L}^{j-1}(-(j-1)q)) \xrightarrow{(\ast)_{i,j}} \wedge^{i+1} W \otimes H^1(\mathcal{L}^{-1}(-(j-1)q)) \\
\downarrow \tilde{\beta}_{i,j} \\
H^1(\wedge^{i+1} \mathcal{M}_W \otimes \mathcal{L}^{j-1}(-q)) \xrightarrow{\delta_{i+1,j}} \wedge^{i+1} W \otimes H^1(\mathcal{L}^{-1})
\end{array}
\]

where \((\ast)_{i,j}\) is the identity map when \( j = 2 \).

By the below Lemma 3.2, \((\ast)_{i,j}\) is injective for \( j \geq 2 \) and for all \( i \). Therefore, \( \tilde{\beta}_{i,j} \) is injective for all \( 0 \leq i \leq p - 1 \) and \( j \geq 2 \) and \((X, \mathcal{L}(-q))\) has property \( N_{p-1} \).

**Lemma 3.2.** Let \( \mathcal{E} = \wedge^{i+1} \mathcal{M}_W \otimes \mathcal{L}^{j-1}((-k+1)q) \). From the exact sequence \( 0 \to \mathcal{O}_X(-q) \to \mathcal{O}_X \to k(q) \to 0 \), we have the following exact sequence:

\[
0 \to \mathcal{E}(-q) \to \mathcal{E} \to \mathcal{E} \otimes k(q) \to 0.
\]

Then \( H^1(X, \mathcal{E}(-q)) \xrightarrow{(\ast)_{i,j,k}} H^1(X, \mathcal{E}) \) is an isomorphism for \( j \geq 2 \) and \( 2 \leq k \leq j \).

**Proof.** It is enough to show that the morphism \( H^0(\mathcal{E}) \to H^0(\mathcal{E} \otimes k(q)) \) is surjective. This surjectivity is equivalent to the fact that \( \mathcal{E} \) is globally generated at \( q \in X \). From the exact sequence

\[
0 \to \wedge^{i+2} \mathcal{M}_W \to \wedge^{i+2} W \otimes \mathcal{O}_X \to \wedge^{i+1} \mathcal{M}_W \otimes \mathcal{L}(-q) \to 0,
\]
we know that $\bigwedge^{i+1} M_W \otimes L(-q)$ is globally generated. Now
\[
\bigwedge^{i+1} M_W \otimes L^{j-1}((-k+1)q) = \bigwedge^{i+1} M_W \otimes (\bigwedge(-q))^{k-1} \otimes L^{j-k}
\]
\[
= \bigwedge^{i+1} M_W \otimes (-q) \otimes (\bigwedge(-q))^{k-2} \otimes L^{j-k}.
\]
Since $\bigwedge^{i+1} M_W \otimes L(-q)$ is globally generated and $L(-q)^{k-2} \otimes L^{j-k}$ is very ample for $j \geq 2$ and $2 \leq k \leq j$ except for $j = k = 2$ (in this case, it is trivial), $E = \bigwedge^{i+1} M_W \otimes L^{j-1}((-k+1)q)$ is globally generated. □

**Corollary 3.3.** Let $X$ be a smooth curve and $L$ a very ample line bundle on $X$ satisfying property $N_p$ for $p \geq 1$. For any $s$ points (including infinitely near points) $\{q_1, q_2, \ldots, q_s\}$ in $X$, if $s \leq p$, then $L(-q_1 - q_2 - \cdots - q_s)$ satisfies $N_{p-s}$. In particular, $L(-q_1 - q_2 - \cdots - q_p)$ is always very ample for any points $q_1, q_2, \ldots, q_p \in X$ and normally generated.

**Corollary 3.4.** Let $X \subset \mathbb{P}^N$ be a smooth linearly normal projective curve. If $X$ has a $(p+2)$-secant $p$-plane, then property $N_p$ fails for $X$.

**Proof.** Assume that $X$ has property $N_p$. Then, by Corollary 3.3, $L(-q_1 - q_2 - \cdots - q_{p-1})$ satisfies property $N_1$ for any points $q_1, q_2, \ldots, q_{p-1} \in X$. But if $X$ has a $(p+2)$-secant $p$-plane, the projection at $(p-1)$ points has a trisecant line. This gives that the projection of $X$ at these $(p-1)$ points fails to hold property $N_1$, which is a contradiction. □

4. Inner projections of higher dimensional projective varieties

For a higher dimensional smooth variety $X$, we need many technical lemmas to prove the main theorem in this section.

Let $X$ be a smooth projective variety of $\mathbb{P}^r$ and $q$ a closed point of $X$. The inner projection $\pi_q : X \dashrightarrow \mathbb{P}^{r-1}$ is a rational map which is well defined outside $q$. Let $\sigma : Bl_q(X) \to X$ be a blowing-up of $X$ at $q$. Then one has the regular morphism $\tilde{\pi}_q : Bl_q(X) \to \mathbb{P}^{r-1}$ with the following diagram:

\[
\begin{array}{ccc}
\tilde{X} = Bl_q(X) & \xrightarrow{\tilde{\pi}_q} & \mathbb{P}^{r-1} \\
\downarrow \sigma & \text{ } & \text{ } \\
X \subset \mathbb{P}^r & \xrightarrow{\pi_q} & X' = \pi_q(X \setminus \{q\}) \subset \mathbb{P}^{r-1}
\end{array}
\]

If $\tilde{\pi}_q : Bl_q(X) \to \mathbb{P}^{r-1}$ is an embedding, then the exceptional divisor $E$ is linearly embedded via $\tilde{\pi}_q$ in $\mathbb{P}^{r-1}$, i.e., $\tilde{\pi}_q(E) = \mathbb{P}^{l-1} \subset \mathbb{P}^{r-1}$, $l = \dim(X)$.

**Proposition 4.1.** Assume that $X$ is a smooth nondegenerate projective variety in $\mathbb{P}^r$ and $q \in X$ is a closed point. Then, the morphism $\tilde{\pi}_q : Bl_q(X) \to \mathbb{P}^{r-1}$ is a closed embedding if and only if $q \notin \text{Trisec}(X)$ where Trisec($X$) is the union of all lines $\ell$ with the property that either $\ell \subset X$ or $\ell \cap X$ is a subscheme of length $\geq 3$.

**Proof.** For details, see [8, pp. 268–269]. □
Let $W = H^0(\tilde{X}, \sigma^*L - E) \subset H^0(X, L) = V$. If $\sigma^*L - E$ is very ample on $\text{Bl}_q(X)$, the morphism $\tilde{\pi}_q : \tilde{X} = \text{Bl}_q(X) \to \mathbb{P}(W)$ is a closed embedding defined by the global sections of $\sigma^*L - E$ and there is a natural exact sequence

$$0 \to \mathcal{M}_W \to W \otimes \mathcal{O}_{\tilde{X}} \xrightarrow{\phi} \sigma^*L - E \to 0,$$

where $\mathcal{M}_W$ is the kernel of $\phi$.

**Lemma 4.2.** $R^1 \sigma_* (\wedge^i \mathcal{M}_W \otimes (\sigma^*L - E)^k) = 0$ for $k \geq 0$ and $i \geq 0$.

**Proof.** Consider the following exact sequence:

$$0 \to \wedge^{i+1} \mathcal{M}_W \otimes (\sigma^*L - E)^{k-1} \to \wedge^{i+1} W \otimes (\sigma^*L - E)^{k-1} \to \wedge^i \mathcal{M}_W \otimes (\sigma^*L - E)^k \to 0.$$

Since $R^i \sigma_* (\sigma^*L \otimes \mathcal{O}_\tilde{X}(-tE)) = \mathcal{L} \otimes R^i \sigma_* (\mathcal{O}_\tilde{X}(-tE)) = 0$ for $i \geq 1$ and $t \geq 0$,

$$R^1 \sigma_*(\wedge^i \mathcal{M}_W \otimes (\sigma^*L - E)^k) = 0 \iff R^2 \sigma_*(\wedge^{i+1} \mathcal{M}_W \otimes (\sigma^*L - E)^{k-1}) = 0 \iff R^3 \sigma_*(\wedge^{i+2} \mathcal{M}_W \otimes (\sigma^*L - E)^{k-2}) = 0 \ldots \iff R^{k+1} \sigma_*(\wedge^{i+k} \mathcal{M}_W) = 0.$$

Therefore it is enough to show that $R^i \sigma_*(\wedge^t \mathcal{M}_W) = 0$ for all positive integers $t$ and $i$. Let $\mathcal{E} = \wedge^t \mathcal{M}_W$ and $\mathcal{E}^i = R^i \sigma_*(\mathcal{E})$. Since $\sigma$ is an isomorphism of $\tilde{X} - E$ onto $X - q$, $R^i \sigma_*(\mathcal{E})|_{X - q} = 0$. Therefore, the sheaves $\mathcal{E}^i$ for $i > 0$ have support at $q$. By the theorem on formal functions [9, p. 277],

$$\hat{\mathcal{E}}^i \cong \lim_{\leftarrow} H^i(E_n, \mathcal{E}|_{E_n}),$$

where $E_n$ is the closed subscheme of $\tilde{X}$ defined by $\mathcal{I}^n$, where $\mathcal{I}$ is the ideal sheaf of $E$ in $\tilde{X}$. There are natural exact sequences

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{O}_{E_{n+1}} \to \mathcal{O}_{E_n} \to 0$$

for each $n$. Since $q$ is a smooth point, we have $\mathcal{I}/\mathcal{I}^2 = \mathcal{O}_E(1)$ and $\mathcal{I}^n/\mathcal{I}^{n+1} \cong S^n(\mathcal{I}/\mathcal{I}^2) \cong \mathcal{O}_E(n)$.

**Claim.** $H^i(E, \mathcal{E} \otimes \mathcal{O}_E(n)) = 0$ for $i > 0$ and all $n > 0$.

If the claim is true, then we conclude from the long exact sequence of cohomology, using induction on $n$, that $H^i(\mathcal{E}|_{E_n}) = 0$ for all $i > 0$ and all $n > 0$. It follows that $\hat{\mathcal{E}}^i = 0$ for $i > 0$. Since $\mathcal{E}^i$ is a coherent sheaf with support at $q$, so $\mathcal{E}^i = 0$. \qed

**Proof of Claim.** Since the exceptional divisor $E$ is a projective space of dimension $l - 1$ where $l = \text{dim}(X)$, we have the following commutative diagram:
where $\mathcal{M}_E = \Omega_{p^{l-1}}(1)$ and $W'$ is a quotient vector space of $W$ by $H^0(\mathcal{O}_E(1))$ and $\dim(W') = r - l$.

From the exact sequence $0 \to \mathcal{M}_E \to \mathcal{M}_W|_E \to W' \otimes \mathcal{O}_E \to 0$, there is a finite filtration of $\wedge^l \mathcal{M}_W|_E$,

$$\wedge^l \mathcal{M}_W|_E = F^0 \supset F^1 \supset \cdots \supset F^l \supset F^{l+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong \wedge^p (\mathcal{M}_E) \otimes \wedge^{l-p} (W' \otimes \mathcal{O}_E)$$

for each $p$. Since $H^i(\wedge^j \Omega(k)) = 0$ unless $i \geq 1$ and $k \geq 1$ by Bott formula, we get $H^i(\wedge^l \mathcal{M}_W|_E) = 0$. For $H^i(\wedge^l \mathcal{M}_W \otimes \mathcal{O}_E(n)) = 0$, use induction on $l$. For $l = 2$, the vanishing is clear since $E = \mathbb{P}^1$ and $\wedge^l \mathcal{M}_W|_E$ is a sum of line bundles on $E$. For $l > 2$, the vanishing follows from the exact sequence:

$$0 \to \wedge^l \mathcal{M}_W \otimes \mathcal{O}_E(n) \to \wedge^l \mathcal{M}_W \otimes \mathcal{O}_E(n+1) \to \wedge^l \mathcal{M}_W \otimes \mathcal{O}_H(n+1) \to 0,$$

where $H$ is a hyperplane in $E$. $\square$

**Lemma 4.3.** $\sigma_*(\wedge^l \mathcal{M}_W \otimes \sigma^* \mathcal{L}^j (-kE))$ is globally generated for $i \geq 0$, $j \geq 1$ and $1 \leq k \leq j$.

**Proof.** Since

$$\sigma_*(\wedge^l \mathcal{M}_W \otimes \sigma^* \mathcal{L}^j (-kE)) = \sigma_*(\wedge^l \mathcal{M}_W \otimes (\sigma^* \mathcal{L} - E)^k \otimes \sigma^* \mathcal{L}^{j-k})$$

$$= \sigma_*(\wedge^l \mathcal{M}_W \otimes (\sigma^* \mathcal{L} - E)^k) \otimes \mathcal{L}^{j-k},$$

it is enough to show that $\sigma_*(\wedge^l \mathcal{M}_W \otimes (\sigma^* \mathcal{L} - E)^k)$ is globally generated.

By taking the pushforward $\sigma_*$ to the exact sequence

$$0 \to \wedge^{i+1} \mathcal{M}_W \otimes (\sigma^* \mathcal{L} - E)^{k-1} \to \wedge^{i+1} W \otimes (\sigma^* \mathcal{L} - E)^{k-1} \to \wedge^i \mathcal{M}_W \otimes (\sigma^* \mathcal{L} - E)^k \to 0$$

...
we have
\[
0 \rightarrow \sigma_*(\wedge^{i+1} M_W \otimes (\sigma^* L - E)^{k-1}) \rightarrow \sigma_*(\wedge^i M_W \otimes (\sigma^* L - E)^{k-1})
\]
\[
\rightarrow \sigma_*(\wedge^i M_W \otimes (\sigma^* L - E)^k) \rightarrow R^1 \sigma_*(\wedge^{i+1} M_W \otimes (\sigma^* L - E)^{k-1}) \rightarrow \ldots.
\]

By Lemma 4.2, \(R^1 \sigma_*(\wedge^{i+1} M_W \otimes (\sigma^* L - E)^{k-1}) = 0\) for \(k \geq 1\). Since \(L\) is very ample, \(\sigma_*(\wedge^{i+1} W \otimes (\sigma^* L - E)^{k-1}) = \wedge^{i+1} W \otimes (L \otimes m_q)^{k-1}\) is globally generated. From the following commutative diagram
\[
\begin{array}{ccc}
\wedge^{i+1} W \otimes H^0(L \otimes m_q)^{(k-1)} \otimes O_X & \longrightarrow & H^0(\wedge^i M_W \otimes (\sigma^* L - E)^k) \otimes O_X \\
\downarrow & & \downarrow \\
\wedge^{i+1} W \otimes (L \otimes m_q)^{(k-1)} & \longrightarrow & \sigma_*(\wedge^i M_W \otimes (\sigma^* L - E)^k)
\end{array}
\]
we are done. □

**Lemma 4.4.** Assume that \(L\) and \(\sigma^* L - E\) are very ample line bundles on \(X\) and \(\tilde{X} = Bl_q(X)\), respectively. Then the natural morphism
\[
H^1(\wedge^i M_W \otimes \sigma^* L^j ((-k - 1)E)) \rightarrow H^1(\wedge^i M_W \otimes \sigma^* L^j (kE))
\]
(4.1)
is injective for all \(i \geq 0, j \geq 1\) and \(1 \leq k \leq j\).

**Proof.** Let \(E = \wedge^i M_W \otimes \sigma^* L^j ((-k - 1)E)\). From the natural exact sequence \(0 \rightarrow E(-E) \rightarrow E \rightarrow E|_E \rightarrow 0\), the injectivity of morphism (4.1) is equivalent to the surjectivity of the morphism \(H^0(\tilde{X}, E) \rightarrow H^0(E|_E)\). Note again that by Lemma 4.2,
\[
R^1 \sigma_*(\wedge^i M_W \otimes \sigma^* L^j ((-k - 1)E)) = R^1 \sigma_*(\wedge^i M_W \otimes (\sigma^* L - E)^{k+1}) \otimes L^{j-k-1} = 0.
\]
Thus, we have an exact sequence
\[
0 \rightarrow \sigma_* E(-E) \rightarrow \sigma_* E \rightarrow \sigma_* (E|_E) \rightarrow 0.
\]
Since \(\sigma_*(E)\) is globally generated by Lemma 4.3, we have surjective morphisms
\[
H^0(\tilde{X}, E) \otimes O_X \rightarrow \sigma_*(E) \rightarrow \sigma_*(E|_E) = H^0(E|_E).
\]
□

**Remark 4.5.** In Lemma 4.4, it can be shown that the given morphism is not always injective for \(k = j + 1\).

**Theorem 4.6.** Let \(X \subset \mathbb{P}(H^0(L))\) be a smooth irreducible variety of \(\dim(X) \geq 2\) with property \(N_p\), \(p \geq 1\). For any \(q \in X \setminus \text{Trsec}(X)\), \(\tilde{\pi}_q(Bl_q(X)) = \tilde{\pi}_q(X \setminus \{q\})\) in \(\mathbb{P}(W)\) is smooth and satisfies property \(N_{p-1}\), i.e., property \(N_{p-1}\) holds for \((Bl_q(X), \sigma^* L - E)\).
Proof. By Proposition 4.1, we know \( \tilde{\pi}_q(\text{Bl}_q(X)) = \pi_q(X \setminus \{q\}) \) in \( \mathbb{P}(W) \) is smooth. From the restriction of the Euler sequence on \( \mathbb{P}(W) \) to \( \tilde{X} = \text{Bl}_q(X) \), we have the following diagram:

\[
\begin{array}{ccccccccc}
0 & 0 & 0 \\
\downarrow & & & \downarrow & & & \downarrow \\
0 & \mathcal{M}_W & \rightarrow & W \otimes \mathcal{O}_{\tilde{X}} & \rightarrow & \sigma^*\mathcal{L}(-E) & \rightarrow & 0 \\
\downarrow & & & \downarrow & & & \downarrow \\
0 & \sigma^*\mathcal{M}_V & \rightarrow & V \otimes \mathcal{O}_{\tilde{X}} & \rightarrow & \sigma^*\mathcal{L} & \rightarrow & 0 \\
\downarrow & & & \downarrow & & & \downarrow \\
0 & \mathcal{O}_{\tilde{X}}(-E) & \rightarrow & \mathcal{O}_{\tilde{X}} & \rightarrow & \mathcal{O}_E & \rightarrow & 0 \\
\downarrow & & & \downarrow & & & \downarrow \\
0 & 0 & 0 \\
\end{array}
\]

The above commutative diagram induces the following diagram:

\[
\begin{array}{ccccccccc}
0 & 0 & 0 \\
\downarrow & & & \downarrow & & & \downarrow \\
0 & \wedge^{i+1}\mathcal{M}_W & \rightarrow & \wedge^{i+1}W \otimes \mathcal{O}_{\tilde{X}} & \rightarrow & \wedge^i\mathcal{M}_W \otimes \sigma^*\mathcal{L}(-E) & \rightarrow & 0 \\
\downarrow & & & \downarrow & & & \downarrow \\
0 & \wedge^{i+1}\sigma^*\mathcal{M}_V & \rightarrow & \wedge^{i+1}V \otimes \mathcal{O}_{\tilde{X}} & \rightarrow & \wedge^i\sigma^*\mathcal{M}_V \otimes \sigma^*\mathcal{L} & \rightarrow & 0 & (4.2) \\
\downarrow & & & \downarrow & & & \downarrow \\
0 & \wedge^i\mathcal{M}_W \otimes \mathcal{O}_{\tilde{X}}(-E) & \rightarrow & \wedge^iW \otimes \mathcal{O}_{\tilde{X}} & \rightarrow & \text{coker}\alpha & \rightarrow & 0 \\
\downarrow & & & \downarrow & & & \downarrow \\
0 & 0 & 0 \\
\end{array}
\]

Twisting through by \( \sigma^*\mathcal{L}^{j-1} \) in diagram (4.2) and taking cohomology, one gets the following diagram:

\[
\begin{array}{ccccccccc}
\wedge^{i+1}W \otimes H^0(\sigma^*\mathcal{L}^{j-1}) & \rightarrow & H^0(\wedge^i\mathcal{M}_W \otimes \sigma^*\mathcal{L}^{j-1}(-E)) & \rightarrow & H^1(\wedge^{i+1}\mathcal{M}_W \otimes \sigma^*\mathcal{L}^{j-1}) \\
\downarrow & & & \downarrow & & & \downarrow \\
\wedge^{i+1}V \otimes H^0(\sigma^*\mathcal{L}^{j-1}) & \rightarrow & H^0(\wedge^i\sigma^*\mathcal{M}_V \otimes \sigma^*\mathcal{L}^{j}) & \rightarrow & H^1(\wedge^{i+1}\sigma^*\mathcal{M}_V \otimes \sigma^*\mathcal{L}^{j-1}) & \rightarrow & \beta_{i,j} \\
\downarrow & & & \downarrow & & \downarrow & \downarrow & \downarrow \\
\wedge^iW \otimes H^0(\sigma^*\mathcal{L}^{j-1}) & \rightarrow & H^0(\text{coker}\alpha \otimes \sigma^*\mathcal{L}^{j-1}) & \rightarrow & H^1(\wedge^i\mathcal{M}_W \otimes \sigma^*\mathcal{L}^{j-1}(-E)) & \rightarrow & \delta_{i,j} \\
\downarrow & & & \downarrow & & & \downarrow \\
0 & H^1(\wedge^i\mathcal{M}_W \otimes \sigma^*\mathcal{L}^{j-1}) & \rightarrow & H^2(\wedge^{i+1}\mathcal{M}_W \otimes \sigma^*\mathcal{L}^{j-1}) & \rightarrow & \omega_{i,j} \\
\end{array}
\]

(4.3)
Observe the following from the above diagram:

1. $\mu_{i,j}$ is always surjective.
2. By hypothesis of property $N_p$ of $(X, \mathcal{L})$ (see Lemma 2.1) and Lemma 2.2, $\alpha_{i,j}$ is surjective for $0 \leq i \leq p$ and $j \geq 2$.
3. $Bl_q(X) \subset \mathbb{P}^{r-1}$ satisfies property $N_{p-1}$ for $\sigma^*\mathcal{L}(-E)$ if $\delta_{i,j}$ is injective for all $1 \leq i \leq p$ and $j \geq 2$ by Lemma 4.4 and the following diagram:

\[
\begin{align*}
H^1(\wedge^i \mathcal{M}_W \otimes (\sigma^*\mathcal{L} - E)^{j-1}) & \longrightarrow \wedge^i W \otimes H^1((\sigma^*\mathcal{L} - E)^{j-1}) \\
& \downarrow \delta_{i,j} \\
H^1(\wedge^i \mathcal{M}_W \otimes \sigma^*\mathcal{L}^{j-1}(-E)) & \longrightarrow \wedge^i W \otimes H^1(\sigma^*\mathcal{L}^{j-1})
\end{align*}
\]

(by Corollary 2.4)

4. For $1 \leq i \leq p$ and $j \geq 2$, the following holds since $\mu_{i,j}$ and $\alpha_{i,j}$ are surjective:

\[
\delta_{i,j} \text{ is injective} \iff \gamma_{i,j} \text{ is surjective} \iff \nu_{i,j} \text{ is surjective} \iff \omega_{i,j} \text{ is injective}.
\]

5. We also have the following commutative diagram of cohomology groups:

\[
\begin{align*}
H^1(\tilde{X}, \wedge^i \mathcal{M}_W \otimes \sigma^*\mathcal{L}^j(-E)) & \xrightarrow{\delta_{i,j+1}} \wedge^i W \otimes H^1(\tilde{X}, \sigma^*\mathcal{L}^j) \\
& \downarrow \omega_{i,j} \quad \downarrow \rho_{i,j} \\
H^1(\tilde{X}, \wedge^i \mathcal{M}_V \otimes \sigma^*\mathcal{L}^j) & \xrightarrow{\beta_{i-1,j+1}} \wedge^i V \otimes H^1(\tilde{X}, \sigma^*\mathcal{L}^j)
\end{align*}
\]

From this diagram, observe the following:

- $\rho_{i,j}$ is always injective.
- $\beta_{i-1,j+1}$ is injective for $1 \leq i \leq p+1$ and $j \geq 1$ by property $N_p$ of $(X, \mathcal{L})$.

Thus for $1 \leq i \leq p+1$ and $j \geq 1$,

$\delta_{i,j+1}$ is injective if and only if $\omega_{i,j}$ is injective.

Note that $\delta_{i,j+1}$ is injective for a sufficient large $j$ since $H^1(\tilde{X}, \wedge^i \mathcal{M}_W \otimes \sigma^*\mathcal{L}^j(-E)) = H^1(X, \sigma_*(\wedge^i \mathcal{M}_W \otimes \mathcal{O}_{\tilde{X}}(-E)) \otimes \mathcal{L}^j) = 0$, $j \gg 0$. Hence the following holds for $1 \leq i \leq p$:
\( \delta_{i,j+1} \) is injective for all \( j \gg 0 \) \( \implies \) \( \omega_{i,j} \) and \( \delta_{i,j} \) are injective
\[ \implies \omega_{i,j-1} \text{ and } \delta_{i,j-1} \text{ are injective} \]
\[ \vdots \]
\[ \implies \omega_{i,2} \text{ and } \delta_{i,2} \text{ are injective}. \]

Therefore it is proved that \( \delta_{i,j} \) is injective for all \( 1 \leq i \leq p \) and \( j \geq 2 \). \( \square \)

Let \( X \) be a smooth projective variety of dimension \( n \) and \( \mathcal{L} \) a very ample line bundle on \( X \). Let \( \sigma : \tilde{X} \rightarrow X \) be the blowing-up at distinct \( s \) points \( q_1, q_2, \ldots, q_s \) in \( X \) with the exceptional divisors \( E_1, \ldots, E_s \). Suppose \( \mathcal{L}' := \sigma^* \mathcal{L}(-E_1 - \cdots - E_s) \) on \( \tilde{X} \) is very ample. Then we have the following corollary from Theorem 4.6 immediately.

**Corollary 4.7.** If \( \mathcal{L} \) satisfies \( N_p \) for \( p \geq 1 \), then \( \mathcal{L}' \) satisfies \( N_{p-s} \) for \( p \geq s \). In particular, \( \sigma^* \mathcal{L}(-E_1 - \cdots - E_p) \) is very ample and normally generated.

We also get the generalization of Corollary 3.4 with extra conditions.

**Corollary 4.8.** Let \( X \subset \mathbb{P}^N \) be a smooth linearly normal variety satisfying \( H^i(X, \mathcal{O}(j)) = 0 \) for all \( 1 \leq i < \dim X \) and \( j \geq 0 \). If \( X \) has a \((p + 2)\)-secant \( p \)-plane \( \Lambda \) such that \( X \cap \Lambda \) is a finite curvilinear scheme, then property \( N_p \) fails for \( X \).

**Proof.** From the given cohomological and curvilinear conditions, we can possibly get the linearly normal smooth curve section containing \((p + 2)\)-secant \( p \)-plane \( \Lambda \) and its graded Betti numbers are same as those of \( X \) by Green theorem [11, 3.b.7]. Thus we are done by Corollary 3.4. \( \square \)

**Corollary 4.9.** Let \( X \subset \mathbb{P}(W) \) be a smooth variety with property \( N^S_p \) and \( t = \text{codim}(W, H^0(\mathcal{L})) \). Then, for any \( q \in X \setminus \text{Trisec}(X) \) we have:

1. An inner projection \( \pi_q(X \setminus \{q\}) \subset \mathbb{P}(W') \) satisfies property \( N^S_{p-1} \).
2. The defining equations of \( \pi_q(X \setminus \{q\}) \) in \( \mathbb{P}(W') \) have degree at most \( t + 2 \) for \( p \geq 2 \).

**Proof.** We can prove this by the exactly same methods as those shown in Theorem 4.6 except taking subsystems of the complete linear system of global sections. For a proof of (2), note that \( t = \text{codim}(W, H^0(\mathcal{L})) = \text{codim}(W', H^0(\sigma^* \mathcal{L} - E)) \). Thus, it follows from Theorem 1.1 in [13] because an inner projection of \( X \) satisfies \( N^S_1 \) property for \( p \geq 2 \). \( \square \)

**Remark 4.10.** We would like to mention that Corollary 4.8 can be proved with no extra cohomological conditions due to Eisenbud, Green, Hulek and Popescu [6, Theorem 1.1].

5. Applications to some examples and questions

The main theorems can be applied to the following examples.
Veronese embedding of \( \mathbb{P}^n \). Let \( v_{2,d} : \mathbb{P}^2 \hookrightarrow \mathbb{P}^r, r = \left( \frac{d+2}{2} \right) - 1 \), be the Veronese embedding associated to the complete linear system \( |O_{\mathbb{P}^2}(d)| \). The image of the embedding \( v_{2,d} \) with \( d \geq 3 \) satisfies property \( N_p \) if and only if \( p \leq 3d - 3 \) (see [17]). Let \( q_1, \ldots, q_m \) be distinct \( m \) points of \( \mathbb{P}^2 \) for \( m \leq 3d - 3 \), and let \( \tilde{X} \) be the blowing-up at \( q_1, \ldots, q_m \) with the exceptional divisors \( E_1, \ldots, E_m \). Noma proved in [16] that the line bundle \( \mathcal{L} := \sigma^*O_{\mathbb{P}^2}(d) \otimes \mathcal{O}_{\tilde{X}}(-E_1 - \cdots - E_m) \) is very ample if and only if, for all \( l \) with \( 1 \leq l \leq m \), any \( l \)-points of \( \{q_1, \ldots, q_m\} \) do not lie on any rational normal curve of degree \( l \) on \( v(\mathbb{P}^2) \subset \mathbb{P}^r \). From Corollary 4.7, \( \mathcal{L} \) satisfies property \( N_{3d-3-m} \) if it is very ample.

For arbitrary \( n \), let \( v_{n,d} : \mathbb{P}^n \hookrightarrow \mathbb{P}^r, r = \left( \frac{d+n}{n} \right) - 1 \), be the Veronese embedding associated to the complete linear system \( |O_{\mathbb{P}^n}(d)| \). It is well known that \( v_{n,d} \) with \( d \geq 2 \) satisfies \( N_p \) for \( p \leq d \) and the line bundle \( \mathcal{L} := \sigma^*O_{\mathbb{P}^n}(d) \otimes \mathcal{O}_{\tilde{X}}(-E_1 - \cdots - E_d) \) is very ample [3, Theorem 1], where \( \tilde{X} \) is the blowing-up at general \( d \) points \( x_1, \ldots, x_d \) with the exceptional divisors \( E_1, \ldots, E_d \). Thus, Corollary 4.7 implies that \( \mathcal{L} \) is normally generated.

On the other hand, it can be shown by using Corollary 4.8 that the image \( v_{n,d}(\mathbb{P}^n) \) fails to hold property \( N_{3d-2} \). This was first proved by Ottaviani and Paoletti [17], and then by Eisenbud, Green, Hulek and Popescu [6, Proposition 3.2].

**Proposition 5.1.** The image \( v_{n,d}(\mathbb{P}^n) \) for \( d \geq 3 \) fails to hold property \( N_{3d-2} \).

**Proof.** Since the image \( v_{n,d}(\mathbb{P}^n) \) is projectively Cohen–Macaulay, the hypotheses of Corollary 4.8 are satisfied. We can also show that the image \( v_{n,d}(\mathbb{P}^n) \) has \( 3d \)-secant \((3d-2)\)-plane (see [6, Proposition 3.2] for details). Therefore, \( v_{n,d}(\mathbb{P}^n) \) for \( d \geq 3 \) fails to hold property \( N_{3d-2} \). □

**Calabi–Yau manifolds.** Let \( X \) be an \( n \)-dimensional smooth projective variety with \( K_X = \mathcal{O}_X \) and let \( \mathcal{L} \) be a very ample line bundle on \( X \). Then \( \mathcal{L}^{\otimes n+p+1} \) satisfies to hold \( N_p \) (see [7]). Assume that \( p \geq 2n \). In [3], for general \( p \) points \( x_1, \ldots, x_p \) with \( p \leq N - 2n - 1 \) where \( N = \dim H^0(X, \mathcal{L}^{\otimes n+p+1}) \), \( \mathcal{L}' := \sigma^*\mathcal{L}^{n+p+1}(-E_1 - \cdots - E_p) \) is very ample. So, we conclude that \( \mathcal{L}' \) is normally generated.

**Rational surfaces.** Gallego and Purnaprajna give a nice sufficient condition for a line bundle \( \mathcal{L} \) on a rational surface \( \tilde{X} \) to hold property \( N_p \).

**Theorem 5.2.** [10] Let \( X \) be a rational surface, and let \( \mathcal{L} \) be an ample line bundle on \( X \). If \( \mathcal{L} \) is base-point-free and \( -K_X \cdot \mathcal{L} \geq p + 3 \), then \( \mathcal{L} \) satisfies property \( N_p \).

If \(-K_X \cdot \mathcal{L} \geq p + 3\), then \(-K_{\tilde{X}} \cdot (\sigma^*\mathcal{L} - E) \geq p + 2\). So, this theorem is consistent with Theorem 4.6. Note also that Del Pezzo surfaces as the blow-ups of \( \mathbb{P}^2 \) are rational surfaces.

**Remark 5.3.** Many works have been done to show that some very ample line bundles on smooth projective varieties satisfy property \( N_p \). For example:

- (Green [11]) Let \( X \) be a smooth projective curve of genus \( g \) and let \( \mathcal{L} \) be a line bundle on \( X \) of degree \( d \). If \( d \geq 2g + 1 + p \), then \((X, \mathcal{L})\) satisfies property \( N_p \).
- (Butler [2], Park [18]) Let \( C \) be a smooth projective curve of genus \( g \). For a vector bundle \( \mathcal{E} \) of rank \( n \) over \( C \), let \( X = \mathbb{P}(\mathcal{E}) \) be the associated projective bundle with tautological line bundle \( \mathcal{H} \) with projection map \( \pi : X \to C \). For a line bundle \( \mathcal{L} = aH + \pi^*B \) on \( X \) with \( a \geq 1 \) and \( B \in \text{Pic} C \), assume that \( \mu^-(\pi_*\mathcal{L}) \geq 2g + 2p \) for some \( 1 \leq p \leq a \). Then Butler
proved that \((X, \mathcal{L})\) satisfies property \(N_p\). For the cases of \(a = 1, n = 2\) and \(n = 3, a = 2\), Park extended Butler’s result to show property \(N_p\) for \(\mu^-(\pi_* \mathcal{L}) \geq 2g + 2p\) with no assumption \(1 \leq p \leq a\).

- (Ein–Lazarsfeld [7]) Let \(X\) be a smooth complex projective variety of dimension \(n\) with the canonical sheaf \(K_X\). For \(A, B \in \text{Pic} X\), assume that \(A\) is very ample and \(B\) is numerically effective. Then, for \(p \geq 0\), \(K_X + (n + p)A + B\) satisfies property \(N_p\) except the case \((X, A, B) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n})\) and \(p = 0\).

Note in these examples that \(H^i(X, \mathcal{O}(j)) = 0\) for all \(1 \leq i < \dim X\) and \(j \geq 0\) if \(\dim(X) \geq 2\). On the other hand, it has been very important to show very ampleness of the form of line bundles \(\sigma^* \mathcal{L}(-m_1 E_1 - \cdots - m_s E_s)\). However, we are interested in the syzygies of the embedding defined by the line bundle \(\sigma^* \mathcal{L}(-m_1 E_1 - \cdots - m_s E_s)\).

Finally, it seems to us that the following questions related to our paper are interesting. Note that if the line bundle \(\sigma^* \mathcal{L}(-2E)\) is very ample then the embedding of \(\tilde{X}\) by \(\sigma^* \mathcal{L}(-2E)\) is the Zariski closure of the tangential projection of \(X\) from the projective tangent space of \(X\) at \(q\). It would be very interesting to get any structural results on these linear systems with multiple base points.

- **Question 1.** What can we say about syzygies of the line bundle \(\sigma^* \mathcal{L}(-m_1 E_1 - \cdots - m_s E_s)\), i.e., of the linear system of multiple base points if it is very ample?

Another question is about the converse of Theorems 3.1 and 4.6.

- **Question 2.** We ask whether the converse of Theorems 3.1 and 4.6 hold or not. More specifically, assume that \(\mathcal{L}\) is a very ample line bundle on a projective variety \(X\). For an embedded projective variety \(X \subset \mathbb{P}(H^0(\mathcal{L})) \simeq \mathbb{P}^r\) by the complete linear system of \(\mathcal{L}\), property \(N_p\) holds for \(X\) if and only if property \(N_{p-1}\) holds for the projection \(\phi_q(X)\) of \(X\) for any point \(q \notin \text{Trisec}(X)\).

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**References**