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The pressure of QED from the two-loop 2PI effective action

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Abstract

We compute the pressure of hot quantum electrodynamics from the two-loop truncation of the 2PI effective action. Since the 2PI resummation guarantees gauge-fixing independence only up to the order of the truncation, our result for the pressure presents a gauge-dependent contribution of $\mathcal{O}(e^4)$. We numerically characterize the credibility of this gauge-dependent calculation and find that the uncertainty due to gauge parameter dependence is under control for $\xi \lesssim 1$. Our calculation also suggests that the choice of Landau gauge may minimize gauge-dependent effects.
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The diagrammatic approach to relativistic quantum field theories heavily relies on the convergence properties of the used expansion scheme. Among the various resummation schemes which have been invented to cure the poor performance of the perturbative expansion [1] in various situations of interest, the loop expansion of the two-particle-irreducible (2PI) effective action implements a ladder resummation, which respects thermodynamical consistency and energy conservation [2,3]. These features make the 2PI scheme attractive for non-equilibrium field theory applications [4]. A prerequisite for a nonequilibrium method to be credible is, however, its reliability in equilibrium. There, it is important to check the convergence of expansion series of the 2PI effective action. To this aim the notoriously ill-behaved pressure has been calculated in a scalar context in Ref. [5] showing a monotonous dependence on the coupling constant as well as a relatively small next-to-leading order correction even at couplings of $\mathcal{O}(1)$.

In the framework of gauge theories, however, the implementation of this approximation scheme suffers from various difficulties. One of these is that thermodynamic observables are gauge-fixing independent only up to the order of the truncation. One can illustrate this issue by studying gauge parameter dependence in the Lorentz covariant gauge. For vanishing background fields, the 2PI effective action is a functional of the fermion, gauge and ghost propagators (respectively denoted by D, G and G_{gh}) which also depends on the gauge-fixing parameter ξ: $\Gamma_{\text{2PI}}[D, G, G_{\text{gh}}; \xi]$. The thermal pressure of the system is obtained by evaluating Γ_{2PI} at its stationary point[1] $D = \bar{D}$, $G = \bar{G}$, $G_{\text{gh}} = \bar{G}_{\text{gh}}$, for a given temperature T, and by subtracting the same calculation at zero temperature:

$$\mathcal{P} = -\frac{T}{V}\Gamma_{\text{2PI}}[\bar{D}, \bar{G}, \bar{G}_{\text{gh}}; \xi]\Big|_{T=0}^{T}. \tag{1}$$

It is then possible to show that the ξ-dependence of $\mathcal{P}$ uniquely comes from the explicit ξ-dependence of Γ_{2PI} and that it disappears if, in Fourier space,

$$\sum_{\mu\nu} q_\mu q_\nu \bar{G}_{\mu\nu}(q) = \xi. \tag{2}$$

This last equation is the BRST identity for the propagator of the exact theory [6], which may break in a truncated resummation. Indeed, within a given truncation of the 2PI effective action, the BRST symmetry usually does not impose the con-

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[1] The barred propagators denote the solution of the stationarity equations: $\delta\Gamma_{\text{2PI}}/\delta D = 0$, $\delta\Gamma_{\text{2PI}}/\delta G = 0$ and $\delta\Gamma_{\text{2PI}}/\delta G_{\text{gh}} = 0$.

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straint (2) above the order of the truncation,[2] leading therefore to ξ-dependent contributions to the pressure.[3]

A certain number of strategies can be put forward in order to try to cope with these inconvenient features. The first possibility is to introduce further approximations, on top of the loop expansion. This is the case of the *approximately self-consistent resummations* introduced in Ref. [10]. Using this method, a gauge-independent determination of the entropy of QCD has been possible and shows a good agreement with lattice results down to temperatures about 2.5 times the transition temperature. There is however no general understanding on how to systematize this approach and evaluate higher orders in a gauge-independent manner.

Another possibility is to stick to the loop expansion of the 2PI effective action but play with the freedom in the choice of field representations. Indeed, the exact theory is invariant under reparametrization of the fields and one could exploit this feature in order to define a loop expansion obeying certain properties. This idea has been discussed in Ref. [11] where it has been applied to the linear sigma model in order to define a systematic loop expansion of the 2PI effective action fulfilling Goldstone's theorem at any order of approximation. Unfortunately no field representation is yet known in gauge theories which would ensure that the BRST identity (2) is fulfilled.

It is finally possible to isolate gauge-independent terms in the expression of the pressure by separating contributions from different perturbative orders. This means a re-expansion of the propagators $\bar{D}$ and $\bar{G}$ in powers of the coupling. The resulting modified resummation scheme did not show a substantial improvement of convergence [12].

A different point of view is based on the experience that the 2PI loop expansion is known to have good convergence properties [5,13]. One can thus expect that contributions above the order of accuracy, and in particular gauge dependences are under control, at least in a large range of coupling values. In this Letter we explore this possibility in QED and compute the pressure (1) from the two-loop truncation of the 2PI effective action using the standard parametrization of the fields. We work in the covariant gauge with arbitrary gauge-fixing parameter ξ, which allows us to study how large gauge-dependent contributions can be.

Before embarking on a numerical evaluation, one has however to pay special attention to a second aspect, namely that of renormalization. The difficulty is related to the fact that truncations of the 2PI effective action only resum particular subclasses of perturbative diagrams for which (perturbative) theorems do not apply. Recently a large effort has been put into extending renormalization theorems to the particular classes of diagrams resummed by the loop expansion of the 2PI effective action. This has been first achieved in the framework of scalar theories [14] as well as scalar theories coupled to a fermionic field [15], and more recently in the framework of QED [7] in the covariant gauge. In this latter case, it is important to emphasize that the renormalization procedure differs substantially from the one in perturbation theory. The reason for this is that, for a given loop truncation of the 2PI effective action and in contrast to what happens in perturbation theory, the photon two- and four-point functions develop longitudinal quantum and thermal corrections. Although these contributions are formally of higher order than the order of the truncation, they bring UV divergences which need to be removed before defining a continuum limit. In Ref. [7] a renormalization procedure involving a new class of counterterms allowed by the gauge symmetry of the theory has been put forward which allows one to deal with this new kind of UV divergences and thus opens the way to practical calculations. In this Letter, we apply these ideas in order to evaluate the pressure of QED from the two-loop 2PI effective action.

Because our purpose is to discuss gauge parameter dependence, it is essential that the considered discretization respects gauge symmetry. In this way, the only source for gauge dependences is the particular truncation we use. For numerical purposes it is also convenient to use lattice rather than dimensional regularization. We thus consider QED on a hypercubic lattice of spacing a. We denote by N_β the number of points on the time direction and N the number of points on each of the spatial directions. The inverse temperature is $\beta = N_\beta a$ and the spatial volume $V = N^3 a^3$. We decompose the lattice action in three pieces: $S = S_g + S_{gf} + S_f$. As gauge-field action, we consider the non-compact action

$$S_g = \frac{1}{4} a^4 \sum_x \sum_{\mu\nu} F_{\mu\nu}(x) F_{\mu\nu}(x), \tag{3}$$

where the field-strength tensor $F_{\mu\nu}(x) = \Delta^f_\mu A_\nu(x) - \Delta^f_\nu A_\mu(x)$ is expressed in terms of the forward derivative[4] $\Delta^f_\mu A_\nu(x) = a^{-1}[A_\nu(x+\hat{\mu}) - A_\nu(x)]$. We use a discretized covariant gauge-fixing term

$$S_{gf} = \frac{1}{2\xi} a^4 \sum_x \sum_{\mu\nu} \Delta^b_\mu A_\mu(x) \Delta^b_\nu A_\nu(x), \tag{4}$$

given in terms of the backward derivative $\Delta^b_\mu A_\nu(x) = a^{-1} \times [A_\nu(x) - A_\nu(x-\hat{\mu})]$ for latter convenience. Finally, the fermionic action is taken to be the naive chiral action

$$S_f = -\frac{1}{2a} a^4 \sum_x \big[\bar{\psi}(x+\hat{\mu})\gamma_\mu U_\mu(x)\psi(x) - \bar{\psi}(x)\gamma_\mu U^+_\mu(x)\psi(x+\hat{\mu})\big], \tag{5}$$

where $U_\mu(x) = \exp(iaeA_\mu(x))$ represents a link variable.

Normally, the interacting two-point function $\bar{D}(x,y)$ or $\bar{G}(x,y)$ corresponds to the correlator of two operators at x and y. On the lattice however, where the fundamental objects

[2] An analysis of similar issues has recently been done in QED [7,8] where it has been shown in particular that, although the 2PI effective action obeys (2PI) Ward identities, these do not impose any constraint on the photon propagator $\bar{G}$. In particular the corresponding polarization tensor is not constrained to be transverse.

[3] More precisely, if one truncates the 2PI effective action at L-loop order, one expects gauge dependences to appear at order e^{2L} [9].

[4] The notation $\hat{\mu}$ stands for the vector of length a along the positive μ direction.

are link variables, it is more convenient to introduce these two-point functions as

$$\bar{D}(x,y)=\langle\psi(x)\bar{\psi}(y)\rangle_c \tag{6}$$

and

$$\bar{G}_{\mu\nu}(x,y)=\langle A_\mu(x)A_\nu\big(y-(1-\delta_{\mu\nu})\hat{\nu}\big)\rangle_c. \tag{7}$$

These definitions maintain the usual translation and reflection symmetries of $\bar{G}$ and $\bar{D}$, as well as the identity $\bar{G}_{\mu\nu}(x,y)=\bar{G}_{\nu\mu}(x,y)$. Notice also that the discretization we consider here, respects the chiral symmetry of the massless fermion: $\bar{D}(x,y)=\sum_\mu\gamma_\mu\bar{D}_\mu(x,y)$. We shall thus consider the 2PI effective action as a functional of D_μ rather than a functional of D.

Since the pressure (1) cannot be determined exactly, we consider the loop expansion of the 2PI effective action as obtained from the Cornwall–Jackiw–Tomboulis formula [3] (a trivial term stemming from the ghosts is included in our numerics but not written explicitly here):

$$\Gamma_{\text{2PI}}[D,G]=-N_f\,\mathrm{Tr}\big[\log D^{-1}+D_0^{-1}D\big]+\frac{1}{2}\mathrm{Tr}\big[\log G^{-1}+G_0^{-1}G\big]+\Gamma_{\text{int}}[D,G], \tag{8}$$

where we have defined $\mathrm{Tr}\,O\equiv a^4\sum_x\sum_i O_{ii}(x,x)=\beta V\sum_i O_{ii}(x=0)$ and we have included the possibility of an arbitrary number of fermionic flavors N_f.[5] The functional $\Gamma_{\text{int}}[D,G]$ is given—up to an overall sign—by all 0-leg 2PI diagrams that one can draw using the two-point functions D and G and the tree level vertices generated by the lattice action. These arise from the expansion of the link variable $U_\mu(x)$, and in turn of S_f, in powers of $A_\mu(x)$. To make sure that the pressure we calculate is correct up to $\mathcal{O}(e^3)$, we have to expand $U_\mu(x)$ to $\mathcal{O}(e^2)$. The vertex obtained from expanding $U_\mu(x)$ to $\mathcal{O}(e^3)$ brings no contribution to the pressure in the case of vanishing background fields, which we assume throughout this work.

Combining the $\mathcal{O}(e)$ and $\mathcal{O}(e^2)$ vertices into two-loop 2PI diagrams, performing the relevant traces and making use of the properties of D, we obtain the following contributions to the interacting part Γ_{int} of the 2PI effective action:

$$\begin{aligned}\frac{\Gamma^a_{\text{int}}}{\beta V}&=e^2N_fa^4\sum_{x,\mu\neq\nu}G_{\mu\nu}(x)\\&\quad\times\big[D_\mu(x)D_\nu(x+\hat{\mu}+\hat{\nu})+D_\nu(x)D_\mu(x+\hat{\mu}+\hat{\nu})\\&\quad+D_\mu(x+\hat{\nu})D_\nu(x+\hat{\mu})+D_\nu(x+\hat{\nu})D_\mu(x+\hat{\mu})\big]\\&\quad+e^2N_fa^4\sum_{x,\mu}G_{\mu\mu}(x)\\&\quad\times\big[2D_\mu(x-\hat{\mu})D_\mu(x+\hat{\mu})+2D_\mu(x)D_\mu(x)\\&\quad-\sum_\nu\big[D_\nu(x-\hat{\mu})D_\nu(x+\hat{\mu})+D_\nu(x)D_\nu(x)\big]\big],\end{aligned} \tag{9}$$

$$\frac{\Gamma^b_{\text{int}}}{\beta V}=ae^2N_f\sum_\mu G_{\mu\mu}(0)\big[D_\mu(\hat{\mu})-D_\mu(-\hat{\mu})\big]. \tag{10}$$

The contribution Γ^a_{int} is the usual fermion loop with a somewhat peculiar photon line (7). The lattice spacing a in Γ^b_{int} manifests that this diagram is a lattice artefact. Both fermion loops are individually quadratically divergent, they together make sure that at the lowest *perturbative* level the photon receives no mass renormalization.

Together with a counterterm contribution $\delta\Gamma_{\text{int}}$ (see below), the expressions (9) and (10) provide the full $\mathcal{O}(e^3)$ interaction part of the 2PI effective action: $\Gamma_{\text{int}}=\Gamma^a_{\text{int}}+\Gamma^b_{\text{int}}+\delta\Gamma_{\text{int}}$. If we now introduce the self-energies

$$\bar{\Sigma}_\mu(x)=\bar{D}^{-1}_\mu(x)-D^{-1}_{0,\mu}(x) \tag{11}$$

and

$$\bar{\Pi}_{\mu\nu}(x)=G^{-1}_{\mu\nu}(x)-G^{-1}_{0,\mu\nu}(x), \tag{12}$$

and use the explicit formula (8) for the 2PI effective action, we can write the stationarity equations defining the interacting two-point functions $\bar{D}$ and $\bar{G}$ as

$$4N_f\bar{\Sigma}_\mu(x)=\frac{1}{a^4\beta V}\frac{\partial\Gamma_{\text{int}}}{\partial D_\mu(x)} \tag{13}$$

and

$$\bar{\Pi}_{\mu\nu}(x)=\frac{2}{a^4\beta V}\frac{\partial\Gamma_{\text{int}}}{\partial G_{\mu\nu}(x)}. \tag{14}$$

The interacting two-point functions $\bar{D}$ and $\bar{G}$ are thus obtained after simultaneously solving Eqs. (11)–(14). As it can easily be checked, in the two-loop approximation that we consider here, Eqs. (13)–(14) do not involve any discretized integral in direct space. On the other hand, Eqs. (11)–(12) can be conveniently solved in momentum space. To this order of the truncation we can thus completely avoid calculating loops by simply Fourier transforming the propagators back and forth in every step of the iterative procedure. We define the Fourier transforms of a generic fermionic (D) or gauge (G) two-point function, respectively, as

$$i^{-1}D_\mu(k)=a^4\sum_x e^{-ik\cdot x}D_\mu(x) \tag{15}$$

and

$$\alpha^{-1}_{\mu\nu}(k)G_{\mu\nu}(k)=a^4\sum_x e^{-ik\cdot x}G_{\mu\nu}(k), \tag{16}$$

where $\alpha_{\mu\nu}(k)=1$ if $\mu=\nu$ and $\alpha_{\mu\nu}(k)=\exp(-ia(k_\mu+k_\nu)/2)$ otherwise. This particular definition of the Fourier transform of G is connected to the fact that the gauge field has to be thought as attached to the midpoints of the lattice links.[6] Solving Eqs. (11)–(12) in Fourier space, needs that we determine the Fourier transforms of the free inverse propagators. After inspection of the free (quadratic) contribution to S, one obtains

$$D^{-1}_{0,\mu}(k)=-\bar{k}_\mu \tag{17}$$

[5] The parameter N_f will be used in what follows in order to eliminate the doublers which appear as a result of discretizing the fermionic action. Since our discretization generates 16 fermion tastes (one pair in each direction) we shall set N_f to 1/16. This is similar, for instance, to the fourth root taken on the staggered fermion determinant in the context of lattice gauge field theory [16].

[6] Even this unusual variant of the fast Fourier transform is available as legacy code [17].

and

$$G^{-1}_{0,\mu\nu}(k) = \hat{k}^2\delta_{\mu\nu} - \left(1-\xi^{-1}\right)\hat{k}_\mu\hat{k}_\nu, \tag{18}$$

with the usual short-hand notations $\bar{k}_\mu a = \sin(k_\mu a)$ and $\hat{k}_\mu a/2 = \sin(k_\mu a/2)$. Eqs. (11)–(14) can be solved for any non-vanishing lattice spacing a, leading to perfectly finite two-point functions $\bar{D}$ and $\bar{G}$. In order to define a proper continuum limit of the latter, as $a \to 0$, one needs however to absorb UV divergences. Renormalization of Eqs. (11)–(14) was considered in Ref. [7] in the context of dimensional regularization and at zero temperature. There, renormalization was achieved by adding a contribution $\delta\Gamma_{\text{int}}$ to the functional Γ_{int}. This contribution carries the counterterms needed for renormalization. In extending this result to lattice regularization, one has to pay attention to the presence of new vertices originating from the expansion of the link variable $U_\mu(x)$ in powers of the field $A_\mu(x)$ (see above). In our present calculation, in addition to the usual vertex coupling A to $\bar{\psi}$ and ψ (which leads to Γ^a_{int}), there is a new vertex coupling A^2 to $\bar{\psi}$ and ψ (which leads to Γ^b_{int}). This new vertex brings an extra factor of a, which is such that the superficial degree of divergence of a given diagram is the same as in dimensional regularization.[7] It follows that we can here apply the same type of analysis of UV divergences as the one used in Ref. [7].

At two-loop order, the shift $\delta\Gamma_{\text{int}}$ is given in lattice regularization by

$$\begin{aligned}\delta\Gamma_{\text{int}} = {} & \frac{\delta g_1}{8}\frac{1}{\beta V}\sum_{k,\mu} G_{\mu\mu}(k)\sum_{q,\nu} G_{\nu\nu}(q) \\ & + \frac{\delta g_2}{4}\frac{1}{\beta V}\sum_{\mu\nu}\sum_k G_{\mu\nu}(k)\sum_q G_{\mu\nu}(q) \\ & + \frac{1}{2}\sum_k\sum_{\mu\nu} G_{\mu\nu}(k) \\ & \times \left[\delta Z_3 \hat{k}^2\delta_{\mu\nu} - (\delta Z_3 - \delta\lambda)\hat{k}_\mu\hat{k}_\nu + \delta M^2\delta_{\mu\nu}\right] \\ & - 4N_f\,\delta Z_2\sum_{k,\mu}\bar{k}_\mu D_\mu(k).\end{aligned} \tag{19}$$

It leads to additional contributions at the level of the self-energies, in particular a longitudinal wave function renormalization ($\delta\lambda$) as well as a photon mass counterterm (δM^2).

The counterterms δg_1 and δg_2 allow to remove subdivergences hidden in Eqs. (13)–(14) and involving four photon legs (see below). After these have been removed, there only remain temperature independent overall divergences that need to be absorbed in the counterterms δZ_2, δZ_3, $\delta\lambda$ and δM^2. Although the exact $\mathcal{O}(4)$ symmetry is broken on the lattice, the tensor structure of the self energies at a fixed scale k is restored in the continuum limit of an isotropic lattice theory. This allows us to use the renormalization conditions introduced in the context of the continuum theory [18]. In particular, one can show that the overall divergences have the structure

$$\bar{\Sigma}^{\text{div}}_\mu(k) = -\sigma\bar{k}_\mu \tag{20}$$

and

$$\bar{\Pi}^{\text{div}}_{\mu\nu}(k) = \pi_M\delta_{\mu\nu} + \pi_T\left(\delta_{\mu\nu}\hat{k}^2 - \hat{k}_\mu\hat{k}_\nu\right) + \pi_L k_\mu k_\nu \tag{21}$$

where σ, π_T, π_L and π_M represent quantities which diverge as $a \to 0$ (quadratically for π_M and logarithmically for the rest of them). Comparing these expressions to those for the counterterms, we find that all divergences can be absorbed by setting

$$\delta Z_2 = -\sigma, \quad \delta Z_3 = -\pi_T, \quad \delta\lambda = -\pi_L \quad \text{and} \quad \delta M^2 = -\pi_M. \tag{22}$$

The set of Eqs. (22) does not fix the finite parts of the counterterms. In order to do so, we fix δZ_2, δZ_3, $\delta\lambda$ and δM^2 through the lattice version of the renormalization conditions given in [7]:

$$\left.\frac{\partial\bar{\Sigma}^\star_3}{\partial\bar{k}_3}\right|_{k^\star} = 0, \quad \left.\frac{\partial\bar{\Pi}^\star_{22}}{\partial\hat{k}^2_3}\right|_{k^\star} = 0, \quad \left.\frac{\partial\bar{\Pi}^\star_{33}}{\partial\hat{k}^2_3}\right|_{k^\star} = 0 \quad \text{and} \quad \bar{\Pi}^\star_{33}\big|_{k^\star} = 0, \tag{23}$$

where $k^\star = (0,0,\mu,0)$ and μ denotes our renormalization scale. The star on the self-energies means that these are considered at a reference temperature $T^\star$. The first two renormalization conditions are similar to those which are used in perturbation theory and completely determine the counterterms δZ_2 and δZ_3. In perturbation theory, where the (lattice) Ward identity for $\bar{\Pi}(k)$

$$0 = \sum_\mu \hat{k}_\mu\bar{\Pi}_{\mu\nu}(k) \tag{24}$$

prevents the appearance of longitudinal corrections to the self energy, the third and fourth conditions in Eq. (23) are trivially satisfied. In our case, however, we need to fix two counterterms ($\delta\lambda$ and δM^2) that cancel longitudinal UV divergences of $\mathcal{O}(e^4)$. A natural way to fix these is to impose the Ward identity on $\bar{\Pi}$ at the renormalization point $k^\star = (0,0,\mu,0)$ and for a given temperature $T^\star$. We do so at $k^\star$ and in a small neighborhood of $k^\star$. In this way, we obtain the third and fourth renormalization conditions of Eq. (23). The arbitrariness of this condition introduces an ambiguity of order $\mathcal{O}(e^4)$.

As already discussed in Ref. [7], when renormalizing the two-point function $\bar{G}$, one has not only to pay attention to longitudinal overall divergences but also to longitudinal subdivergences which involve four-photon legs. Again, if no truncation is considered, these subdivergences automatically cancel since they reproduce the exact four-photon function which is transverse. However, for a given truncation of the 2PI effective action, this cancellation of divergences is only true up to the order of the truncation. Above, new divergences appear which need to be absorbed by means of the counterterms δg_1 and δg_2. The particular structure of these divergences has been worked out in Ref. [7] for the case of dimensional regularization. The result is that, in order to absorb the four-photon divergences, one needs to impose, at the renormalization point, the transversality of a four-point function defined by means of a set of Bethe–Salpeter

[7] More precisely, one has $\delta = 4 - E_A - (3/2)E_\psi$, where E_A and E_ψ, respectively denote the number of external photon and fermion legs of the diagram at hand.

equations. Here, we extend this result to the case of lattice regularization.

The Bethe–Salpeter equations can be written as a closed set of equations for a four-point function $\bar{V}_{\mu\nu,\sigma\rho}(p,k)$ involving four photon legs and a four-point function $\bar{W}_{ij,\sigma\rho}(p,k)$ involving two photon and two fermion legs [7]. Similarly to what we did with the propagator $\bar{D}$, we turn the Dirac indices i, j into one Lorentz index μ: $\sum_\mu \bar{W}_{\mu,\sigma\rho}\gamma_{\mu ij} = \bar{W}_{ij,\sigma\rho}$. Given that $k^\star = (0,0,\mu,0)$, the renormalization conditions fixing δg_1 and δg_2, as given in Ref. [7], read

$$\bar{V}^\star_{2233}(k^\star,k^\star) = 0 \quad \text{and} \quad \bar{V}^\star_{3333}(k^\star,k^\star) = 0. \tag{25}$$

In order to impose these renormalization conditions, we do not need to solve the set of Bethe–Salpeter equations for arbitrary values of the momenta and arbitrary configurations of Lorentz indices. Indeed, the set of equations remains closed if we fix one of the momenta to $k = k^\star$ and two of the Lorentz indices to $\sigma = \rho = 3$. We thus consider equations for $\bar{V}_{\mu\nu}(p) = \bar{V}_{\mu\nu 33}(p,k^\star)$ and $\bar{W}_\alpha(p) = \bar{W}_{\alpha,33}(p,k^\star)$. Introducing the notations

$$A_{\sigma\rho}(p) = \delta(p-k^\star)\delta_{\sigma 3}\delta_{\rho 3}, \tag{26}$$

$$V_{\mu\nu}(p) = \bar{G}_{\mu\alpha}(p)\bar{V}_{\alpha\beta}(p)\bar{G}_{\beta\nu}(p), \tag{27}$$

$$W_\mu(p) = -2\,\bar{D}_\mu(p)\sum_\rho \bar{W}_\rho(p)\bar{D}_\rho(p) + \bar{W}_\mu(p)\sum_\rho \bar{D}_\rho(p)\bar{D}_\rho(p), \tag{28}$$

we may write the corresponding set of Bethe–Salpeter equation as

$$\bar{V}_{\mu\nu}(p) = -\frac{\delta_{\mu\nu}}{2}\frac{\delta g_1}{\beta V}\sum_{q,\rho}\big[V_{\rho\rho}(q) - 2A_{\rho\rho}(q)\big] - \frac{\delta g_2}{\beta V}\sum_q\big[V_{\mu\nu}(q) - 2A_{\mu\nu}(q)\big] - \sum_{q,\rho}\frac{\partial \Pi_{\mu\nu}(p)}{\partial D_\rho(q)}W_\rho(q), \tag{29}$$

$$\bar{W}_\mu(p) = -\sum_{q,\rho\sigma}\frac{\partial \Sigma_\mu(p)}{\partial G_{\rho\sigma}(q)}\big[V_{\rho\sigma}(q) - 2A_{\rho\sigma}(q)\big] - \sum_{q,\rho}\frac{\partial \Sigma_\mu(p)}{\partial D_\rho(q)}W_\rho(q). \tag{30}$$

We simulaneously solve these pair of equations together with Eqs. (11)–(14) at the renormalization temperature $T^\star$ by adjusting the counterterms after each step of iteration so that the renormalization conditions (23) and (25) are fulfilled.[8] Using the obtained values for the counterterms, we can solve for the physical two-point functions $\bar{D}$ and $\bar{G}$ at any other temperature T, which admit a proper continuum limit.

Plugging these values into the CJT formula (8) truncated at two-loop order gives us a non-perturbative approximation to the QED pressure, compatible with perturbation theory up to order $\mathcal{O}(e^3)$. Notice that, even with all the above counterterms included, there is a quartic divergence remaining in the pressure. This divergence is temperature independent and can be removed by a *cosmological constant* renormalization. The renormalization condition is usually given by the requirement of zero vacuum pressure. Here we do not renormalize or evaluate the model at zero temperature. We determine the counterterms in the equations of motion at $T^\star$. Then, using these counterterms we evaluate the pressure at $T^\star$ and $T^\star/2$. Assuming a $\sim T^4$ scaling with the temperature, we determine the pressure as the difference of the divergent pressure values as obtained from the formula of the effective action, divided by $(15/16)(T^\star)^4$. The assumed scaling of temperature is broken due to the presence of the renormalization scale. This effect introduces an error of $\mathcal{O}(e^4)$ which is above the actual accuracy of our calculation.

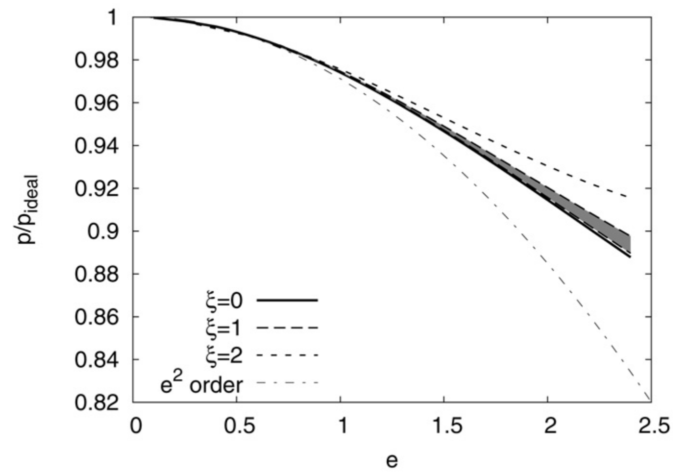

Fig. 1. Two-loop QED pressure as a function of the coupling e and for different values of the gauge-fixing parameter ξ. The plain line corresponds to $\xi = 0$ (Landau gauge), long-dashed lines to $\xi = 1$ (Feynman gauge) and short-dashed lines to $\xi = 2$. The sensitivity with respect to the renormalization scale μ is illustrated in the case of the Feynman gauge. We also plot the perturbative $\mathcal{O}(e^2)$ result for comparison.

In order to improve numerical stability, we take into account the following points. Calculating the pressure difference involves the subtraction of two quartically divergent contributions. Instead, we carry out the spatial part of the trace in $\Gamma_{2\mathrm{PI}}$ *after* performing the subtraction. An other important alteration to the equations above is the exclusion of the spatially homogeneous lattice mode on the level of the 2PI effective action. This is necessary to avoid instabilities as $e \to 0$, since the finite photon mass contribution behaves as $\sim e^4$.

In Fig. 1 we plot the QED pressure in the two-loop 2PI approximation, for a wide range of coupling values ($0 \leqslant e \leqslant 2.4$) and for various values of the gauge-fixing parameter. As discussed in Ref. [9] the higher the gauge-fixing parameter is, the less convergent the 2PI loop expansion becomes. It is thus meaningless to consider our calculation for too high values of ξ and, as suggested in Ref. [9], we restrict our calculations to values of the gauge-fixing parameter ranging from $\xi = 0$ (Landau gauge) to $\xi = 2$. For small values of the coupling, our results are almost insensitive to the gauge-fixing parameter and nicely re-

[8] As expected, the numerical values of δM^2, $\delta\lambda$, δg_1 and δg_2 scale as $\sim e^4$.

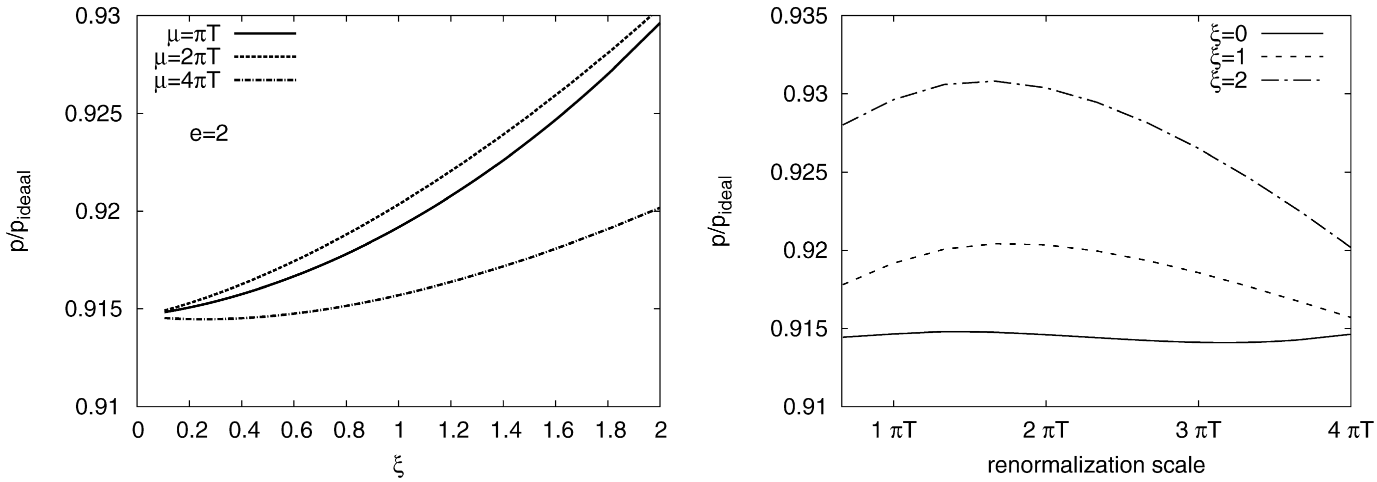

Fig. 2. Renormalization scale dependence of the two-loop pressure.

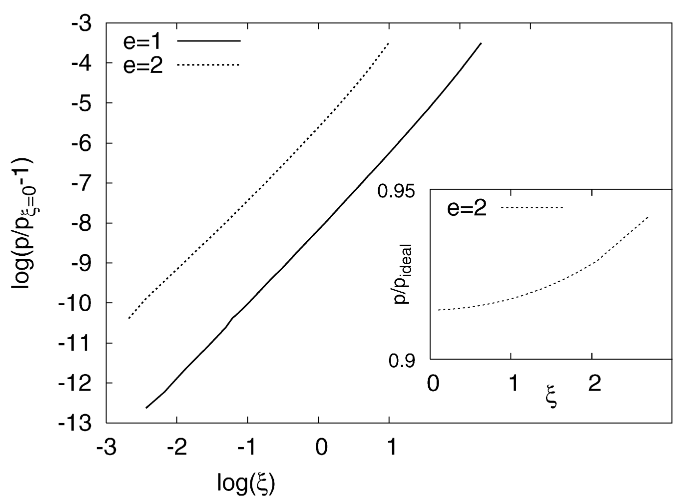

Fig. 3. Gauge-fixing parameter dependence of the two-loop pressure.

produce the perturbative result to order $\mathcal{O}(e^2)$. This comes as no surprise since the two-loop 2PI approximation contains all diagrams contributing to the pressure to order $\mathcal{O}(e^2)$. In principle, the same diagrams also contain the $\mathcal{O}(e^3)$ contribution. However, due to finite size effects, the latter is not accounted for by our numerics. For this reason, our results are compared to perturbation theory to order $\mathcal{O}(e^2)$. Numerically we find a good agreement with perturbation theory up to $e \sim 1$ which is precisely where the perturbative expansion usually breaks down. A more complete comparison would involve a substantial improvement of our code in order to reach the scaling regime where the $\mathcal{O}(e^3)$ contribution can be accessed through infinite volume extrapolation of our lattice results. Since our goal is not to test lattice perturbation theory in the infinite volume limit, but to explore the gauge dependence of the 2PI resummation scheme for which the $\mathcal{O}(e^3)$ is irrelevant in the approximation at hand, we ignore finite size effects at this stage.

For large values of the coupling, our calculation becomes a priori sensitive to two types of uncertainties. First of all, renormalization is done by imposing renormalization conditions at a certain momentum $k^\star = (0, 0, \mu, 0)$ which introduces an artificial dependence on the scale μ. Moreover the truncation of the 2PI effective action introduces gauge parameter dependences starting at order $\mathcal{O}(e^4)$. These two types of uncertainties can be taken as a way to estimate the error of the calculation.

The dependence with respect to the renormalization scale μ is illustrated in Fig. 1 for the case $\xi = 1$ (Feynman gauge) where μ is varied in the interval $\pi T \leqslant \mu \leqslant 4\pi T$ as it is usually done in calculations at finite temperature. A study of the μ-dependence as the gauge-fixing parameter is varied and for a given value of the coupling ($e = 2$) is depicted in Fig. 2. Notice that, at fixed gauge-fixing parameter ξ, the μ-dependence is not monotonous. However $\mu = 2\pi T$ roughly represents the value at which the pressure reaches it maximum value, in this range. We notice that the uncertainty due to scale dependence is not particularly severe, which indicates the good convergence behavior of the 2PI approach. Moreover this uncertainty is $\sim 1\%$ for $\xi = 2$ and decreases considerably down to its minimum value reached for $\xi = 0$, which makes the Landau gauge a particularly interesting choice among all possible gauges. We also notice that, in general, choosing a higher renormalization scale flattens the gauge dependence towards the Landau gauge value.

The second source of uncertainties is gauge dependence. As already mentioned a calculation for high values of the gauge-fixing parameter makes little sense. In the considered range of gauge parameter values, the error due to gauge dependence is of the order of or less than 1–1.5%. The Landau gauge plays again a special role since it corresponds to the value of ξ for which the pressure is the less sensitive to gauge parameter dependence. Indeed, independently of the value of the coupling, one has $p_\xi - p_{\xi=0} \sim \xi^2$ as $\xi \to 0$, as it is clear on the logarithmic plot of Fig. 3.

In conclusion, our calculation shows, in covariant gauge, a relatively small error coming from gauge parameter dependence. The parametric suppression of the gauge parameter dependence has already been shown in Ref. [9]. We have now established that the so far unknown coefficients of this parametric dependence do not spoil this behavior. The gauge dependence can also be regarded as a feature, which opens a way to error estimates without the need for considering higher order diagrams.

We think, that the 2PI effective action can be regarded as an efficient resummation technique for gauge theories, where the actual choice of gauge-fixing has an impact on the quality of the resummation. As for the particular calculation presented here, the Landau gauge is the preferred choice. This finding can be interpreted as a manifestation of the generic idea, that Landau gauge minimizes the presense of non-physical gauge contributions, which is often exploited in QCD [19].

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References

[1] U. Kraemmer, A. Rebhan, Rep. Prog. Phys. 67 (2004) 351;
J.P. Blaizot, E. Iancu, A. Rebhan, Nucl. Phys. A 698 (2002) 404;
J.P. Blaizot, E. Iancu, A. Rebhan, Phys. Rev. D 68 (2003) 025011;
J.O. Andersen, M. Strickland, Ann. Phys. 317 (2005) 281.
[2] J.M. Luttinger, J.C. Ward, Phys. Rev. 118 (1960) 1417;
G. Baym, Phys. Rev. 127 (1962) 1391;
C. De Dominicis, P.C. Martin, J. Math. Phys. 5 (1964) 14.
[3] J.M. Cornwall, R. Jackiw, E. Tomboulis, Phys. Rev. D 10 (1974) 2428.
[4] J. Berges, AIP Conf. Proc. 739 (2005) 3.
[5] J. Berges, S. Borsanyi, U. Reinosa, J. Serreau, Phys. Rev. D 71 (2005) 105004.
[6] M. Le Bellac, Thermal Field Theory, Cambridge Univ. Press, 1996.
[7] U. Reinosa, J. Serreau, JHEP 0607 (2006) 028.
[8] U. Reinosa, J. Serreau, arXiv: 0708.0971 [hep-th].
[9] A. Arrizabalaga, J. Smit, Phys. Rev. D 66 (2002) 065014.
[10] J.-P. Blaizot, E. Iancu, A. Rebhan, Phys. Rev. Lett. 83 (1999) 2906;
J.-P. Blaizot, E. Iancu, A. Rebhan, Phys. Lett. B 470 (1999) 181;
J.-P. Blaizot, E. Iancu, A. Rebhan, Phys. Rev. D 63 (2001) 065003.
[11] S. Leupold, Phys. Lett. B 646 (2006) 155.
[12] J.O. Andersen, M. Strickland, Phys. Rev. D 71 (2005) 025011.
[13] J.-P. Blaizot, A. Ipp, A. Rebhan, U. Reinosa, Phys. Rev. D 72 (2005) 125005.
[14] H. van Hees, J. Knoll, Phys. Rev. D 65 (2002) 025010;
H. van Hees, J. Knoll, Phys. Rev. D 65 (2002) 105005;
J.P. Blaizot, E. Iancu, U. Reinosa, Phys. Lett. B 568 (2003) 160;
J.P. Blaizot, E. Iancu, U. Reinosa, Nucl. Phys. A 736 (2002) 149;
J. Berges, S. Borsányi, U. Reinosa, J. Serreau, Ann. Phys. 320 (2005) 344.
[15] U. Reinosa, Nucl. Phys. A 772 (2006) 138.
[16] C. Bernard, Phys. Rev. D 73 (2006) 114503.
[17] M. Frigo, S.G. Johnson, Proc. IEEE 93 (2) (2005) 216.
[18] T. Reisz, Commun. Math. Phys. 117 (1988) 79;
T. Reisz, Nucl. Phys. B 318 (1989) 417.
[19] L. von Smekal, R. Alkofer, A. Hauck, Phys. Rev. Lett. 79 (1997) 3591;
J.M. Pawlowski, D.F. Litim, S. Nedelko, L. von Smekal, Phys. Rev. Lett. 93 (2004) 152002.