A Note on Upper Bounds on the Maximum Modulus of Subdominant Eigenvalues of Nonnegative Matrices

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ABSTRACT

The main aim of this note is to suggest a way of selecting the vector $a^T$ in a theorem of Brauer for use in finding upper bounds of the maximum modulus of subdominant eigenvalues of a nonnegative irreducible matrix. Upper bounds thus obtained for some matrices in a paper of Rothblum and Tan are compared with those obtained by theorems in that paper.

INTRODUCTION

Let $P$ be an $n \times n$ real matrix. The spectrum of $P$, denoted by $\sigma(P)$, is the set of eigenvalues of $P$. The spectral radius of $P$, denoted by $\rho(P)$, is defined by $\rho(P) = \max \{ |\lambda| : \lambda \in \sigma(P) \}$. If $P$ is an $n \times n$ nonnegative irreducible matrix, where $n \geq 2$, then an eigenvalue of $P$ which is different from $\rho(P)$ is called a subdominant eigenvalue of $P$, and the maximum modulus of subdominant eigenvalues of $P$, denoted by $\xi(P)$, is called the coefficient of ergodicity of $P$.

Let $\| \|$ be a norm on $\mathbb{R}^n$. For an $n \times n$ nonnegative irreducible matrix $P$ with spectral radius $\rho$, and an $n \times n$ diagonal matrix $D$ with positive diagonal elements, the coefficients $\tau_\| \| (P)$ and $\tau_{D\| \|} (P)$ are defined respectively by

$$
\tau_\| \| (P) = \max_{\| x \| = 1} \| x^T P \| \quad \text{and} \quad \tau_{D\| \|} (P) = \tau_\| \| (D^{-1}PD),
$$

\(^1\)We apply a norm $\| \|$, formally defined on $\mathbb{R}^n (\mathbb{C}^n)$, to elements in $R^{1 \times n} (C^{1 \times n})$ by having $\| y \| = \| y^T \|$ for a corresponding row vector $y$.

where \( w \in \mathbb{R}^n \) is a positive right eigenvector of \( P \) corresponding to the eigenvalue \( \rho \). For \( 1 \leq p \leq \infty \), \( \tau_p \| \cdot \| (P) \) and \( \tau_p^D \| \cdot \| (P) \) will be denoted respectively by \( \tau_p(P) \) and \( \tau_p^D(P) \) if the norm \( \| \cdot \| \) is the \( l_p \) norm.

Let \( B = (b_{ij}) \) be an \( n \times n \) real matrix. To a norm \( \| \cdot \| \) on \( \mathbb{R}^n \) there corresponds a matrix norm, which, also denoted by \( \| \cdot \| \), is defined by \( \|B\| = \max\{\|x^T B\|: \|x\| \leq 1; x \in \mathbb{R}^n \} \). The Frobenius norm of \( B \) is defined by \( \|B\|_F = (\sum_{j=1}^n \sum_{i=1}^n b_{ij}^2)^{1/2} \). It is known that \( \rho(B) \leq \|B\| \) and \( \rho(B) \leq \|B\|_F \).

In this note we are concerned with two results in Rothblum and Tan [2].

Let \( P \) be an \( n \times n \) nonnegative irreducible matrix, and \( w \) be a positive right eigenvector of \( P \) corresponding to the eigenvalue \( \rho = \rho(P) \). Inequality (5.2) and equation (5.3) of Theorem 5.1 of Brauer state that if \( a \in \mathbb{R}^n \), then

\[
\xi(P) \leq \rho(P - wa^T),
\]

and if \( a^T w = \rho \), then

\[
\xi(P) = \rho(P - wa^T).
\]

Inequalities (5.8) of Theorem 5.5 of Rothblum and Tan state that if \( \| \cdot \| \) is a norm on \( \mathbb{R}^n \), then

\[
\xi(P) \leq T\| \cdot \| (P) \leq \|P - wa^T\|.
\]

The main aim of this note is to show that it is always possible to choose the vector \( a^T \) in such a way that to each \( a^T \) so chosen, there corresponds a submatrix \( S \) of \( P - wa^T \) with the property that \( \xi(P) = \rho(S) \). Thus for \( p \in \{1, F, \infty\} \), we have\(^2\)

\[
\xi(P) \leq \|S\|_p \leq \|P - wa^T\|_p.
\]

In Theorem 1 we propose a way of using (2) to obtain an upper bound for \( \xi(P) \). Comparisons show that the bounds obtained in this way for the two matrices in Rothblum and Tan [2] are either as good as or better than those given by \( T\| \cdot \| (P) \) for some \( \| \cdot \| \) on \( \mathbb{R}^n \).

\(^2\)It can be shown, see Stewart [3], that \( \|B\|_\infty = \max\{\sum_{j=1}^n |b_{ij}|: i = 1, \ldots, n\} \), \( \|B\|_1 = \max\{\sum_{i=1}^n |b_{ij}|: j = 1, \ldots, n\} \).
MAIN RESULT

**Theorem 1.** Let $P$ be an $n \times n$ nonnegative irreducible matrix, where $n \geq 2$, and $w = (w_1, \ldots, w_n)^T$ be a positive right eigenvector of $P$ corresponding to the eigenvalue $\rho = \rho(P)$. Let $I = \{ i : w_i \neq 0 \}$. For each $i \in I$, define $R_i = P - w_i w_i^T P_i$, where $P_i$ is the $i$th row of $P$. Let $S_i$ be the $(n-1) \times (n-1)$ matrix obtained from $R_i$ by erasing its $i$th row and column. If $f : R^{(n-1) \times (n-1)} \rightarrow \mathbb{R}$ is a function from the set of $(n-1) \times (n-1)$ real matrices into $\mathbb{R}$ satisfying $\rho(S) \leq f(S)$ for each $S \in R^{(n-1) \times (n-1)}$, then

$$\xi(P) \leq \min \{ f(S_i) : i \in I \}.$$

In particular, if $M_i = \min \{ \|S_i\|_1, \|S_i\|_F, \|S_i\|_\infty \}$, and $m = \min \{ M_i : i \in I \}$, then $\xi(P) \leq m$.

**Proof.** For each $i \in I$, $R_i = P - w_i w_i^T$, where $a^T = w_i^T P_i$. Since $a^T w - w_i^T P_i w = w_i^T (Pw) = w_i^T \rho(P) w_i = \rho(P)$, it follows from (2) that $\rho(R_i) = \rho(P)$. Since the $i$th row of $R_i$ is the zero vector, $\rho(R_i) = \rho(S_i)$, and the theorem follows from the facts that $\rho(S_i) \leq M_i$ and $\rho(S_i) \leq f(S_i)$ for each $i \in I$. 

COMPARISONS

The following examples were taken from Rothblum and Tan [2].

(i) Let

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 5 & 4 \\ 0 & 3 & 0 \end{bmatrix},$$

$$\tau_1(P) = 2.4 \quad \tau_\infty(P) = 2.2857$$

$$\tau_F(P) = 2.0169 \quad \tau_F(P^T) = 2.1818$$

$$\tau_\infty(P^T) = 2.5714 \quad \tau_F(P^T) = 2.0144$$

Let $D_1 = \text{diag}(1, 7, 3)$ and $D_2 = \text{diag}(2, 7, 4)$. Then

$$\tau_1^{D_1}(P) = \tau_1^{D_2}(P^T) = 2, \quad \tau_\infty^{D_1}(P) = \tau_\infty^{D_2}(P) = 2.$$
The eigenvalues of $P$ are 0, $-2$, and 7. From Theorem 1, $m = 2$ for $D_1^{-1}PD_1$.

(ii) Let

$$
P = \begin{bmatrix}
12 & 6 & 6 \\
3 & 3 & 18 \\
8 & 8 & 8
\end{bmatrix},
$$

$$\tau_1(P) = 12, \quad \tau_\infty(P) = 12,$$

$$\tau_2(P) = 11.36, \quad \tau_F(P) = 11.66,$$

$$\tau_1(P^T) = 11.15, \quad \tau_\infty(P^T) = 14.12,$$

$$\tau_2(P^T) = 11.38.$$

The eigenvalues of $P$ are 5, $-6$, and 24. From Theorem 1, $m = 8.37$.

The author is indebted to the referee for a better proof of the author's result. This proof brings out more clearly the significance of the result and reveals an important way of selecting the vector $a^T$ of a theorem of A. Brauer for use in finding an upper bound of $\xi(P)$, in view of inequalities (3), (4) and the upper bounds of $\xi(P)$ of the matrix $P$ in example (ii).

REFERENCES


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