Assessment of material uncertainties in solid foams based on local homogenization procedures

Carla Beckmann, Jörg Hohe *

Fraunhofer-Institut für Werkstoffmechanik IWM, Wöhlerstr. 11, 79108 Freiburg, Germany

Abstract

The present study is concerned with a numerical prediction and assessment of uncertainties in the macroscopic material properties of solid foams. The material properties are determined by means of a homogenization analysis considering a large scale representative volume element. The microstructure for the representative volume element is determined randomly using a Voronoï tesselation in Laguerre geometry with prescribed cell size distribution. For assessment of the scatter in the effective material response, the homogenization scheme is applied to subsets of the large scale representative volume element. By this means, an interrelation between the local microstructural characteristics and the local mesoscopic material response is established. In a first approach, the individual cells of the foam microstructure are employed as testing volume elements. As an alternative, a moving window technique is applied. The sets of homogenization results obtained by both approaches are evaluated by stochastic methods. For the local effective properties, a distinct scatter is observed. The results in both cases reveal that the local porosity is the most important influence parameter. For the microstructures investigated, only weak local correlations of the effective stiffnesses with a rapid spatial decay of the correlation is observed.

1. Introduction

In the recent time, solid foams gain increasing importance as construction materials in many fields in modern lightweight construction and other technological fields. Solid foams can be processed from a variety of base materials including polymeric, metallic and even ceramic materials. Their main advantage in lightweight construction is their rather low specific weight due to their high void volume fraction. For energy absorption systems, their high compressibility at approximately constant resulting stresses is utilized. Furthermore, solid foams are employed in thermal and acoustic insulation or heat exchanging applications. Compared to regular cellular structures such as honeycomb cores in sandwich construction, solid foams have the benefit that they can easily be processed to any desired shape.

The main disadvantage of solid foams in structural application is their random, disordered microstructure. The irregular microstructure may result in a distinct scatter and unpredictability of the macroscopic or "effective" material properties. Especially for metallic and ceramic foams with large cell sizes, the cell size as the characteristic microstructural length might be close to the order of magnitude of the smallest characteristic macroscopic length.

Examples are the core thickness in sandwich construction or the diameter of hollow columns in automotive body structures, which might be filled with solid foams for an enhancement of their stiffness or crash performance. In these cases, the numerical or experimental assessment of the structural performance requires the knowledge of the scatter to be expected in the effective material response of the foamed materials.

In the industrial design process, the assessment of the structural response nowadays is mostly performed by numerical methods. In this context, materials with distinct microstructure such as solid foams are preferably treated in terms of effective properties rather than using direct models of the respective microstructure, for reasons of numerical efficiency. The macroscopic, effective properties can either be determined experimentally or be predicted numerically in homogenization analyses. The latter approach is especially useful in the design and optimization of microstructured materials in order to fit with any kind of prescribed requirements. For application of algorithmic optimization procedures, numerical approaches are essential.

The homogenization analysis of foamed and other cellular microstructure has been subject of numerous studies, since the pioneering work of Gent and Thomas (1963) appeared. Most of the early studies are based on idealized periodic foam models such as Thomson (1887) tetrakaidecahedral cell model, Christensen (1987) dodecahedron or the cubic model employed by Gibson...
and Ashby (1997) for parametric studies. Among these repetitive single cell models, the Kelvin foam is the model involving the lowest surface-to-volume ratio, although it is outperformed by Weaire and Phelan (1994) multiple cell model. Christensen's model is the only simple polyhedral single cell model satisfying the fivefold symmetry condition for isotropy. Using these foam models, a variety of deterministic studies on the effective material properties of solid foams have been provided, among which only the early analytical studies by Dement'ev and Tarakanov (1970), Patel and Finnie (1970) as well as by Warren and Kraynik (1988) should be mentioned here.

Analyses based on idealized periodic foam models in general yield good approximations of the mean effective properties. Nevertheless, the scatter and the uncertainty of the effective material response cannot be accessed. Experimental studies on metallic foams by Ramamurty and Paul (2004) or Despois et al. (2006) reveal significant uncertainties in the effective properties of disordered large-cell foams. For the effective properties, total scatter bands widths of more than half of the respective mean value may occur. Although the scatter decreases with increasing specimen sizes due to self-averaging effects, numerical studies by Kanaun and Tkachenko (2006), van der Burg et al. (1997) and Zhu et al. (2000) indicate that sample sizes in the range of several hundreds of cells might be required in order to obtain stable results.

To account for the uncertainty in the microstructural geometry and topology and thus the uncertainty in the effective material response, different probabilistic approaches have been proposed in the literature. Three basic groups of approaches can be identified. Among others, Cuitiño and Zheng (2003) as well as Warren and Kraynik (1997) deal with uncertainties in the cell orientation of idealized single cell models using averaging techniques to eliminate artificial anisotropies. In a similar manner, (Fortes and Ashby, 1999) consider a single strut of an open cell foam together with a probability distribution for its spatial orientation. The second group of approaches uses large scale computational models for the random cellular microstructure, usually generated by a Voronoi (1908) process in order to account not only for random cell orientations but also for cell shape and size variations (e.g. van der Burg et al., 1997). The third group of approaches is also based on the repeated analysis of medium scale “testing volume elements” generated either by a Voronoi (1908) process or a perturbation of a regular cellular microstructure. In this context, either direct Monte–Carlo analyses or analyses based on pre-defined sets of the random variables forming a discretization of their space have been used. Examples are the contributions by Gan et al. (2005), Schraad and Harlow (2006), Zhu et al. (2000) or the present authors Hohe and Beckmann (2011), Hohe and Beckmann (2012). The results of the testing volume element analyses are evaluated by stochastic methods.

In the present study, an alternative route is employed to obtain the stochastic information. In a first step, a large scale, statistically representative volume element for the microstructure is considered. The computational model for the microstructure is generated randomly using a Voronoi (1908) process in Laguerre geometry (Fan et al., 2004; Fazekas et al., 2002). The microstructural response of the representative volume element under a number of reference strain states is analyzed numerically by means of the finite element method. Subsequently, the entire representative volume element is divided into substructures. The local stress–strain response of the substructures is determined by applying the homogenization equations to these subsets of the representative volume element. The smallest feasible microstructure related substructures are the individual cells. Hence, they are employed as testing volume elements. The homogenization results obtained thereon are evaluated by means of stochastic methods. This approach generalises a previous study on scatter effects in two-dimensional model foams (Harden-acke and Hohe, 2009) to three dimensional cellular solids. Recently, (Mangipudi and Onck, 2011) have used a similar procedure for determination of local stresses and strains in strain localization zones during the failure of two-dimensional model foams.

As an alternative to the microstructure related definition of the testing volume elements, a space related “moving window” technique is employed. This technique has the advantage, that all testing volume elements are of a uniform size and shape with a unique spacing throughout the microstructure. Both approaches are applied to an analysis of the scatter in the elastic stiffness components of a closed cell foam, using an aluminum foam as an example. The results show a distinct scatter of the results on the lowest microstructural level, which decreases with increasing size of the testing volume elements. The effective properties feature a strong correlation with the local relative density. In consistence with previous studies, the local relative density proves to be the most important microstructural variable causing scatter in the effective material properties. The cell size distribution has an indirect effect on the uncertainty of the effective properties since on the level of the individual cells, the local relative density is usually correlated with the local cell size. Compared to the multiple analysis of medium scale testing volume elements, the local homogenization of substructures of a large scale, statistically representative volume element has the advantage that all possible interactions between neighboring cells are included in a natural manner. Furthermore, this type of approach enables the analysis of spatial correlations between the effective properties of neighboring testing volume elements and their spatial decay which is not accessible in the repeated analysis of medium scale volume elements.

2. Numerical analysis

2.1. Homogenization

Consider a body Ω consisting of a cellular material (Fig. 1). Let Ω be bounded by an external boundary ∂Ω = ∂Ωf ∪ ∂Ωd with prescribed displacements u|∂Ωf = u| and prescribed tractions t|∂Ωd = t| on ∂Ωd. In addition, the body Ω may be loaded by distributed prescribed body forces F. For reasons of numerical efficiency in the analysis of its structural response, the body Ω has to be substituted with a similar body Ω′ with similar external boundaries ∂Ω′f and ∂Ω′d and similar prescribed boundary conditions u|′ = u|, t|′ = t| applied on the external boundaries and inside of Ω′. In contrast to Ω, the substitute body Ω′ consists of a quasi homogeneous, “effective” medium with yet unknown properties.

For the determination of the properties of the effective medium, a representative volume element ΩRVE for the given microstructure of Ω and a similar volume element ΩRVE consisting of the effective medium are considered. The effective material properties are determined such that the mechanical response of both volume elements, ΩRVE and ΩRVE′, is equivalent on the intermediate mesoscopic level of structural hierarchy. For definition of the mesoscopic equivalence, different approaches have been proposed in the literature. Within the present study, a simple volume average definition is used, requiring that the volume averages

\[ F_\gamma = \frac{1}{V_{\text{RVE}}} \int_{V_{\text{RVE}}} F_\gamma \, dV = \frac{1}{V_{\text{RVE}}} \int_{\partial V_{\text{RVE}}} F_\gamma \, dV = \bar{F}_\gamma \]

\[ P_\gamma = \frac{1}{V_{\text{RVE}}} \int_{V_{\text{RVE}}} P_\gamma \, dV = \frac{1}{V_{\text{RVE}}} \int_{\partial V_{\text{RVE}}} P_\gamma \, dV = \bar{P}_\gamma \]

of the components \( F_\gamma = u_\gamma + \delta_\gamma \) of the deformation gradient and the components \( P_\gamma \) of the first Piola–Kirchhoff stress tensor have to be equal. Hence, the effective material law is determined by deforming the representative volume element ΩRVE′ by a number of indepen-
dent reference deformation states and computing the corresponding local stresses. Subsequently, the corresponding effective deformations $F_0$ and effective stresses $P_0$ according to Eqs. (1) and (2). From the interrelations of the effective deformation and stress components $F_0$ and $P_0$ for the effective medium, the properties of the effective medium can be derived. As an alternative to the effective deformation and first Piola–Kirchhoff stress components $F_0$ and $P_0$, any other pair of strain and stress measures derived from them can be used to describe the material response of the body. In the case of an intersection of the surrounding sphere for a randomly positioned nucleus is copied to the neighborhood of the representative volume element has to be sufficiently large in order to account for all possible microstructural features such as cells with different sizes and shapes and their possible interactions in a statistically representative manner. On the other hand, the size of the representative volume element has to be much smaller than the smallest characteristic length $l$ of the macroscopic body $\Omega$ or the characteristic length, over which significant stress or strain variations inside $\Omega$ might develop, whichever length is smaller. Hence, the condition

$$L \gg l \gg d$$

for the size of the representative volume element has to be satisfied for an accurate determination of effective material properties (Fig. 1).

### 2.2. Computational foam model

For the homogenization analysis of solid foams, adequate computational models for the microstructure are required. Within the present study, the geometry and topology of the microstructure are generated randomly using a Voronoi (1908) process in Laguerre geometry (Fan et al., 2004; Fazekas et al., 2002) in a periodic enhancement. For this purpose, a brick shaped representative volume element $\Omega^{RVE}$ with dimensions $l$, with respect to the coordinate axes $x_i$ is considered. Within the representative volume element, $n$ randomly positioned nuclei $p$ are defined at the (random) locations $x_i^{nuc(p)}$. Each nucleus is supplied with a surrounding sphere with a prescribed radius $r^{(p)}$ which is not permitted to intersect with the corresponding spheres belonging to any other nuclei $q$ at $x_i^{nuc(q)}$ with $q \neq p$. The radii $r^{nuc(p)}$ are defined such that the sphere volumes obey a prescribed probability distribution $F(V^{nuc})$ of their volumes $V_{x_i^{nuc(p)}}$. In order to obtain a periodic computational model for the cellular microstructure, each freshly defined nucleus is copied to the neighborhood of the representative volume element at a spatial position shifted by $\pm l$ (or zero) so that a total of $27n$ nuclei is defined within $\Omega^{RVE}$ and its direct neighborhood. In the case of an intersection of the surrounding sphere for a freshly defined nucleus at $x_i^{nuc(p)}$ with the spheres for previously defined nuclei $q$ during their successive generation, the respective nucleus is deleted and re-defined at an alternative random position. The process is repeated until all $n$ nuclei within $\Omega^{RVE}$ together with the corresponding $26n$ nuclei in its direct neighborhood are defined in a non-conflicting manner.

The Voronoi cells $\Omega^{(p)}$ belonging to the individual nuclei $p$ located at the spatial point $X^{(p)}$ are defined as the sets of spatial points $x_i$ which are closer to the respective nucleus $p$ than to all other nuclei $q$. Hence,

$$\Omega^{(p)} = \{ x_i | x_i \in \mathbb{R}^3, r_i^{(p)}(x_i, X^{(p)}) \leq r_i^{(p)}(x_i, X^{(q)}), q \neq p \}$$

where the distance $r_i^{(p)}(x_i, X^{(p)})$ in Laguerre geometry is related to the Euclidean distance $r_i^{(p)}(x_i, X^{(p)})$ by

$$r_i^{(p)}(X^{(p)}) = \left( (r_i^{(p)}(X^{(p)}))^2 - (r_i^{(p)})^2 \right)^{1/2}$$

(Fig. 2). In previous analyses by Fan et al. (2004) and Fazekas et al. (2002) as well as by the present authors (Hardenacke and Hohe, 2010; Hohe and Beckmann, 2011), the Voronoi process in Laguerre geometry defined by Eqs. (4) and (5) has been found to provide reliable and realistic computational foam models. Especially, it outperforms the standard ($F$-) Voronoi process obtained from Eqs. (4) and (5) by reducing the sphere radii $r^{(p)}$ to zero as well as the $\delta$-Voronoi process in terms of the surface-to-volume ratio of the individual cells.
For the finite element analyses of the response of the microstructure, the cell walls of the computational foam models are meshed with three-node tetrahedral finite shell elements. A standard, displacement based formulation with six degrees of freedom per node is employed. A geometrically nonlinear formulation is employed in order to account for large deflection and finite membrane strains in the case of local microstructural instabilities under compressive loads. For simplicity, cell walls intersecting with the external surfaces of the representative volume element $\Omega_{\text{RVE}}$ are modelled entirely on either the positive or the negative external surface of $\Omega_{\text{RVE}}$. Hence, non-plane (although piecewise parallel) external surfaces of $\Omega_{\text{RVE}}$ are obtained.

Along the cell wall intersection lines, the use of shell elements implies an inconsistency, since the adjacent cell walls and thus the corresponding shell elements intersect at an angle $\phi \neq 180^\circ$. Hence, the shells describing the adjacent cell walls overlap. On the other hand, the overlapping ranges in general will be small as long as low density foams are considered, where the cell wall thickness is much smaller than the characteristic dimensions of the individual cells. Hence, the modelling error provided by the partial overlapping of the shell volumes out of the shell’s reference surfaces can be neglected without considerable errors.

The analyses are performed under displacement controlled boundary conditions. In this context, an interrelation between the displacements $u_i$ of the displacements on the external surfaces of the representative volume element and the components $F_{ij}$ of the effective deformation gradient can easily be derived by substituting the components $F_{ij}$ of the local deformation gradient for $\Omega_{\text{RVE}}$ in Eq. (1) with their definition in terms of the displacement gradients and the volume integral into a surface integral using Green’s theorem. Hence,

$$
\frac{1}{\Omega_{\text{RVE}}} \int_{\partial \Omega_{\text{RVE}}} u_i n_j \, dA = F_{ij} - \delta_{ij} \tag{6}
$$

where $n_i$ are the components of the outward normal unit vector on the external surfaces $\partial \Omega_{\text{RVE}}$ of the representative volume element. Considering homogeneously distributed reference deformation cases $F_{ij}$ on the mesoscopic level, periodic boundary conditions are applied to corresponding surfaces $\partial \Omega_{\text{RVE}}$ and $\partial \Omega_{\text{RVE}}$ on opposite sides of the representative volume element. Together with the bricklike shape of the representative volume element, the boundary conditions

$$
\frac{u_i^{(+)} - u_i^{(-)}}{l_j} = F_{ij} - \delta_{ij} \tag{7}
$$

are obtained. The boundary conditions (7) are applied between the displacement components $u_i^{(+)}$ and $u_i^{(-)}$ of all pairs of corresponding nodes on the external surfaces of $\Omega_{\text{RVE}}$. In addition, six further boundary conditions have to be applied to constrain the translational and rotational rigid body motions of the representative volume element. Due to the restraints on the rotational degrees of freedom, the skew part of the deformation gradient $F_{ij}$ cannot be considered. For the investigations of the present study, this restriction does not imply any loss in generality, since in the subsequent analyses, only the symmetric parts of the deformation gradient as objective measures of strain are required in the considered reference deformation states. Nevertheless, if necessary, an extension of the consideration to the skew part of the effective deformation gradient by skipping the rotation restraints is straightforward.

Examples for finite element models of the microstructure with different cell numbers $n$ are presented in Fig. 3. Notice that although the isolated cell walls in the model with 64 cells have no visible connection with the remainder of the microstructure, they are properly connected with the main model through the periodic boundary conditions (7).

### 2.3. Local homogenization

#### 2.3.1. Probabilistic analysis

The determination of the effective material response based on the procedure described in Sections 2.1 and 2.2 requires that the condition (3) for the size of the representative volume element is satisfied. Results provided by Kanaun and Tkachenko (2006), van der Burg et al. (1997) and Zhu et al. (2000) indicate, that the necessary number of cells in $\Omega_{\text{RVE}}$ may be in the range of 100 and beyond in order to account for all possible microstructural effects and thus to ensure that $\Omega_{\text{RVE}}$ is statistically representative. On the other hand, especially for metallic foams, characteristic cell sizes can be in the range of several millimeters. For closed-cell aluminum foams, e.g. Kolluri et al. (2007) report an average cell diameter of 3.8 mm. Thus, edge lengths $l_i$ in the range of 15 to 18 mm are required for a cubic representative volume element containing between 100 and 200 individual cells. This size is in the same order of magnitude of (or at least not significantly below) the smallest characteristic length for many technical applications such as sandwich core or hollow beam thicknesses. Hence, the condition (3) is satisfied for this case and thus, no well defined representative volume element in terms of this condition exists. Although, representative volume elements $\Omega_{\text{RVE}}$ for the respective microstructure satisfying the minimum size requirement might be identified, the effective material properties determined on their analysis are related to mesoscopic length scales beyond the respective minimum lengths of the macroscopic structure. Thus, the self-averaging effect for the material cannot be exploited and the variability and uncertainty in the properties for a quasi homogeneous effective medium have to be considered even on the macroscale (Fig. 1).

For their numerical characterization, probabilistic homogenization schemes are required.

For determination of the scatter to be expected in the properties of the substitute, effective medium most available studies in literature employ small scale, statistically non representative “testing volume elements” $\Omega_{\text{RVE}}$, satisfying the size condition $L \gg l$ at least in an approximate manner. By generation of a population of $m$ testing volume elements $\Omega_{\text{RVE}}$ with different random microstructures and subsequent homogenization, a statistically representative set of effective property data is obtained, which enables a stochastic assessment of the effective material response either in terms of the basic stochastic parameters such as the expectation value, variance and standard deviation

$$
\mathcal{E}(y') = \sum_{k=1}^{m} y_k \cdot p(y'_k) \tag{8}
$$

$$
\mathcal{V}(y') = \sum_{k=1}^{m} (\mathcal{E}(y') - y'_k)^2 \cdot p(y'_k) \tag{9}
$$

$$
\mathcal{S}(y') = (\mathcal{V}(y'))^{1/2} \tag{10}
$$

where $p(y'_k)$ is the individual probability for occurrence of the microstructure of the respective testing volume element $\Omega_{\text{RVE}}$ and thus for the occurrence of the effective property $y'$ based thereon. Alternatively, the probability and probability density distributions

$$
\mathcal{F}(y'_{(k)}) = \sum_{i=1}^{k} p(y'_i) \tag{11}
$$

$$
f(y'_{(k)}) = \frac{\partial \mathcal{F}(y')}{\partial y'} \bigg|_{y' = y'_{(k)}} \tag{12}
$$

can be considered directly, defining the accumulated probability that the random effective property $y'$ has a value of $y'_i$ or less and the corresponding probability density respectively.
2.3.2. Microstructure related homogenization

The repeated analysis of independent small scale testing volume elements involves the problem that the underlying microstructures may not account for all possible interaction effects, especially rare ones. Furthermore, this method is not capable for an analysis of the correlation between the effective properties at neighboring spatial positions in the random field formed by the random effective property $C_{22}^{y}$, $C_{33}^{y}$. Therefore, a different approach is proposed in the present study. Instead of a set of small scale non-representative testing volume elements a large scale, statistically representative volume element $X_{RVE}$ is considered, satisfying the minimum size requirement for being statistically representative for the investigated cellular material. In order to obtain the varying material properties $p(y)$ on a lower length scale, the representative volume element is subdivided into sub-structures. For cellular solids, the smallest feasible sub-structures are the individual cells. Hence, they are employed as testing volume elements $X_{TVE}^{(k)}$ in the random homogenization analysis (Fig. 4).

Using this concept, the random effective stress and strain components for an applied macroscopic deformation state are determined by application of the homogenization Eqs. (1) and (2) to the testing volume elements $\Omega_{RVE}^{(k)}$. Hence, the components $\mathbf{F}_{ij}^{y}$ of the effective deformation gradient are obtained by

$$\mathbf{F}_{ij}^{y}(\Omega_{RVE}^{(k)}) = \frac{1}{V_{RVE}^{(k)}} \int_{\partial \Omega_{RVE}^{(k)}} u_{ij} n_{i} dA + \delta_{ij}$$

(13)

considering the cell wall midplanes as the boundaries $\partial \Omega_{RVE}^{(k)}$ of the testing volume elements. Considering the finite element discretization of the cell walls, Eq. (13) can be transformed to

$$\mathbf{F}_{ij}^{y}(\Omega_{RVE}^{(k)}) = \frac{1}{V_{RVE}^{(k)}} \sum_{e=1}^{e_{RTVE}} \mathbf{X}_{eff(e)}^{(k)}$$

(14)

where

$$\mathbf{X}_{eff(e)}^{(k)} = \int_{\partial \mathbf{W}_{TVE(e)}} u_{ij} n_{i} dA$$

(15)

is the contribution of the individual elements. When using linear, displacement based triangular finite elements as in the present study, Eq. (15) can be rewritten as

$$\mathbf{X}_{eff(e)}^{(k)} = \frac{1}{3} \left( u_{ij}^{(1)} + u_{ij}^{(2)} + u_{ij}^{(3)} \right) n_{j} A_{elm}^{(e)}$$

(16)
where $u_i^{(e)}$ are the nodal displacements at the three nodes of the element $e$ whereas $A_i^{(em)(e)}$ denotes its area (Fig. 4). In a similar manner, the effective stresses $P_i^{E}$ for the testing volume element $TVE_{(k)}$ are determined by

$$P_i^{E}(TVE_{(k)}) = \frac{1}{V_{TVE}} \int_{\partial V_{TVE}} ho_i^{(e)} dA$$

(17)

where

$$\rho_i^{(e)} = \int_{\partial V_{TVE}} P_i dA$$

(18)

considering that the internal void volume of the cells is stress free. Notice that the integral in Eq. (18) is taken only for the lower half of the cell wall located within the testing volume element $TVE_{(k)}$ whereas the upper half of the cell wall belongs to the neighboring cell. For the employed constant strain triangular elements, Eq. (18) can be simplified to

$$\rho_i^{(e)} = A_i^{(em)(e)} \int_{\partial V_{TVE}} P_i dA$$

(19)

where the remaining integral with respect to the local thickness direction $x_i^{(e)}$ is evaluated numerically using the stress results at the integration points located through the thickness of the finite element.

Using Eqs. (14) to (19), the effective deformation and stress components $F_i$ and $P_i^E$ are easily obtained from the nodal displacements of the cell wall finite elements and the local stresses – transformed into the global coordinate system – at their integration points. Since the testing volume elements $TVE_{(k)}$ are embedded into a statistically representative foam model, the homogenization results account for all possible microstructural features – even rare ones – in a natural manner. The correlation between the effective properties at neighboring spatial positions in the random effective medium are accessible without limitations.

2.3.3. Moving window approach

When using the microstructure related homogenization procedure proposed in Section 2.3.2, care has to be taken in the stochastic assessment since the volumes $V_{TVE}$ of the individual testing volume elements $TVE_{(k)}$ are not equal. Hence, the individual probabilities $p_i^{(y)}$ for occurrence of the homogenization result $y$ in Eqs. (8) to (12) is not uniform. This problem can be avoided by a definition of testing volume elements $TVE_{(k)}$ as fixed subsets of the Euclidean space, irrespectively of the microstructure therein. Furthermore, a space related definition of the testing volume elements would be advantageous for an easier evaluation of the interrelation of the homogenization results $y$ for neighboring testing volume elements.

For this purpose, a moving window technique is employed. The testing volume elements $TVE_{(k)}$ are spherical elements with uniform radius $r^{TVE}$. The testing volume elements $TVE_{(k)}$ are defined on a spatially regular grid with a similar spacing of $\Delta l$ in all three spatial directions $x_i$ according to Fig. 5. Overlapping testing volume elements are permitted, since this approach basically constitutes a technique for assigning material properties to equally spaced points – here the sphere centers – considering the situation at these points and within a fixed spatial neighborhood.

Due to the space related definition, the external boundaries of the testing volume elements $TVE_{(k)}$ are located predominantly within the void interior of the cells. Hence, a definition of displacement fields on the testing volume element surfaces, as required in Eq. (13), is problematic. To circumvent this problem, the effective deformation and stress components $F_i(TVE_{(k)})$ and $P_i^E(TVE_{(k)})$ for the space related testing volume elements are derived from the effective stress and deformation components $F_i(TVE_{(k)})$ and $P_i^E(TVE_{(k)})$ obtained in the microstructure related approach defined in Section 2.3.2 assuming a uniform distribution of $F_i(TVE_{(k)})$ and $P_i^E(TVE_{(k)})$ through the respective microstructure related testing volume elements $TVE$. Based on this assumption, the effective deformation and stress components for the alternative testing volume elements are defined by

$$F_i(TVE_{(k)}) = \frac{1}{V_{TVE}} \int_{\text{int} \; TVE} F_i(TVE_{(k)}) \tilde{w}(r^k) \; dV$$

(20)

$$P_i^E(TVE_{(k)}) = \frac{1}{V_{TVE}} \int_{\text{int} \; TVE} P_i^E(TVE_{(k)}) \tilde{w}(r^k) \; dV$$

(21)

where

$$\tilde{w}(r^k) = \frac{w(r^k)}{r^k}$$

(22)

is the normalized form of a weight function $w(r^k)$ depending on the Euclidean distance $r^k$ of the respective spatial point to the reference point of the respective testing volume element $TVE_{(k)}$. By an appropriate choice of the weight function in the assignment of the effective material properties to the reference point, the importance of specific regions within the testing volume element – as e.g. the direct neighborhood of the reference point – can be weighted higher than the importance of other regions. By choosing $w(r^k) = 1$, all spatial positions within $TVE_{(k)}$ are provided with a uniform weight. Alternatively, the original effective properties $F_i(TVE_{(k)})$ and $P_i^E(TVE_{(k)})$ at positions closer to the reference point in the center of the testing volume elements $TVE_{(k)}$ can be considered to be more important by choosing $w(r^k)$ as a function decaying with increasing distance $r^k$ to the sphere center. Within the present study either a uniform weight function $w(r^k) = 1$ or a decaying function of the Gaussian type

$$w(r^k) = \frac{1}{s(2\pi)^{1.2}} e^{-0.5(\frac{r^k}{s})^2}$$

(23)

are employed, where the parameter $s$ is chosen such that $w(r^{TVE}) = 10^{-6}$.

3. Example

As an illustrative example, the probabilistic local homogenization procedures defined in Sections 2.3.2 and 2.3.3 are applied to an assessment in the material uncertainty of an elastic closed cell aluminum foam. The cell wall material behavior is defined by a Young’s modulus of $E = 70$ GPa and a Poisson’s ratio of $\nu = 0.3$. The cell wall thickness $c^{w}$ is assumed to be uniform and is chosen in such a manner that the relative density of the representative
volume element $\Omega_{\text{RVE}}$ is $\rho/\rho_0 = 0.08$. In the analyses, a cubic representative volume element $\Omega_{\text{RVE}}$ with a uniform edge length of $l_{\text{RVE}} = 10 \text{ mm}$ consisting of 256 cells is considered. Thus, the expectation value of the cell size is $\mathbb{E}[V] = 3.91 \text{ mm}^3$. Although this cell size is in the lower limit for standard aluminum foams, similar results would be obtained for all other absolute cell sizes since no direct length dependence of the considered properties occurs. The standard deviation $\mathbb{S}(V)$ of the cell size is assumed to be 40% of its expectation value in order to obtain a distinctively disordered microstructure.

The cell size distribution is assumed to be of the logarithmic normal type with the probability density distribution

$$f(V_{\text{enc}}) = \frac{1}{V_{\text{enc}} \sigma (2\pi)^{3/2}} \exp \left(-\frac{(\ln V_{\text{enc}} - \mu)^2}{2\sigma^2}\right)$$

(24)

for the volumes $V_{\text{enc}}$ of the spheres surrounding the individual nuclei in the Voronoi process in Laguerre geometry employed for generation of the computational foam model (Section 2.2). The position and shape parameters $\mu$ and $\sigma$ are chosen such that the resulting cell size distribution $F(V)$ features the pre-defined expectation value and standard deviation.

The computational foam model is presented in Fig. 3 (model with 256 cells). The finite element model consists of 238936 elements and 116289 nodes involving 697734 degrees of freedom. Since models of this size require a large numerical effort, an alternative approach is used, where a total of 16 representative volume elements with a volume of $V_{\text{RVE}} = 62.5 \text{ mm}^3$ consisting of 16 cells each are considered to form a 256 cell sample for the stochastic evaluation. In addition, a set of 4 medium scale representative volume elements consisting of 64 cells each with a volume of $V_{\text{RVE}} = 250 \text{ mm}^3$ is analyzed. Since the numerical effort for the repeated analysis of small scale models increases linearly with the number of models whereas it increases in a higher order manner (depending on the type of numerical solution algorithm), the repeated analysis of small scale models is numerically more efficient than a single analysis of a large scale model with the same total number of cells and an equivalent mesh density.

The local material response of the testing volume elements is assessed in terms of the effective stiffness components $\mathbb{C}_{ij}$ according to

$$\begin{pmatrix}
    \sigma_{11} \\
    \sigma_{22} \\
    \sigma_{33} \\
    \sigma_{12} \\
    \sigma_{13} \\
    \sigma_{23}
\end{pmatrix} =
\begin{pmatrix}
    C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
    C_{22} & C_{23} & C_{24} & C_{25} & C_{26} & C_{27} \\
    C_{33} & C_{34} & C_{35} & C_{36} & C_{37} & C_{38} \\
    C_{44} & C_{45} & C_{46} & C_{47} & C_{48} & C_{49} \\
    C_{55} & C_{56} & C_{57} & C_{58} & C_{59} & C_{60} \\
    C_{66} & C_{67} & C_{68} & C_{69} & C_{70} & C_{71}
\end{pmatrix}
\begin{pmatrix}
    \varepsilon_{11} \\
    \varepsilon_{22} \\
    \varepsilon_{33} \\
    \varepsilon_{12} \\
    \varepsilon_{13} \\
    \varepsilon_{23}
\end{pmatrix}$$

(25)

defining an anisotropic linear elastic material. The stiffness components $\mathbb{C}_{ij}$ can be derived from the components $\mathbb{C}_{ijkl}$ of the elasticity tensor in the general form $\sigma_{ij} = \mathbb{C}_{ijkl} \varepsilon_{kl}$ by combining the first and the second pair of indices of $\mathbb{C}_{ijkl}$ to a new index employing the rules $11 \rightarrow 1, 22 \rightarrow 2, 33 \rightarrow 3, 23 \rightarrow 4, 13 \rightarrow 5$ and $12 \rightarrow 6$.

Table 1

<table>
<thead>
<tr>
<th>$R(\rho/\rho_0, V)$</th>
<th>1 $\times$ 256 cells</th>
<th>4 $\times$ 64 cells</th>
<th>16 $\times$ 16 cells</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R(\rho/\rho_0, V)$</td>
<td>−0.9133</td>
<td>−0.9380</td>
<td>−0.9459</td>
</tr>
</tbody>
</table>

4. Microstructure related assessment of material uncertainties

4.1. Raw data

In a first approach, the microstructure related local homogenization scheme defined in Section 2.3.2 is employed for assessment of the uncertainties in the local material response of the model foam defined in Section 3. In Fig. 6, the distributions of the cell size $V$ and the relative density $\rho/\rho_0$ are presented for the model with 256 cells. In this context, a non-uniform relative density of the individual cells is obtained, since the cell volume varies whereas the cell wall thickness is uniform.

The probability distribution $F(V)$ of the cell size is shown in Fig. 6(a), whereas the discrete probability distribution $F(\rho/\rho_0)$ of the relative density is plotted in Fig. 6(b). Thereby, each data point represents the density of a single cell. Due to the fact that there are only few large and small cells in the model, the amount of available data in the upper and lower tails of the distributions is less dense. In cases where the tails are important, it would be possible to generate an adequate number of representative volume elements with very small and large cells respectively in order to increase the data in the tails systematically.
The discrete probability distributions are approximated by logarithmic normal distributions \((24)\). The shape and position parameters \(r\) and \(l\) of the continuous distributions are determined such that the expectation value \(E(y)\) and the variance \(V(y)\) for the discrete distributions of the considered variable \(y\) and their continuous counterparts are identical. The resulting continuous probability distributions and the corresponding probability density distributions are also presented in Figs. 6(a) and (b). The computed discrete distributions and their continuous approximations are in an excellent agreement, indicating that the Voronoi process in Laguerre geometry employed for generation of the computational foam model provides an excellent control of the cell size distribution of the computational foam model.

The relation between the relative density and the cell volume is shown in Fig. 6(c). The local relative density varies between \(\rho/\rho_0 \approx 0.05\) and 0.13. The cell volume ranges from \(V^c \approx 1\) to 12, where the cell volume increases with decreasing relative density. Thereby, the scatter band is rather narrow indicating a high correlation between the relative density and the cell volume. The corresponding correlation coefficients

\[
R(y^1, y^2) = \frac{E(y^1, y^2) - E(y^1)E(y^2)}{(V(y^1)V(y^2))^{1/2}}
\]

with \(y^1 = \rho/\rho_0\) and \(y^2 = V^c\) are presented in Table 1. In all cases, the single analysis of a large scale representative volume element \(\Omega_{RVE}\) with 256 cells as well as the analyses of sets of several medium or small scale volume elements, correlation coefficients close to \(R(\rho/\rho_0, V^c) = -1\) are obtained, indicating a rather strong inverse correlation of the cell volume \(V^c\) with the local relative density \(\rho/\rho_0\).

In Fig. 7, the raw data for the effective normal, coupling and shear stiffness components \(C_{ij}\) are plotted in dependence on the local relative density \(\rho/\rho_0\). The first row of subfigures (Figs. 7) is related to the single analysis of the large scale representative volume element \(\Omega_{RVE}\) with 256 cells whereas the second and third rows of subfigures are related to the analysis of 4 medium scale volume elements with 64 cells each (Figs. 7) and small scale volume elements with 16 cells each (Figs. 7) respectively. The normal effective stiffness components \(C_{11}, C_{22}\) and \(C_{33}\) are presented in the first line of subfigures whereas the second and third lines of subfigures are related to the normal coupling stiffness components \(C_{23}, C_{13}\) and

![Fig. 7. Microstructure related assessment – raw data.](image-url)
as well as to the effective shear stiffness components $\mathcal{T}_{14}$, $\mathcal{T}_{15}$ and $\mathcal{T}_{16}$ respectively.

In all three approaches, no distinct differences are observed between the results related to the individual spatial directions $x_i$. The three normal stiffness components $\mathcal{T}_{11}$, $\mathcal{T}_{22}$ and $\mathcal{T}_{33}$ are found within a unique scatter band. In a similar manner, unique scatter bands are obtained for the three shear stiffness components $\mathcal{T}_{14}$, $\mathcal{T}_{15}$ and $\mathcal{T}_{16}$ as well as for the three normal coupling stiffness components $\mathcal{T}_{24}$, $\mathcal{T}_{25}$ and $\mathcal{T}_{26}$. Hence, as to be expected, the computed response of the random foam microstructure is approximately isotropic, provided that a statistical representative set of testing volume elements $\Omega^{TVE}$ is considered. Except a few outliers, the local relative density $\rho/\rho_0$ proves to be the main influence parameter affecting the local effective stiffness of the cellular microstructure under investigation.

A comparison of the effective stiffness components obtained by the single analysis of a large scale representative volume element with 256 cells and the repeated analyses of small and medium scale volume elements ($16 \times 16$ cells and $4 \times 64$ cells) yields no distinct difference between the results based on the different approaches. Hence, all statistical significant interaction effects are recaptured even with small scale volume elements. Nevertheless, the range of the local relative density captured by the small scale volume elements with $16$ cells (Figs. 7(c), (f) and (i)) is slightly narrower than the corresponding range for the larger models due to the fact that the occurrence of rather large or small cells is suppressed due to the limited number of cells in the individual volume elements.

4.2. Probabilistic assessment

The raw data bases for the stiffness components are evaluated statistically according to Sections 2.3.1 and 2.3.2. In Fig. 8, the probability distributions $\mathcal{F}(\mathcal{T}_{ij})$ and the corresponding probability density distributions $f(\mathcal{T}_{ij})$ are presented for the models with 256, 64 and 16 cells respectively. Due to the statistical isotropy of the foamed material, the individual stiffness components $\mathcal{T}_{ij}$ are evaluated in the groups of the normal stiffness coefficients $\mathcal{T}_{\text{nrm}}$ ($\mathcal{T}_{11}$, $\mathcal{T}_{22}$ and $\mathcal{T}_{33}$), the normal coupling stiffness coefficients $\mathcal{T}_{\text{cpl}}$ ($\mathcal{T}_{14}$, $\mathcal{T}_{15}$ and $\mathcal{T}_{16}$) and the shear stiffness coefficients $\mathcal{T}_{\text{sh}}$ ($\mathcal{T}_{14}$, $\mathcal{T}_{15}$ and $\mathcal{T}_{16}$). The results are based on the cell size distribution and the corresponding probability distribution of the relative density presented in Figs. 6(a) and (b) with an expectation value of $\mathcal{E}(\rho/\rho_0) \approx 0.08$ and a standard deviation of $\sigma(\rho/\rho_0) \approx 0.013$.

For the probability distributions $\mathcal{F}(\mathcal{T}_{ij})$, both, the discrete distributions computed directly by means of the local homogenization scheme as well as continuous approximations in terms of a logarithmic normal distribution (24) are presented. As before, the coefficients of the logarithmic normal approximation of the discrete probability distributions are determined such that the expectation value and the variance of the respective discrete probability distribution and its approximation are identical. For the probability density distributions $f(\mathcal{T}_{ij})$, only the continuous approximations are presented since the determination of the probability density distribution as the numerical derivative of the discrete probability distributions results in rather noisy functions which would be meaningless without distinct smoothing.

As expected in the discussion on the raw data base (Fig. 7), almost identical results are obtained whether a single foam model with 256 cells is considered or whether the 256 cell raw data base is determined from the multiple analysis of small and medium scale models ($4 \times 64$ cells and $16 \times 16$ cells). In all three approaches for generation of the raw data base, statistically identical results are obtained, where both, the mean values as well as the statistical distributions of the respective local effective properties are equal. Obviously, already the set of the smallest model containing 16 cells only provides an adequate representation of all possible microstructural features leading to scatter in the effective material properties, at least in a statistically representative manner considering all 16 foam models in the data set. Hence, the numerically more efficient multiple analysis of small to medium scale volume elements may be employed for generation of the raw data base without loss in generality for the statistical results.

For the local normal stiffness components, a median stiffness of $\mathcal{T}_{\text{nrm}} = 2.72$ GPa and a standard deviation of $\sigma(\mathcal{T}_{\text{nrm}}) = 0.56$ GPa is obtained. The probability distribution $\mathcal{F}(\mathcal{T}_{\text{nrm}})$ is distinctly

Fig. 8. Microstructure related assessment – probability distributions.
asymmetric with respect to its lower and upper tails. The scatter band between cumulative probabilities of $F(C_{nm}) = 0.5\%$ and $99.5\%$, containing the inner 99\% of the data ranges from $C_{nm} = 2.03$ GPa and 4.83 GPa and thus from $C_{nm, med} = 1.23S(C_{nm})$ to $C_{nm, med} = 3.79S(C_{nm})$. This results reveals that the probability distribution for the normal stiffness features a distinct skewness and thus is essentially of a non Gaussian type. Therefore, a stochastic assessment of the material uncertainty in terms of its expectation value and standard deviation alone might be insufficient. Similar results are obtained for the probability distributions of the coupling and shear stiffness components $C_{cpl}$ and $C_{shr}$. Their median values are $C_{cpl, med} = 1.20$ GPa and $C_{shr, med} = 0.76$ GPa respectively with standard deviations of $S(C_{cpl}) = 0.21$ GPa and $S(C_{shr}) = 0.19$ GPa. The inner 99\% intervals are defined by $C_{cpl} \in [0.89, 1.94]$ GPa and $C_{shr} \in [0.59, 1.59]$ GPa respectively. Due to the narrower scatter bands, higher peaks are obtained for the probability density distributions $f(C_{cpl})$ and $f(C_{shr})$ compared to the probability density distribution $f(C_{nm})$ of the normal stiffness (Figs. 8(d) to (f)).

The probability distributions $F(C)$ for the effective stiffness components are qualitatively similar to the underlying probability distribution $F(p/p_0)$ of the relative density due to the strong correlation between effective stiffness and relative density (Fig. 7). For a quantitative assessment of the correlation between relative density and local effective stiffness, the correlation coefficients $R(C_{ij}, p/p_0)$ are determined. The results are listed in Table 2. For the normal stiffness, a rather strong correlation with the relative density is observed, indicated by an average correlation coefficient of $R(C_{nm}, p/p_0) \approx 0.88$. A similar result is obtained for the coupling stiffness components. For the shear stiffness components, a slightly less distinct correlations with an average of $R(C_{cpl}, p/p_0) \approx 0.84$ is obtained. Nevertheless, even this value is not significantly below the results for the other stiffness components. The distinct correlation between local effective stiffness and relative density reveals that a scatter in the local relative density caused by spatial variations in the microstructure might be the most important parameter controlling the scatter in the effective material properties of solid foams.

5. Moving window analysis

5.1. Raw data

The microstructure related assessment of the local effective properties performed in Section 4 enables the establishment of a relation between the uncertainty in the microstructural geometry and topology of solid foams and the uncertainty in the corresponding effective properties. Nevertheless, the microstructure related assessment has the disadvantage that the individual data points in the raw data base are related to testing volume elements $\Omega^{TVE}$ with variable size and shape. To circumvent this problem, a moving window approach is proposed in Section 2.3.3. In the present section, this approach is applied to the example specified in Section 3. As a base for the computations, the results obtained in Section 4 on the single analysis of the 256 cell foam model are employed. Within this representative volume element, the testing volume elements (moving windows) are positioned on a square grid of $10 \times 10 \times 10$ equally spaced locations. Spherical testing volume elements with diameters of $r^{TVE} = 1$ mm and 4 mm are employed.

In a competitive approach, the analyses based on the small testing volume elements are performed with a Gaussian weight function (22) and – alternatively – with a constant weight function $w(r^2)$ in Eqs. (20) and (21).

In a preliminary analysis, the distribution of the relative density for the testing volume elements $\Omega^{TVE}$ is studied since this quantity has been found to be the most important microstructural parameter in the microstructure related probabilistic homogenization (Section 4.2). The results for the probability and probability density distributions are presented in Fig. 9. Depending on the different definitions of the effective properties assigned to the different testing volume element (or window) positions, different probability distributions are obtained for the relative density $p/p_0$. Whereas the median relative density is not affected by the definition of the effective properties, an increasing size of the testing volume elements results in a steeper probability distributions $F(p/p_0)$ and thus probability density distributions $f(p/p_0)$ with higher peak values. The width of the corresponding scatter bands decreases with increasing testing volume element size $r^{TVE}$ due to increasing self-averaging effects with increasing averaging volume (Fig. 9). Gaussian weight function with $r^{TVE} = 1$ mm and 4 mm). For the same reason, the use of a Gaussian instead of a constant weight function $w(r^2)$ in the determination of the effective properties results in a steeper probability distribution $F(p/p_0)$ since the Gaussian function (22) supplies spatial points far from the center of the testing volume element with small weights so that the contribution of the ranges close to the testing volume element boundary becomes less significant than for a homogeneously distributed weight function. Thus the use of the Gaussian function reduces the effective averaging volume in the determination of the effective properties for the respective testing volume element.

The corresponding raw data for the effective stiffness components $C_{ij}$ are presented in Fig. 10. Similar as in Fig. 7 for the microstructure related approach, the first line of subfigures is related to the normal stiffness components $C_{11}$, $C_{22}$ and $C_{33}$ whereas the second and third lines are related to the normal coupling components $C_{12}$, $C_{13}$ and $C_{23}$ and the shear stiffness components $C_{44}$, $C_{55}$ and $C_{66}$ respectively. The first column of subfigures is related to the small testing volume elements with a constant weight function $w(r^2)$ in Eqs. (20) and (21) whereas the second and third rows are related to Gaussian weight functions in conjunction with small and large testing volume elements ($r^{TVE} = 1$ mm and $r^{TVE} = 4$ mm) respec-

![Fig. 9. Moving window analysis – relative density distributions.](image-url)
tively. Except for a few outliers, the effective stiffness components are again found to be strongly influenced by the local relative density. The corresponding correlation coefficients $R(C_{ij}/\rho/\rho_0)$ presented in Table 3 are smaller than the correlation coefficients obtained in the microstructure related assessment (Table 2). No significant effect of the testing volume element size and the type of weight function is observed for the correlation of effective stiffness and local relative density.

5.2. Probability distributions

The probability distributions $\mathcal{F}(C_{ij})$ for the effective stiffness components are presented in Fig. 11 together with the corresponding probability density distributions $f(C_{ij})$. Similar as in the microstructure related assessment, the individual normal, coupling and shear stiffness components related to the individual spatial directions are analyzed in combined form due to the statistically isotropic material response on the macroscopic level. Again, the discrete probability distributions are approximated by continuous distributions of the logarithmic normal type (24), providing a good estimate of the direct numerical results.

<table>
<thead>
<tr>
<th>$r^{TVE}$</th>
<th>$r^{TVE}$</th>
<th>$r^{TVE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{11}$</td>
<td>$0.8971$</td>
<td>$0.8971$</td>
</tr>
<tr>
<td>$C_{22}$</td>
<td>$0.8862$</td>
<td>$0.8862$</td>
</tr>
<tr>
<td>$C_{33}$</td>
<td>$0.9026$</td>
<td>$0.9026$</td>
</tr>
<tr>
<td>$C_{23}$</td>
<td>$0.8457$</td>
<td>$0.8457$</td>
</tr>
<tr>
<td>$C_{13}$</td>
<td>$0.8674$</td>
<td>$0.8674$</td>
</tr>
<tr>
<td>$C_{12}$</td>
<td>$0.8674$</td>
<td>$0.8674$</td>
</tr>
<tr>
<td>$C_{44}$</td>
<td>$0.8372$</td>
<td>$0.8372$</td>
</tr>
<tr>
<td>$C_{55}$</td>
<td>$0.8674$</td>
<td>$0.8674$</td>
</tr>
<tr>
<td>$C_{66}$</td>
<td>$0.8374$</td>
<td>$0.8374$</td>
</tr>
</tbody>
</table>

Compared to their counterparts based on the microstructure related approach (Fig. 8(a)), the discrete probability distributions $\mathcal{F}(C_{ij})$ for the effective normal, coupling and shear stiffness components based on the moving window approach feature a smoother appearance, especially towards their lower tails. Their median
values are slightly larger than in the microstructure related assessment whereas their skewness and thus their asymmetry is slightly reduced, although still present. Considering the small testing volume element size ($r_{TVE} = 1$ mm) with a constant weight function, a median normal stiffness of $C_{nrm}^{med} = 2.88$ GPa and a 99% scatter band with $C_{nrm} \in [2.04, 4.30]$ GPa are obtained for the normal stiffness. Compared to the corresponding values of $C_{nrm}^{med} = 2.72$ GPa and $C_{nrm} \in [2.03, 4.83]$ GPa for the microstructure related approach, the median is shifted towards the center of the 99% scatter band. The width of the 99% scatter band is reduced, although still present. Considering the small testing volume element sizes and the two different definitions of the weight function $w(r^2)$ in Eqs. (20) and (21) the median values of the probability distributions for the normal, coupling and shear stiffness components respectively are found to be equal. On the other hand, the slopes of the respective probability distributions $f(C_{nrm}), f(C_{cpl})$ and $f(C_{shr})$ and thus the peaks of the corresponding probability density distributions $F(C_{nrm}), F(C_{cpl})$ and $F(C_{shr})$ are found to increase with increasing testing volume element size $r_{TVE}$ (Figs. 11(b), (c), (e) and (f) respectively). As for the distribution of the relative density discussed in Section 5.1, the reason for this effect is the increasing self averaging effect with increasing sample size. The use of a constant weight function $w(r^2)$ instead of the strongly inhomogeneous Gaussian function (22) results in an increase of the volume inside the testing volume element which is supplied with a significant weight. Thus, the effective volume for the averaging procedure in Eqs. (20) and (21) increases. Consequently, steeper slopes are obtained in the probability distributions in Fig. 11(a) compared to Fig. 11(b) based on the Gaussian weight function.

5.3. Autocorrelation

In addition to the advantage that all effective components are based on testing volume elements of equal size and shape, the moving window approach has the advantage that the testing volume elements are located on a regular, equally spaced grid, enabling a simple analysis of the correlation of the effective properties of neighboring testing volume elements. The autocorrelation is accessed in terms of Moran’s index $I$. It is based on a neighborhood matrix $W$ with the components

$$w_{(k)} = \begin{cases} 1 & \text{if } l \text{ is one of the 26 neighboring windows of } k; \\ 0 & \text{otherwise}. \end{cases}$$

(27)

defining the neighborhood of the testing volume element $\Omega_{TVE}^{k}$. For simplicity, the neighborhood matrix $W$ is normalized row by row

$$W_{(l)} = \frac{w_{(l)}}{\sum_{l \in W} w_{(l)}}$$

(28)

where $n$ is the total number of testing volume elements $\Omega_{TVE}^{k}$. With (28), the local Moran’s index is defined by

$$I_{(k)}(C_y) = \frac{(\bar{C}_y - C_y) \sum_{l = 1}^{n} w_{(l)} (\bar{C}_y - C_y)}{\sqrt{\sum_{l = 1}^{n} w_{(l)} (\bar{C}_y - C_y)^2}}$$

(29)

where $C_y$ is the effective stiffness component $C_y$ for testing volume element $\Omega_{TVE}^{k}$.

Based on the local Moran’s index (29), the global Moran’s index is defined by

$$I(C_y) = \frac{1}{n} \sum_{k = 1}^{n} I_{(k)}(C_y)$$

(30)

as the average of the local Moran’s Indices $I_{(k)}(C_y)$ for the individual testing volume elements $\Omega_{TVE}^{k}$.
The results for the local Moran's indices \( I_{lok}(\overline{C}_1) \) of the effective normal stiffness component \( \overline{C}_1 \), are presented in Fig. 12 concerning the analyses with a Gaussian weight function and testing volume element sizes with \( r_{TVE} = 1 \) mm and 4 mm. For information, the absolute value of the normal stiffness \( \overline{C}_1 \) and the relative density \( \rho/\rho_0 \) for the respective cases are added. The individual subfigures are related to clipping planes through the representative volume element parallel to the \( x_1-x_2 \)-plane at different \( x_2 \)-values.

For the smaller testing volume element size of \( r_{TVE} = 1 \) mm, a global Moran's index is \( I(\overline{C}_1) = 0.0619 \) whereas \( I(\overline{C}_1) = 0.1288 \) is obtained for the larger testing volume element size with \( r_{TVE} = 4 \) mm. The larger Moran's index in the latter case indicates a more distinct autocorrelation for the larger testing volume elements. The increase in the autocorrelation with the testing volume element size \( r_{TVE} \) is caused by the fact that for larger neighboring testing volume elements, the overlap volume covered by both volume elements increases. Nevertheless, in both cases, the spatial autocorrelation of the effective stiffness components is weak compared to known effects in other materials such as fiber reinforced composites with long fibers.

Considering the results of the local Moran's index \( I_{lok}(\overline{C}_1) \), Fig. 12 reveals that areas with both, a negative Moran's index as well as with a positive Moran's Index are present. Hence, spatial autocorrelation apparently exists in some parts of the model foam. Comparing the results for the local Moran's index with the results for the effective stiffness and the relative density at the respective spatial positions shows that in most cases the Moran's index \( I_{lok}(\overline{C}_1) \) is high if the corresponding effective stiffness component \( \overline{C}_1 \) and the local relative density \( \rho/\rho_0 \) are low. Low local relative densities are in general related to larger cell sizes and thus larger areas provided with the respective local homogenization result. Examples for such ranges in case of the small testing volume element size with \( r_{TVE} = 1 \) mm are the spatial positions \( x_i = (0.5, 7.5, 2.5), (3.5, 5.5, 4.5), (3.5, 5.5, 5.5), (4.5, 5.5, 5.5), (5.5, 6.5, 5.5) \) and \( (4.5, 7.5, 8.5) \). In the analysis with the larger testing volume element size \( r_{TVE} = 4 \) mm, both, the global Moran's index and the spreading of the local Moran's Indices are higher. Hence, more distinct effects are obtained. As it can be observed in the fourth to sixth row of subfigures in Fig. 12, all cases where the local Moran's index exceeds \( I_{lok}(\overline{C}_1) = 1.5 \) the corresponding stiffness component \( \overline{C}_1 \) is lower than 2.5 GPa with a local relative density of \( \rho/\rho_0 < 0.07 \).

In order to study the correlation length for the spatial correlation of the effective stiffness components \( \overline{C}_1 \) in more detail, the correlation coefficient \( R(\Delta \overline{C}_1, r) \) for the correlation of the difference in the effective stiffness of two spatial points with their distance is considered. In Fig. 13(a), (b) and (c), the results are presented for the normal, coupling and shear stiffnesses \( \overline{C}_{nrm}, \overline{C}_{cpl} \) and \( \overline{C}_{she} \) respectively. In all cases, the correlation coefficient decreases rapidly with increasing distance. For distances of \( r_{TVE} \geq 1.5 \) mm, which is the average cell radius in the considered model foam, almost vanishing correlation coefficients are obtained. Evidently, in the present cases, no distinct correlation between the effective properties of neighboring cells is present.

6. Comparison

In Section 5.2, a preliminary comparison of the results based on the microstructure related assessment and the moving window approach revealed that both procedures to a large extent provide matching results. In order to discuss the deviation between the two proposed methods in more detail, the differences \( F_{mIC}(y) - F_{win}(y) \) between the probability distribution \( F_{mIC}(y) \) obtained by the microstructure related approach for the effective property \( y \) and its counterpart \( F_{win}(y) \) obtained by the moving window approach are determined. The results for the local relative density \( \rho/\rho_0 \) are presented in Fig. 14 whereas Figs. 15(a) to (c) are related to the differences in the probability distributions for the effective normal, coupling and shear stiffness coefficients \( \overline{C}_{nrm}, \overline{C}_{cpl} \) and \( \overline{C}_{she} \) respectively. In all cases, the microstructure related results are based on the single analysis of the large scale representative volume element with 256 cells whereas for the moving window approach testing volume elements \( \overline{C}_{win} \) with radii of \( r_{TVE} = 1 \) mm and 4 mm with a Gaussian weight function \( w(r^2) \) as well as testing volume elements with \( r_{TVE} = 1 \) mm and a constant weight function are considered.

For the probability distributions \( F(\rho/\rho_0) \) of the local relative density, the differences between the microstructure related approach and the moving window approach with the three different testing volume element definitions considered are qualitatively similar (Fig. 14). For all three moving window definitions, the maximum differences to the microstructure related approach are observed for relative density in the range of \( \rho/\rho_0 = 0.07 \) to 0.08. In the microstructure related approach, this relative density corresponds to cells with a cell size in the range of the average cell size, considering the strong correlation of the local relative density with the cell size (see Fig. 6(c)). For the moving window approach, the local relative density is obtained as the average of the local relative densities of the ranges within the respective window. Due to the low probability for finding two or more cells with large size and thus low relative density as direct neighbors, the probability for occurrence of small relative densities for individual windows decreases compared to the microstructure related approach. Consequently, the cumulative probabilities \( F_{win}(\rho/\rho_0) \) for small relative densities \( \rho/\rho_0 \) based on the moving window approach are lower than in the microstructure related approach. If the local relative density comes into the range of its overall average, the probability distribution \( F_{win}(\rho/\rho_0) \) based on the moving window approach features a steeper increase than its counterpart \( F_{mIC}(\rho/\rho_0) \) based on the microstructure related approach. In the moving window approach, the probability for obtaining medium relative densities for individual testing volume elements is higher than in the microstructure related approach since medium relative densities are not only obtained for cells with cell sizes close to the average cell size but may also be obtained for windows covering portions of large cells (and thus small relative density) together with portions of small cells featuring a high local relative density. As a consequence of the steeper increase of the cumulative probability \( F_{win}(\rho/\rho_0) \) in the range of the overall average of the relative density, the distinct drop in the differences \( F_{win}(\rho/\rho_0) - F_{win}(\rho/\rho_0) \) for \( \rho/\rho_0 > 0.075 \) is obtained in Fig. 14.

For the differences \( F_{mIC}(\overline{C}_1) - F_{win}(\overline{C}_1) \) in the probability distributions for the effective stiffness components presented in Fig. 15, the same trends as for the differences in the probability distributions of the local relative density are observed. This result reveals again that the probability distribution of the local relative density to a large extent governs the scatter in the effective mechanical properties of solid foams.

7. Conclusions

The present study is concerned with the development of methods for the numerical prediction of the uncertainty in the effective material properties of solid foams based on the known uncertainty in the microstructural constitutive parameters. For the analysis of the scatter, a probabilistic homogenization procedure is proposed, where a large number of small scale testing volume elements is considered instead of a large scale, statistically representative volume element as usually employed in classical deterministic homogenization procedures. In contrast to representative volume elements, the individual testing volume elements themselves are not statistically representative for the microstructure of the
considered microheterogeneous material. Nevertheless, the entire set of testing volume elements has to be representative for the respective material in a statistical sense. The (local) effective properties of the individual testing volume elements are determined by...
standard homogenization analyses of their mechanical response. As a result, a raw data base for the effective properties is obtained, providing a statistically representative set of effective material data. Evaluating the raw data base for the individual effective properties by means of stochastic method yields the probability distributions for the effective properties.

In a competitive analysis, two different approaches are proposed for the definition of the testing volume elements. In a microstructure related approach, the individual cells of a large scale computational foam model as the smallest feasible microstructural elements are employed as testing volume elements. Deforming the entire foam model by a number of independent reference deformation states results in a number of local stress and strain states for the individual testing volume elements, which are used as input for the homogenization analysis. Since the testing volume elements are embedded into a large scale, statistically representative microstructure, they account for all possible interaction effects with the surrounding microstructure in a natural manner. As an alternative, a spatial definition of the testing volume elements in terms of a moving window approach is employed, where pre-defined spatial volumes form the testing volume elements, for which the effective material properties are determined by a weighted averaging of the material properties determined for the ranges within the respective testing volume element obtained in the microstructure related approach. The computational foam model is determined randomly by means of a Voronoi process in Laguerre geometry. By a proper definition of the foam model in terms of the probability distributions of the random variables defining the uncertain microstructure such as the cell size distribution or the distribution of the local relative density etc. the probabilistic homogenization procedure enables the establishment of the interrelation of the probability distributions of the random variables defining the microstructure and the probability distributions of its effective properties.

As an example, the proposed methods are applied to an analysis of the effective elastic properties of a closed cell aluminum foam. It is found that the local relative density is the most important microstructural parameter to control the scatter of the effective material response. As already observed in previous studies, the cell size distribution is found to have only limited direct effects, nevertheless, strong indirect effects are present through the strong correlation of the cell size with the local relative density since an increasing size of individual cells in general correlates with a decreasing local relative density of the effective material. In the present example, only a weak spatial autocorrelation of the effective properties for neighboring spatial positions is observed. Even on the level of the individual pores as the smallest feasible testing volume elements, the effective material response of neighboring microstructural elements is found to be almost independent from each other. Nevertheless, this observation is restricted to the considered sim-
ple computational foam model and may not be generalized. Especially for foams featuring clusters of small cells embedded into ranges consisting of larger cells, a more distinct autocorrelation has to be expected.

The methods proposed in the present study can easily be applied to a probabilistic analysis of these effects as well as to further effects causing scatter in the effective properties such as cell wall imperfections, damaged cell walls, non-homogeneous cell wall thickness etc. The proposed methods provide a reliable tool for establishing the link between the probability for occurrence of all of these effects and the resulting uncertainty in the effective material response. The results of the stochastic homogenization analysis can subsequently be used in a stochastic finite element analysis of the structural response of engineering components and structures consisting partially or in full of the considered cellular material. The stochastic finite element analysis then allows a reliable assessment of the uncertainties in the response of the considered structure due to the uncertainty in the material behaviour.

Acknowledgement

The present work has been funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Grant No. Ho1852/6-2. The financial support is gratefully acknowledged.

References


Thomson, W. (Lord Kelvin), 1887. On the division of space with minimum partitional area. Phil. Mag. 24, 503–514.**