Global asymptotic stability of a higher order rational difference equation✩

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Abstract

In this note, we consider the following rational difference equation:

\[ x_{n+1} = \frac{f(x_{n-r_1}, \ldots, x_{n-r_k})g(x_{n-m_1}, \ldots, x_{n-m_l}) + 1}{f(x_{n-r_1}, \ldots, x_{n-r_k}) + g(x_{n-m_1}, \ldots, x_{n-m_l})}, \quad n = 0, 1, \ldots, \]

where \( f \in C((0, +\infty)^k, (0, +\infty)) \) and \( g \in C((0, +\infty)^l, (0, +\infty)) \) with \( k, l \in \{1, 2, \ldots\} \), \( 0 \leq r_1 < \cdots < r_k \) and \( 0 \leq m_1 < \cdots < m_l \), and the initial values are positive real numbers. We give sufficient conditions under which the unique equilibrium \( \bar{x} = 1 \) of this equation is globally asymptotically stable, which extends and includes corresponding results obtained in the recent literature.

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1. Introduction

The study of properties of rational difference equations has been an area of intense interest in recent years (for example, see [1–5]). In [6], Li discussed the global asymptotic stability of

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rational difference equation
\[ x_{n+1} = \frac{x_n x_{n-1} + 1}{x_n + x_{n-1}}, \quad n = 0, 1, \ldots, \] (E1)
where the initial values \( x_0, x_1 \in R_+ = (0, +\infty). \)

In [7,8], Li studied the global asymptotic stability of the following two nonlinear difference equations:
\[ x_{n+1} = \frac{x_n x_{n-1} x_{n-2} + x_{n-2} + x_{n-3} + a}{x_n x_{n-1} x_{n-2} + x_{n-2} x_{n-3} + 1 + a}, \quad n = 0, 1, \ldots, \] (E2)
and
\[ x_{n+1} = \frac{x_n x_{n-1} x_{n-2} + x_{n-2} + x_{n-3} + a}{x_n x_{n-1} x_{n-2} + x_{n-2} x_{n-3} + 1 + a}, \quad n = 0, 1, \ldots, \] (E3)
where \( a \in [0, +\infty) \) and the initial values \( x_0, x_1, x_2 \in R_+ \).

Recently, Berenhaut and Stevic [9] studied the global asymptotic stability of the following rational difference equation:
\[ x_{n+1} = \frac{x_{n-k} x_{n-l} x_{n-m} + x_{n-k} + x_{n-l} + x_{n-m}}{x_{n-k} x_{n-l} + x_{n-k} x_{n-m} + x_{n-l} x_{n-m} + 1}, \quad n = 0, 1, \ldots, \] (E4)
where \( 0 \leq k < l < m \) and the initial conditions \( x_{-m}, \ldots, x_0 \in R_+ \).

The main theorem in this note is motivated by the above studies. In this paper, we consider the following nonlinear difference equation:
\[ x_{n+1} = \frac{f(x_{n-r_1}, \ldots, x_{n-r_k})g(x_{n-m_1}, \ldots, x_{n-m_l}) + 1}{f(x_{n-r_1}, \ldots, x_{n-r_k}) + g(x_{n-m_1}, \ldots, x_{n-m_l})}, \quad n = 0, 1, \ldots, \] (1)
where \( f \in C(R_+^k, R_+) \) and \( g \in C(R_+^l, R_+) \) with \( k, l \in \{1, 2, \ldots\} \), \( 0 \leq r_1 < \cdots < r_k \) and \( 0 \leq m_1 < \cdots < m_l \), and the initial values are positive real numbers.

2. Main result

In the sequel, write \( a^* = \max\{a, 1/a\} \) for any \( a \in R_+ \).

**Lemma 1.** Let \( a, b, c \in R_+ \) satisfy \( c = (ab + 1)/(a + b) \), then \( c^* = (a^* b^* + 1)/(a^* + b^*) \).

**Proof.** From (1) we have
\[ c - 1 = \frac{(a - 1)(b - 1)}{a + b}. \] (2)
If \( a \geq 1 \) and \( b \geq 1 \), then \( a = a^* \) and \( b = b^* \). By (2) we have \( c \geq 1 \), which implies
\[ c^* = c = \frac{a b + 1}{a + b} = \frac{a^* b^* + 1}{a^* + b^*}. \]
If \( a \geq 1 \) and \( b < 1 \), then \( a = a^* \) and \( b = 1/b^* \). By (2) we have \( c \leq 1 \), which also implies
\[ c^* = 1/c = \frac{a + b}{a b + 1} = \frac{a^* + 1/b^*}{a^*/b^* + 1} = \frac{a^* b^* + 1}{a^* + b^*}. \]
In a similar fashion, we may show that \( c^* = \frac{a^* b^* + 1}{a^* + b^*} \) if \( a < 1 \) and \( b \geq 1 \) or \( a < 1 \) and \( b < 1 \). This completes the proof. \( \Box \)
Lemma 2. Let $a, b, c \in R_+$ satisfy $c = (ab + 1)/(a + b)$, then $c^* \leq \min\{a^*, b^*\}$.

Proof. By (1) it follows that
\[ c = \frac{ab + 1}{a + b} \leq \frac{a^*b^* + bb^*}{a + b} = b^* \]
and
\[ 1/c = \frac{a + b}{ab + 1} \leq \frac{a^*b^* + b^*}{ab + 1} = b^*. \]
Thus we have $c^* \leq b^*$. In a similar fashion, we may show that $c^* \leq a^*$. This completes the proof.

Now we formulate and prove the main result of this note.

Theorem 1. Let $f, g$ satisfy the following two conditions:

(H1) $[f(u_1, u_2, \ldots, u_k)]^* = f(u^*_1, u^*_2, \ldots, u^*_k)$ and $[g(u_1, u_2, \ldots, u_l)]^* = g(u^*_1, u^*_2, \ldots, u^*_l)$.

(H2) $f(u^*_1, u^*_2, \ldots, u^*_k) \leq u^*_1$.

Then $\bar{x} = 1$ is the unique positive equilibrium of Eq. (1) which is globally asymptotically stable.

Proof. Let $\{x_n\}_{n=0}^\infty$ be a solution of Eq. (1) with initial conditions $x_{-m}, x_{-m+1}, \ldots, x_0 \in R_+$, where $m = \max\{r_k, m_l\}$. From (1), (H1), (H2), Lemmas 1 and 2 it follows that for any $n \geq 0$,
\[1 \leq x_{n+1}^* \leq f \left( \frac{x_{n-r_1}^* + \ldots + x_{n-r_k}^*}{x_{n-r_1}^* + \ldots + x_{n-r_k}^*} + 1 \right) \leq f \left( x_{n-r_1}^* + \ldots + x_{n-r_k}^* \right) \leq x_{n-r_1}^*. \]
from which we get that for any $n \geq 0$ and $0 \leq i \leq r_1$,
\[1 \leq x_{i+(n+1)(r_1+1)}^* \leq x_{i+n(r_1+1)}^*. \]
Let $\lim_{n \to \infty} x_{i+n(r_1+1)}^* = A_i$ for any $0 \leq i \leq r_1$, then $A_i \geq 1$ ($0 \leq i \leq r_1$). Write $M = \max\{A_0, A_1, \ldots, A_{r_1}\}$ and $A_{i+n(r_1+1)}^* = A_i$ for any integer $n$ ($0 \leq i \leq r_1$). Then there exists $0 \leq j \leq r_1$ such that
\[\lim_{n \to \infty} x_{j+n(r_1+1)}^* = M.\]
By (3) we have
\[x_{j+n(r_1+1)}^* \leq f \left( x_{j+n(r_1+1)}^* \cdot x_{j+(n+1)(r_1+1)-1-r_2}^* \cdots x_{j+(n+1)(r_1+1)-1-r_k}^* \right) \leq x_{j+n(r_1+1)}^*. \]
It follows
\[M = f(M, A_{j-1-r_2}, \ldots, A_{j-1-r_k}) = M. \]
By (1), (H1) and (H2), we have
\[ M = \frac{f(M, A_{j-1-r_2}, \ldots, A_{j-1-r_k}) g(A_{j-1-m_1}, A_{j-1-m_2}, \ldots, A_{j-1-m_l}) + 1}{f(M, A_{j-1-r_2}, \ldots, A_{j-1-r_1}) + g(A_{j-1-m_1}, A_{j-1-m_2}, \ldots, A_{j-1-m_l})} = mg(A_{j-1-m_1}, A_{j-1-m_2}, \ldots, A_{j-1-m_l}) + 1 = M + g(A_{j-1-m_1}, A_{j-1-m_2}, \ldots, A_{j-1-m_l})\]
from which it follows \( M = 1 \). This implies \( A_1 = 1 \) for \( 0 \leq i \leq r_1 \) and \( \lim_{n \to \infty} x_n^* = 1 \). Since \( 1/x_n^* \leq x_n \leq x_n^* \), we obtain \( \lim_{n \to \infty} x_n = 1 \). By (4) it follows
\[ 1 = \frac{f(1, 1, \ldots, 1) g(1, 1, \ldots, 1) + 1}{f(1, 1, \ldots, 1) + g(1, 1, \ldots, 1)}. \]
Thus \( \bar{x} = 1 \) is the unique positive equilibrium of Eq. (1).

For any \( 1 > \varepsilon > 0 \), choose \( \delta = \varepsilon/(1+\varepsilon) \) and let \( \{x_n\}_{n=-m}^\infty \) be a solution of Eq. (1) with initial conditions \( x_{-m}, x_{-m+1}, \ldots, x_0 \in (1-\delta, 1+\delta) \). Then for any \( 1 \leq i \leq 0 \), we have that \( x_i < 1+\varepsilon \) and \( 1/x_i \leq 1/(1-\delta) = 1+\varepsilon \). By (3) it follows that for any \( n \geq 0 \),
\[ 1 \leq x_{n+1}^* \leq x_{n-r_1} < 1 + \varepsilon. \]
Thus we get that for any \( n \geq 0 \),
\[ 1 - \varepsilon < \frac{1}{1 + \varepsilon} \leq x_{n+1}^* \leq x_{n+1} < 1 + \varepsilon. \]

Which implies that \( \bar{x} = 1 \) is globally asymptotically stable. This completes the proof. \( \square \)

3. Example

In this section, we shall give an application of Theorem 1.

Example 1. Let \( k \geq 2 \) and \( f_1(u) = u \) (\( u > 0 \)). For any \( 2 \leq j \leq k \), let
\[ f_j(u_1, \ldots, u_j) = \frac{f_{j-1}(u_1, u_2, \ldots, u_{j-1}) u_j + 1}{f_{j-1}(u_1, u_2, \ldots, u_{j-1}) + u_j}. \]
Consider equation
\[ x_{n+1} = f_k(x_{n-r_1}, \ldots, x_{n-r_k}), \quad n = 0, 1, \ldots, \]
where \( 0 \leq r_1 < \cdots < r_k \) and the initial conditions \( x_{-r_k}, \ldots, x_0 \in R_+ \). Then \( \bar{x} = 1 \) is the unique positive equilibrium of Eq. (5) which is globally asymptotically stable.

Proof. From Lemma 1, it follows that
\[ [f_2(u_1, u_2)]^* = \frac{u_1^* u_2^* + 1}{u_1^* + u_2^*} = f_2(u_1^*, u_2^*). \]
Inductively, we obtain that for any \( 2 \leq j \leq k \),
\[ [f_j(u_1, \ldots, u_j)]^* = \frac{[f_{j-1}(u_1, u_2, \ldots, u_{j-1})]^* u_j + 1}{[f_{j-1}(u_1, u_2, \ldots, u_{j-1})]^* + u_j^*} = \frac{f_{j-1}(u_1^*, u_2^*, \ldots, u_{j-1}^*) u_j^* + 1}{f_{j-1}(u_1^*, u_2^*, \ldots, u_{j-1}^*) + u_j^*} = f_j(u_1^*, \ldots, u_j^*). \]
It is obvious that \( f_1(v)^* = v^* = f_1(v^*) \). Thus the conditions \((H_1)\) and \((H_2)\) hold. By Theorem 1 we know that \( \bar{x} = 1 \) is the unique positive equilibrium of Eq. (5) which is globally asymptotically stable. □

**Remark 1.** Let \( k = 2, r_1 = 0 \) and \( r_2 = 1 \), Eq. (5) reduces to Eq. \((E1)\).

**Remark 2.** Let \( k = 3 \), Eq. (5) reduces to Eq. \((E4)\).

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**References**


