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Large deviations for stochastic differential equations driven by G-Brownian motion[☆]

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Abstract

A joint large deviation principle for G-Brownian motion and its quadratic variation process is presented. The rate function is not a quadratic form due to quadratic variation uncertainty. A large deviation principle for stochastic differential equations driven by G-Brownian motion is also established.

MSC: 60F10; 60H10; 60J65

Keywords: Large deviations; G-Brownian motion; G-stochastic differential equation

1. Introduction

Peng [11] introduced G-Brownian motion. The expectation $\mathbb{E}[\cdot]$ associated with the G-Brownian motion is a sublinear expectation which is called G-expectation. The stochastic calculus with respect to the G-Brownian motion has been established (cf. [11,14,12]). The existence and uniqueness of the solution for stochastic differential equations driven by G-Brownian motion in the space $M_G^2(0,T)$ have also been obtained by the contracting mapping theorem (cf. [14]). The Hölder continuity and the homeomorphic property of the solution for stochastic differential equations driven by G-Brownian motion are established in [9].

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The aim of this paper is to study large deviations for G-Brownian motion and stochastic differential equations driven by G-Brownian motion.

The paper is organized as follows. In Section 2, we present a joint large deviation principle for G-Brownian motion and its quadratic variation process and obtain a representation of the rate function. The large deviations for stochastic differential equations driven by G-Brownian motion are given in Section 3. The moment estimates for G-stochastic integrals play an important role in this paper. A brief introduction and some general results on large deviations for G-capacity are in Appendix.

We conclude this section with some notations on G-expectation.

For convenience, we briefly recall some basic conceptions and results about G-Brownian motion and G-stochastic integrals (see [6,11,14,12] for details). Let Ω denote the space of all $\mathbb{R}^d$-valued continuous paths $\omega : [0, +\infty) \ni t \longmapsto \omega_t \in \mathbb{R}^d$, with $\omega_0 = 0$, equipped with the distance $\rho(\omega^1, \omega^2) := \sum_{n=1}^{\infty} 2^{-n} (\max_{t\in[0,n]} |\omega_t^1 - \omega_t^2| \wedge 1)$. For each $T > 0$, set

$$L_{\text{ip}}(\mathcal{F}_T) := \{\varphi(\omega_{t_1}, \omega_{t_2}, \ldots, \omega_{t_n}) : n \geq 1, t_1, \ldots, t_n \in [0, T], \varphi \in \text{lip}(\mathbb{R}^{d\times n})\},$$

where $\text{lip}(\mathbb{R}^{d\times n})$ is the set of bounded Lipschitz continuous functions on $\mathbb{R}^{d\times n}$. Define $L_{ip}(\mathcal{F}) := \bigcup_{n=1}^{\infty} L_{ip}(\mathcal{F}_n) \subset C_b(\Omega) \cdot \mathbb{S}^d$ denotes the space of $d \times d$ symmetric matrices. Γ is a given nonempty, bounded and closed subset of $\mathbb{R}^{d\times d}$ which is the space of all $d \times d$ matrices. Set $\Sigma := \{\gamma\gamma^\tau, \gamma \in \Gamma\} \subset \mathbb{S}^d$ and assume that Σ is a bounded, convex and closed subset. For $A = (A_{ij})_{i,j=1}^d \in \mathbb{S}^d$ given, set

$$G(A) = \frac{1}{2} \sup_{\gamma\in\Gamma} \text{tr}[\gamma\gamma^\tau A]. \tag{1.1}$$

For each $\varphi \in \text{lip}(\mathbb{R}^d)$, define

$$\mathbb{E}(\varphi) = u(1, 0)$$

where $u(t, x)$ is the viscosity solution of the following G-heat equation:

$$\frac{\partial u}{\partial t} - G(D^2 u) = 0, \quad \text{on } (t, x) \in [0, \infty) \times \mathbb{R}^d, \qquad u(0, x) = \varphi(x), \tag{1.2}$$

and $D^2 u$ is the Hessian matrix of u, i.e., $D^2 u = (\partial^2_{x_i x_j} u)_{i,j=1}^d$. Then $\mathbb{E} : \text{lip}(\mathbb{R}^d) \mapsto \mathbb{R}$ is a sublinear expectation. This sublinear expectation is also called G-normal distribution on $\mathbb{R}^d$ and denoted by $N(0, \Sigma)$ (cf. [13]).

Let $\mathcal{H}$ be a vector lattice of real functions defined on Ω such that $L_{ip}(\mathcal{F}) \subset \mathcal{H}$ and if $X_1, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in \text{lip}(\mathbb{R}^n)$. Let $\mathbb{E}[\cdot] : \mathcal{H} \mapsto \mathbb{R}$ be a sublinear expectation on $\mathcal{H}$. A d-dimensional random vector X with each component in $\mathcal{H}$ is said to be G-normal distributed under the sublinear expectation $\mathbb{E}[\cdot]$ if for each $\varphi \in \text{lip}(\mathbb{R}^d)$,

$$u(t, x) := \mathbb{E}(\varphi(x + \sqrt{t} X)), \quad t \geq 0, \ x \in \mathbb{R}^d$$

is the viscosity solution of the G-heat equation (1.2). $\mathbb{E}[\cdot]$ is called to be a G-expectation if the d-dimensional canonical process $\{B_t(\omega) = \omega_t, t \geq 0\}$ is a G-Brownian motion under the sublinear expectation, that is, $B_0 = 0$ and

(i) For any $s, t \geq 0$, $B_t \sim B_{t+s} - B_s \sim N(0, t\Sigma)$.

(ii) For any $m \geq 1, 0 = t_0 < t_1 < \cdots < t_m < \infty$, the increment $B_{t_m} - B_{t_{m-1}}$ is independent from $B_{t_1}, \ldots, B_{t_{m-1}}$, i.e., for each $\varphi \in \mathrm{lip}(\mathbb{R}^{d\times m})$,

$$\mathbb{E}(\varphi(B_{t_1}, \ldots, B_{t_{m-1}}, B_{t_m} - B_{t_{m-1}})) = \mathbb{E}(\psi(B_{t_1}, \ldots, B_{t_{m-1}})) \tag{1.3}$$

where $\psi(x_1, \ldots, x_{m-1}) = \mathbb{E}(\varphi(x_1, \ldots, x_{m-1}, B_{t_m} - B_{t_{m-1}}))$. For any $\mathbf{a} = (a_1, \ldots, a_d)^\tau \in \mathbb{R}^d$, $B_t^{\mathbf{a}} := \sum_{i=1}^d a_i B_t^i$ is a one-dimensional $G_{\mathbf{a}}$-Brownian motion, where B^i denotes the ith coordinate of the G-Brownian motion B. Define

$$G_{\mathbf{a}}(\beta) = \frac{1}{2}\sup_{\gamma\in\Gamma} \mathrm{tr}(\beta\gamma\gamma^\tau \mathbf{a}\mathbf{a}^\tau) = \frac{1}{2}(\sigma_{\mathbf{a}\mathbf{a}^\tau}\beta^+ - \sigma_{-\mathbf{a}\mathbf{a}^\tau}\beta^-), \quad \beta \in \mathbb{R}, \tag{1.4}$$

and

$$\sigma_{\mathbf{a}\mathbf{a}^\tau} = \sup_{\gamma\in\Gamma} \mathrm{tr}(\gamma\gamma^\tau \mathbf{a}\mathbf{a}^\tau), \qquad \sigma_{-\mathbf{a}\mathbf{a}^\tau} = -\sup_{\gamma\in\Gamma} \mathrm{tr}(-\gamma\gamma^\tau \mathbf{a}\mathbf{a}^\tau).$$

Let $\mathbb{E}$ be a G-expectation on $\mathcal{H}$. The topological completion of $L_{ip}(\mathcal{F}_T)$ (resp. $L_{ip}(\mathcal{F})$) under the Banach norm $\mathbb{E}[|\cdot|]$ is denoted by $L_G^1(\mathcal{F}_T)$ (resp. $L_G^1(\mathcal{F})$). $\mathbb{E}[\cdot]$ can be extended uniquely to a sublinear expectation on $L_G^1(\mathcal{F})$. We denote by $\mathbb{E}$ the extension.

Let P be the Wiener measure on Ω. Let $\mathcal{A}_{0,\infty}^\Gamma$ be the collection of all Γ-valued $\{\mathcal{F}_t, t \geq 0\}$-adapted processes on the interval $[0, +\infty)$, i.e., $\{\theta_t, t \geq 0\} \in \mathcal{A}_{0,\infty}^\Gamma$ if and only if θ_t is $\mathcal{F}_t := \sigma(\omega_s, s \leq t)$ measurable and $\theta_t \in \Gamma$ for each $t \geq 0$, and let P_θ be the law of the process $\left\{\int_0^t \theta_s d\omega_s, t \geq 0\right\}$ under the Wiener measure P.

We denote $\mathcal{P} = \{P_\theta : \theta \in \mathcal{A}_{0,\infty}^\Gamma\}$ and define

$$\bar{C}(A) := \sup_{\theta\in\mathcal{A}_{0,\infty}^\Gamma} P_\theta(A), \quad A \in \mathcal{B}(\Omega). \tag{1.5}$$

Then $\mathcal{P}$ is tight and $\bar{C}(\cdot)$ is a Choquet capacity (see Theorem 1 in [6]). For each $X \in L^0(\Omega) := \{X : X \in \mathcal{B}(\Omega)\}$ (the space of all Borel measurable real functions on Ω) such that $E_{P_\theta}(X)$ exists for each $\theta \in \mathcal{A}_{0,\infty}^\Gamma$, set

$$\bar{\mathbb{E}}(X) := \sup_{\theta\in\mathcal{A}_{0,\infty}^\Gamma} E_{P_\theta}(X). \tag{1.6}$$

Then (Theorem 59 in [6]) for all $X \in L_G^1(\mathcal{F})$,

$$\mathbb{E}X = \bar{\mathbb{E}}X. \tag{1.7}$$

The quadratic variation process $\langle B^{\mathbf{a}}\rangle_t$ of the process $B^{\mathbf{a}}$ is defined by

$$\langle B^{\mathbf{a}}\rangle_t = (B_t^{\mathbf{a}})^2 - 2\int_0^t B_s^{\mathbf{a}} \mathrm{d}B_s^{\mathbf{a}}. \tag{1.8}$$

$\{\langle B^{\mathbf{a}}\rangle_t, t \geq 0\}$ is an increasing process with $\langle B^{\mathbf{a}}\rangle_0 = 0$. For each fixed $s \geq 0$,

$$\langle B^{\mathbf{a}}\rangle_{t+s} - \langle B^{\mathbf{a}}\rangle_s = \langle (B^s)^{\mathbf{a}}\rangle_t,$$

where $B_t^s = B_{t+s} - B_s, t \geq 0, (B^s)_t^{\mathbf{a}} = (\mathbf{a}, B_t^s)$, and $(x, y) = \sum_{i=1}^d x_i y_i$ for $x, y \in \mathbb{R}^d$. Set $(\langle B\rangle_t)_{ij} = \langle B^i, B^j\rangle_t$. Then by Corollary 5.3.19 in [12],

$$\langle B\rangle_t \in t\Sigma = \{t \times \gamma\gamma^\tau; \gamma \in \Gamma\}. \tag{1.9}$$

Therefore, for any $0 \leq s \leq t$

$$\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s \leq \sigma_{\mathbf{a}\mathbf{a}^\tau}(t-s). \tag{1.10}$$

Throughout this paper, we assume that there exist constants $0 < \underline{\sigma} \leq \overline{\sigma} < \infty$ such that

$$\Gamma \subset \{\gamma \in \mathbb{R}^{d\times d}; \underline{\sigma} I_{d\times d} \leq \gamma\gamma^\tau \leq \overline{\sigma} I_{d\times d}\}. \tag{1.11}$$

2. Large deviation for G-Brownian motion and its quadratic variation

In this section we prove the LDP for G-Brownian motion and its quadratic variation process by using sub-additive method (cf. [1]) and give a representation of the rate function by the Varadhan integral lemma.

2.1. Finite-dimensional large deviations

The following lemma is a simple fact about sub-additive functions.

Lemma 2.1. *Let $f : \mathbb{Z}_+ \to \mathbb{R}^1 \cup \{+\infty\}$ be a sub-additive function (i.e. $f(m+k) \leq f(m) + f(k)$, for all $m, k \in \mathbb{Z}_+$). If there exists some $m_0 \in \mathbb{R}_+$ such that for any $m \geq m_0$, $f(m) < +\infty$, then $\lim_{m\to\infty} \frac{f(m)}{m}$ exists.*

Lemma 2.2. *For any $t \in (0, T]$, $N \in \mathbb{N}$ and open convex subset $A \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d\times d})$, define*

$$f_{A,t}(N) = -\log \bar{C}\left(\frac{1}{N}(B_{Nt}, \langle B\rangle_{Nt}) \in A\right).$$

Then $f_{A,t}(\cdot)$ is a sub-additive function and $\lim_{N\to\infty} \frac{1}{N} f_{A,t}(N)$ exists.

Proof. For $1 \leq k \leq N$, set

$$X_k = (B_{kt} - B_{(k-1)t}, \langle B\rangle_{kt} - \langle B\rangle_{(k-1)t})$$

and

$$\bar{X}_N = \frac{1}{N}\sum_{k=1}^N X_k, \qquad \bar{X}^M_{M+N} = \frac{1}{N}\sum_{k=1}^N X_{M+k}.$$

Then

$$\frac{1}{N}(B_{Nt}, \langle B\rangle_{Nt}) = \frac{1}{N}\sum_{k=1}^N X_k,$$

and $\bar{X}_{M+N} = \frac{M}{M+N}\bar{X}_M + \frac{N}{M+N}\bar{X}^M_{M+N}$ which implies $\{\bar{X}_M \in A\} \cap \{\bar{X}^M_{M+N} \in A\} \subset \{\bar{X}_{M+N} \in A\}$ since A is a convex subset.

Because $A \subset \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d\times d})$ is an open subset, we can choose a sequence of functions $f_n \in L_{ip}(\mathbb{R}^d \times \mathbb{R}^{d\times d})$ such that $0 \leq f_n \uparrow I_A$. Then, by the definition of $\bar{C}$ and Theorem 59 in [6], we have

$$\begin{aligned}\bar{C}(\bar{X}_M \in A, \bar{X}^M_{M+N} \in A) &= \lim_{n\to\infty} \bar{\mathbb{E}}(f_n(\bar{X}_M) f_n(\bar{X}^M_{M+N}))\\ &= \lim_{n\to\infty} \mathbb{E}(f_n(\bar{X}_M) f_n(\bar{X}^M_{M+N}))\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \mathbb{E} f_n(\bar{X}_M)\mathbb{E} f_n(\bar{X}^M_{M+N}) \\
&= \lim_{n\to\infty} \bar{\mathbb{E}} f_n(\bar{X}_M)\bar{\mathbb{E}} f_n(\bar{X}^M_{M+N}) \\
&= \bar{C}(\bar{X}_M \in A)\bar{C}(\bar{X}^M_{M+N} \in A) \\
&= \bar{C}(\bar{X}_M \in A)\bar{C}(\bar{X}_N \in A).
\end{aligned}
$$

Consequently,

$$\bar{C}(\bar{X}_M \in A)\bar{C}(\bar{X}_N \in A) \le \bar{C}(\bar{X}_{M+N} \in A), \tag{2.1}$$

which proves that $f_{A,t}$ is sub-additive.

In view of Lemma 2.1, it remains to prove that either $f_{A,t} \equiv +\infty$ or $f_{A,t}(N) < +\infty$ for sufficiently large N. To this end, suppose that $\bar{C}(\bar{X}_M \in A) > 0$ for some M. Then $A \neq \phi$. Since $\bar{c}(A) := \bar{C}(\bar{X}_M \in A)$ is also a capacity on $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d\times d})$, there exists a compact subset $K \subset A$ such that $\bar{C}(\bar{X}_M \in K) > 0$. Let F be the closed convex hull of K. Then $F \subset A$ is also compact and $\bar{C}(\bar{X}_M \in F) > 0$. Set $L = \sup_{x\in F} |x|$ and choose $\delta > 0$ such that $\mathrm{dist}(F, A^c) > 2\delta$. Next, select $N_0 > M$ such that $\alpha = \min_{1\le k\le M} \bar{C}\left(\left|\frac{k}{N_0} X_k\right| < \delta\right) > 0$ and $\frac{k}{N_0} L \le \delta$ for all $0 < k \le M$. Then, for $N \ge M$, by the same method as in the proof of (2.1), we have

$$\bar{C}(\bar{X}_N \in A) \ge \bar{C}\left(\frac{Mq_N}{N}\bar{X}_{Mq_N} \in F^\delta\right)\bar{C}\left(\left|\frac{r_N}{N}\bar{X}_{r_N}\right| < \delta\right),$$

where $q_N = [N/M]$ and $r_N = N - [N/M]M$. Note that for $N \ge N_0$, $\bar{X}_{Mq_N} \in F$ implies that $\mathrm{dist}(\frac{Mq_N}{N}\bar{X}_{Mq_N}, F) \le \frac{r_N}{N} L < \delta$. Then, we have

$$\bar{C}\left(\frac{Mq_N}{N}\bar{X}_{Mq_N} \in F^\delta\right) \ge \bar{C}(\bar{X}_{Mq_N} \in F) \ge \left(\bar{C}(\bar{X}_M \in F)\right)^{q_N}.$$

Hence, for $N \ge N_0$, $\bar{C}(\bar{X}_N \in A) \ge \alpha\left(\bar{C}(\bar{X}_M \in F)\right)^{q_N} > 0$. □

We now define:

$$\lambda_t(x) = -\lim_{\delta\to 0}\lim_{N\to\infty}\frac{1}{N}\log\bar{C}\left(\frac{1}{N}(B_{Nt}, \langle B\rangle_{Nt}) \in B(x,\delta)\right), \tag{2.2}$$

where $B(x,\delta) = \{y, |x-y| < \delta\}$. Then it is easy to get that $\lambda_t(x)$, $x \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$ is a lower semicontinuous and convex function.

Lemma 2.3. *For any open set $O \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d\times d})$,*

$$\liminf_{N\to\infty}\frac{1}{N}\log\bar{C}\left(\frac{1}{N}(B_{Nt}, \langle B\rangle_{Nt}) \in O\right) \ge -\inf_{x\in O}\lambda_t(x)$$

and for any compact subset $K \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d\times d})$,

$$\limsup_{N\to\infty}\frac{1}{N}\log\bar{C}\left(\frac{1}{N}(B_{Nt}, \langle B\rangle_{Nt}) \in K\right) \le -\inf_{x\in K}\lambda_t(x).$$

Proof. Let $O \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d\times d})$ be an open subset. Given $x \in O$, we can choose an open convex B such that $x \in B \subset O$. Then, we have

$$-\lambda_t(x) \le \lim_{N\to\infty}\frac{1}{N}\log\bar{C}\left(\frac{1}{N}(B_{Nt}, \langle B\rangle_{Nt}) \in B\right)$$

$$\leq \liminf_{N\to\infty} \frac{1}{N} \log \bar{C}\left(\frac{1}{N}(B_{Nt}, \langle B\rangle_{Nt}) \in O\right).$$

Hence

$$\liminf_{N\to\infty} \frac{1}{N} \log \bar{C}\left(\frac{1}{N}(B_{Nt}, \langle B\rangle_{Nt}) \in O\right) \geq -\inf_{x\in O} \lambda_t(x).$$

Next, let $K \in \mathcal{B}\left(\mathbb{R}^d \times \mathbb{R}^{d\times d}\right)$ be a compact subset. For any $\varepsilon > 0$, for each $x \in K$, choose $\delta_x > 0$ such that

$$\lim_{N\to\infty} \frac{1}{N} \log \bar{C}\left(\frac{1}{N}(B_{Nt}, \langle B\rangle_{Nt}) \in B(x, \delta_x)\right) \leq -\lambda_t(x) + \varepsilon.$$

Select finite points $x_1, \ldots, x_l \in K$ such that $K \subset \bigcup_{i=1}^{l} B(x_i, \delta_{x_i})$. Then

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N} \log \bar{C}\left(\frac{1}{N}(B_{Nt}, \langle B\rangle_{Nt}) \in K\right) \\
&\quad \leq \limsup_{N\to\infty} \frac{1}{N} \log\left(N\left(\max_{1\leq i\leq l} \bar{C}\left(\frac{1}{N}(B_{Nt}, \langle B\rangle_{Nt}) \in B(x_i, \delta_{x_i})\right)\right)\right) \\
&\quad \leq \max_{1\leq i\leq l} \limsup_{N\to\infty} \frac{1}{N} \log \bar{C}\left(\frac{1}{N}(B_{Nt}, \langle B\rangle_{Nt}) \in B(x_i, \delta_{x_i})\right) \\
&\quad \leq -\inf_{x\in K} \lambda_t(x) + \varepsilon
\end{aligned}$$

which implies

$$\limsup_{N\to\infty} \frac{1}{N} \log \bar{C}\left(\frac{1}{N}(B_{Nt}, \langle B\rangle_{Nt}) \in K\right) \leq -\inf_{x\in K} \lambda_t(x). \quad \square$$

Lemma 2.4 (*Exponential Inequality*). *Let $B = \{B_t = (B_t^1, \ldots, B_t^d), t \geq 0\}$ be a d-dimensional G-Brownian motion. Then for any $T_1 < T_2$ and $r > 0$,*

$$\bar{C}\left(\sup_{T_1\leq t\leq T_2} |B_t - B_{T_1}| \geq r\right) \leq d\mathrm{e}^{-\frac{r^2}{\bar{\sigma}(T_2-T_1)d}}.$$

Proof. By the upper-expectation representation (1.7) of the G-expectation, it is easy to get

$$\bar{C}\left(\sup_{T_1\leq t\leq T_2} |B_t - B_{T_1}| \geq r\right) = \sup_{\theta\in\mathcal{A}_{0,\infty}^{\Gamma}} E_{P_\theta}\left(\sup_{T_1\leq t\leq T_2} \left|\int_{T_1}^{t} \theta_s \mathrm{d}B_s\right| \geq r\right).$$

Now one can get the conclusion of this lemma by the maximum inequality of martingale. $\square$

Lemma 2.5. *Let $m \geq 1$ and $0 = t_0 < t_1 < \cdots < t_m = T$ be fixed. For $N \in \mathbb{N}, A \in \mathcal{B}((\mathbb{R}^d \times \mathbb{R}^{d\times d})^m)$, define*

$$\bar{C}_{N,m}(A) := \bar{C}\left(\frac{1}{N}\left(\left(B_{Nt_1}, \langle B\rangle_{Nt_1}\right), \ldots, \left(B_{Nt_m} - B_{Nt_{m-1}}, \langle B\rangle_{Nt_m} - \langle B\rangle_{Nt_{m-1}}\right)\right) \in A\right).$$

Then for any open subset $O \in \mathcal{B}((\mathbb{R}^d \times \mathbb{R}^{d\times d})^m)$,

$$\liminf_{N\to\infty} \frac{1}{N} \log \bar{C}_{N,m}(O) \geq -\inf_{x\in O} \lambda_{t_1,\ldots,t_m}(x)$$

and for any closed subset $F \in \mathcal{B}((\mathbb{R}^d \times \mathbb{R}^{d\times d})^m)$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \bar{C}_{N,m}(F) \le -\inf_{x\in F} \lambda_{t_1,\dots,t_m}(x),$$

where $\lambda_{t_1,\dots,t_m}(x) = \sum_{i=1}^m \lambda_{t_i-t_{i-1}}(x_i)$, $x = (x_1,\dots,x_m)$, $x_i \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$.

Proof. We first prove the lower bound of large deviations (LLD). For any $x \in O$ with $\lambda_{t_1,\dots,t_m}(x) < \infty$, take $U := \Pi_{i=1}^m U_i \subset O$ such that $x = (x_1,\dots,x_m) \in U$, where U_i is open subset of $\mathbb{R}$. Choose a sequence of functions $0 \le g_{i,l} \in L_{ip}(\mathbb{R}^d \times \mathbb{R}^{d\times d})$ such that $0 \le g_{i,l} \uparrow I_{U_i}, l \to \infty$. By Theorems 57 and 59 in [6], and Proposition 16 in [14], we have

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N} \log \bar{C}_{N,m}(U)\\
&\quad = \liminf_{N\to\infty} \lim_{l\to\infty} \frac{1}{N} \log \bar{\mathbb{E}}\left(\Pi_{i=1}^m g_{i,l}\left((B_{Nt_i} - B_{Nt_{i-1}}, \langle B\rangle_{Nt_i} - \langle B\rangle_{Nt_{i-1}})/N\right)\right)\\
&\quad = \liminf_{N\to\infty} \lim_{l\to\infty} \frac{1}{N} \log \mathbb{E}\left(\Pi_{i=1}^m g_{i,l}\left((B_{Nt_i} - B_{Nt_{i-1}}, \langle B\rangle_{Nt_i} - \langle B\rangle_{Nt_{i-1}})/N\right)\right)\\
&\quad = \liminf_{N\to\infty} \lim_{l\to\infty} \frac{1}{N} \sum_{i=1}^m \log \mathbb{E}\left(g_{i,l}\left((B_{Nt_i} - B_{Nt_{i-1}}, \langle B\rangle_{Nt_i} - \langle B\rangle_{Nt_{i-1}})/N\right)\right)\\
&\quad = \liminf_{N\to\infty} \sum_{i=1}^m \frac{1}{N} \log \bar{C}\left((B_{Nt_i} - B_{Nt_{i-1}}, \langle B\rangle_{Nt_i} - \langle B\rangle_{Nt_{i-1}})/N \in U_i\right).
\end{aligned}$$

Hence, by Lemma 2.3,

$$\liminf_{N\to\infty} \frac{1}{N} \log \bar{C}_{N,m}(O) \ge -\sum_{i=1}^m \lambda_{t_i-t_{i-1}}(x_i) = -\lambda_{t_1,\dots,t_m}(x).$$

Now we show the weak upper bound of large deviations (w-ULD). For any $x = (x_1,\dots,x_m) \in (\mathbb{R}^d \times \mathbb{R}^{d\times d})^m$, $\delta > 0$, and $1 \le i \le n$, by Lemma 2.3, there exists a neighborhood $U_i(x_i)$ of x_i such that

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N} \log \bar{C}((B_{Nt_i} - B_{Nt_{i-1}}, \langle B\rangle_{Nt_i} - \langle B\rangle_{Nt_{i-1}})/N \in U_i(x_i))\\
&\quad \le \begin{cases} -\lambda_{t_i-t_{i-1}}(x_i) + \delta, & \text{if } \lambda_{t_i-t_{i-1}}(x_i) < +\infty\\ -1/\delta, & \text{otherwise.}\end{cases}
\end{aligned}$$

Set $U_x = U_1(x_1) \times U_2(x_2) \times \cdots \times U_m(x_m)$. Then U_x is a neighborhood of x. By the same method as in the proof of the LLD above, we have

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N} \log \bar{C}_{N,m}(U_x)\\
&\quad = \limsup_{N\to\infty} \frac{1}{N} \sum_{i=1}^m \log \bar{C}((B_{Nt_i} - B_{Nt_{i-1}}, \langle B\rangle_{Nt_i} - \langle B\rangle_{Nt_{i-1}})/N \in U_i(x_i))\\
&\quad \le \begin{cases} -\lambda_{t_1,\dots,t_m}(x) + m\delta, & \text{if } \lambda_{t_i-t_{i-1}}(x_i) < +\infty \text{ for all } 1 \le i \le m\\ -1/\delta, & \text{otherwise}\end{cases}
\end{aligned}$$

which implies that for any compact subset K,

$$\limsup_{N\to\infty} \frac{1}{N} \log \bar{C}_{N,m}(K) \le -\inf_{x\in K} \lambda_{t_1,\dots,t_m}(x).$$

Finally, let us prove the exponential tightness. It is enough to prove that for any $T_1 < T_2$,

$$\lim_{l\to\infty}\limsup_{N\to\infty}\frac{1}{N}\log\bar{C}\left(|(B_{NT_2}-B_{NT_1},\langle B\rangle_{NT_2}-\langle B\rangle_{NT_1})|\geq lN\right)=-\infty. \tag{2.3}$$

Since $|\langle B^i,B^j\rangle_t-\langle B^i,B^j\rangle_s|\leq\bar{\sigma}|t-s|$ for all $1\leq i,j\leq d$, (2.3) is a consequence of Lemma 2.4. □

Theorem 2.1. *Let $m\geq 1$ and $0=t_0<t_1<\cdots<t_m=T$ be fixed. For $\varepsilon>0, A\in\mathcal{B}\left(\left(\mathbb{R}^d\times\mathbb{R}^{d\times d}\right)^m\right)$, define*

$$\tilde{C}_{\varepsilon,m}(A)=\bar{C}\left(\left(\left(\varepsilon B_{\frac{t_1}{\varepsilon}},\varepsilon\langle B\rangle_{\frac{t_1}{\varepsilon}}\right),\ldots,\left(\varepsilon B_{\frac{t_m}{\varepsilon}},\varepsilon\langle B\rangle_{\frac{t_m}{\varepsilon}}\right)\right)\in A\right).$$

Then for any closed subset $F\in\mathcal{B}\left((\mathbb{R}^d\times\mathbb{R}^{d\times d})^m\right)$,

$$\limsup_{\varepsilon\to 0}\varepsilon\log\tilde{C}_{\varepsilon,m}(F)\leq-\inf_{x\in F}\sum_{i=1}^m\lambda_{t_i-t_{i-1}}(x_i-x_{i-1})$$

and for any open subset $O\in\mathcal{B}\left((\mathbb{R}^d\times\mathbb{R}^{d\times d})^m\right)$,

$$\liminf_{\varepsilon\to 0}\varepsilon\log\tilde{C}_{\varepsilon,m}(O)\geq-\inf_{x\in O}\sum_{i=1}^m\lambda_{t_i-t_{i-1}}(x_i-x_{i-1}),$$

where $x=(x_1,\ldots,x_m)$ and $x_0=0$.

Proof. By Lemma 2.5, we only need to prove that for any $\delta>0$,

$$\limsup_{\varepsilon\to 0}\frac{1}{[1/\varepsilon]}\log\bar{C}\left(\sup_{t\in[0,T]}\left|\left(\varepsilon B_{t/\varepsilon}-\frac{1}{[1/\varepsilon]}B_{[1/\varepsilon]t},\varepsilon\langle B\rangle_{t/\varepsilon}-\frac{1}{[1/\varepsilon]}\langle B\rangle_{[1/\varepsilon]t}\right)\right|\geq\delta\right)$$
$$=-\infty. \tag{2.4}$$

where $[x]$ denotes the largest integer less than or equal to x. Since $|\langle B^i,B^j\rangle_t-\langle B^i,B^j\rangle_s|\leq\bar{\sigma}|t-s|$ for all $1\leq i,j\leq d$, and

$$\begin{aligned}
&\bar{C}\left(\sup_{t\in[0,T]}\left|\left(\varepsilon B_{t/\varepsilon}-\frac{1}{[1/\varepsilon]}B_{[1/\varepsilon]t},\varepsilon\langle B\rangle_{t/\varepsilon}-\frac{1}{[1/\varepsilon]}\langle B\rangle_{[1/\varepsilon]t}\right)\right|\geq\delta\right)\\
&\leq\sum_{1\leq i\leq j\leq d}\bar{C}\left(\frac{1}{[1/\varepsilon]}\sup_{t\in[0,T]}\left|\langle B^i,B^j\rangle_{t/\varepsilon}-\langle B^i,B^j\rangle_{[1/\varepsilon]t}\right|\geq\delta/4d^2\right)\\
&\quad+\sum_{1\leq i\leq j\leq d}\bar{C}\left(\left(\frac{1}{[1/\varepsilon]}-\varepsilon\right)\sup_{0\leq t\leq T}\left|\langle B^i,B^j\rangle_{t/\varepsilon}\right|\geq\delta/4d^2\right)\\
&\quad+d\bar{C}\left(\left(\frac{1}{[1/\varepsilon]}-\varepsilon\right)\sup_{0\leq t\leq T}\left|B^1_{t/\varepsilon}\right|\geq\delta/4d\right)\\
&\quad+d\bar{C}\left(\frac{1}{[1/\varepsilon]}\sup_{0\leq t\leq T}\left|B^1_{t/\varepsilon}-B^1_{[1/\varepsilon]t}\right|\geq\delta/4d\right),
\end{aligned}$$

we can get (2.4) from Lemma 2.4. □

Corollary 2.1. *Let $m \geq 1$ and $0 = t_0 < t_1 < \cdots < t_m = T$ be fixed. For $\varepsilon > 0$, $A \in \mathcal{B}\left((\mathbb{R}^d)^m\right)$, define*

$$\check{C}_{\varepsilon,m}(A) = \bar{C}\left(\left(\varepsilon B_{\frac{t_1}{\varepsilon}}, \ldots, \varepsilon B_{\frac{t_m}{\varepsilon}}\right) \in A\right).$$

Then for any closed subset $F \in \mathcal{B}\left((\mathbb{R}^d)^m\right)$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \check{C}_{\varepsilon,m}(F) \leq - \inf_{x\in F} \check{I}_{t_1,\ldots,t_m}(x)$$

and for any open subset $O \in \mathcal{B}\left((\mathbb{R}^d)^m\right)$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \check{C}_{\varepsilon,m}(O) \geq - \inf_{x\in O} \check{I}_{t_1,\ldots,t_m}(x),$$

where

$$\check{I}_{t_1,\ldots,t_m}(x) = \inf_{y\in(\mathbb{R}^{d\times d})^m} \sum_{i=1}^{m} \lambda_{t_i - t_{i-1}}((x_i - x_{i-1}, y_i - y_{i-1})) \tag{2.5}$$

and $x_0 = 0$, $y_0 = 0$.

2.2. Exponential tightness

Lemma 2.6. *For any $0 < \beta < 1/2$, there exists a positive constant c such that*

$$\sup_{0<\varepsilon\leq 1} \bar{\mathbb{E}}\left(\exp\left\{c \sup_{s,t\in[0,T]} \frac{\varepsilon|B_{t/\varepsilon} - B_{s/\varepsilon}|^2}{|s-t|^{2\beta}}\right\}\right) < \infty.$$

Proof. This can be obtained from Theorem 3.1 in [9]. Now we give a direct proof. Without loss of generality we assume B_t is one-dimensional and $\bar{\sigma} = 1$. For each $\theta \in \mathcal{A}^{\Gamma}_{0,\infty}$, set $A^\theta_t = \int_0^t |\theta_s|^2 \mathrm{d}s$. Then for some constant $C > 0$,

$$\begin{aligned}\sup_{s,t\in[0,T]} \frac{\left|\int_s^t \theta_u \mathrm{d}B_u\right|}{|s-t|^{2\beta}} &= \sup_{s,t\in[0,T]} \frac{|B_{A^\theta_t} - B_{A^\theta_s}|}{|s-t|^{2\beta}} \\ &= \sup_{s,t\in[0,T]} \frac{|B_{A^\theta_t} - B_{A^\theta_s}|}{|A^\theta_t - A^\theta_s|^{2\beta}} \frac{|A^\theta_t - A^\theta_s|^{2\beta}}{|s-t|^{2\beta}} \leq C \sup_{s,t\in[0,T]} \frac{|B_t - B_s|}{|s-t|^{2\beta}}.\end{aligned}$$

Therefore, by (1.7), we have

$$\begin{aligned}\bar{\mathbb{E}}\left(\exp\left\{c \sup_{s,t\in[0,T]} \frac{\varepsilon|B_{t/\varepsilon} - B_{s/\varepsilon}|^2}{|s-t|^{2\beta}}\right\}\right) &\leq E_P\left(\exp\left\{c \sup_{s,t\in[0,T]} \frac{\varepsilon|B_{t/\varepsilon} - B_{s/\varepsilon}|^2}{|s-t|^{2\beta}}\right\}\right) \\ &= E_P\left(\exp\left\{c \sup_{s,t\in[0,T]} \frac{|B_t - B_s|^2}{|s-t|^{2\beta}}\right\}\right).\end{aligned}$$

Since

$$E_P(|B_t - B_s|^{2m}) = |t-s|^m (2m-1)!!,$$

we have that for any $\delta > 0$ with $4\delta e < 1$, for all $s, t \in [0, T]$,

$$E_P\left(\exp\left\{\frac{\delta|B_t - B_s|^2}{|t-s|}\right\}\right) \le \frac{2}{1-4\delta e}$$

which implies from (3.1) in [9] that the conclusion of the lemma holds. □

For $0 \le \alpha < 1$ given and $m \ge 1$, for each $\psi \in C_0([0, T], \mathbb{R}^m)$, set

$$\|\psi\|_\alpha = \sup_{s,t\in[0,T]} \frac{|\psi(t) - \psi(s)|}{|s-t|^\alpha}$$

and

$$C_0^\alpha([0, T], \mathbb{R}^m) = \left\{\psi \in C_0([0, T], \mathbb{R}^m);\ \lim_{\delta\to 0} \sup_{|s-t|<\delta} \frac{|\psi(t) - \psi(s)|}{|s-t|^\alpha} = 0, \|\psi\|_\alpha < \infty\right\}.$$

Then $(C_0^\alpha([0, T], \mathbb{R}^m), \|\cdot\|_\alpha)$ is a separable Banach space.

Theorem 2.2. *Let $0 \le \alpha < 1/2$ and $0 \le \beta < 1$. Let B be G-Brownian motion and let $\langle B\rangle = \langle B, B\rangle = (\langle B^i, B^j\rangle)_{1\le i,j\le d}$ be its quadratic variation process. Then*

$$\left\{\bar{C}\left(\left(\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}\right)_{0\le t\le T} \in \cdot\right)\right\}$$

is exponentially tight in $(C_0^\alpha([0, T], \mathbb{R}^d), \|\cdot\|_\alpha) \times (C_0^\beta([0, T], \mathbb{R}^{d\times d}), \|\cdot\|_\beta)$.

Proof. Take $0 \le \alpha < \alpha' < 1/2$ and $0 \le \beta < \beta' < 1$. Set

$$K_l = \Bigg\{(f, g) \in (C_0^\alpha([0, T], \mathbb{R}^d), \|\cdot\|_\alpha) \times (C_0^\beta([0, T], \mathbb{R}^{d\times d}), \|\cdot\|_\beta);$$
$$\sup_{s,t\in[0,T]} \frac{|f(t) - f(s)|}{|t-s|^{\alpha'}} \le l,\ \sup_{s,t\in[0,T]} \frac{|g(t) - g(s)|}{|t-s|^{\beta'}} \le l\Bigg\}.$$

Then by Lemma 4.1 in [9], K_l is compact in $(C_0^\alpha([0, T], \mathbb{R}^d), \|\cdot\|_\alpha) \times (C_0^\beta([0, T], \mathbb{R}^{d\times d}), \|\cdot\|_\beta)$ for all $l < \infty$. Since

$$\bar{C}\left(\left(\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}\right)|_{t\in[0,T]} \in K_l^c\right) \le \bar{C}\left(\sup_{s,t\in[0,T]} \frac{\varepsilon|B_{t/\varepsilon} - B_{s/\varepsilon}|}{|t-s|^{\alpha'}} \ge \frac{l}{2}\right)$$
$$+ \bar{C}\left(\sup_{s,t\in[0,T]} \frac{\varepsilon|\langle B\rangle_{t/\varepsilon} - \langle B\rangle_{s/\varepsilon}|}{|t-s|^{\beta'}} \ge \frac{l}{2}\right).$$

From $|\langle B^i, B^j\rangle_t - \langle B^i, B^j\rangle_s| \le \bar{\sigma}|t-s|$, we have

$$\lim_{l\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \bar{C}\left(\sup_{s,t\in[0,T]} \frac{\varepsilon|\langle B\rangle_{t/\varepsilon} - \langle B\rangle_{s/\varepsilon}|}{|t-s|^{\beta'}} \ge \frac{l}{2}\right) = -\infty.$$

By Lemma 2.6 and Chebyshev's inequality, we have

$$\lim_{l\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \bar{C}\left(\sup_{s,t\in[0,T]} \frac{\varepsilon|B_{t/\varepsilon} - B_{s/\varepsilon}|}{|t-s|^{\alpha'}} \ge \frac{l}{2}\right) = -\infty.$$

Therefore,

$$\lim_{l\to+\infty}\limsup_{\varepsilon\to 0}\varepsilon\log\bar{C}\left(\left(\varepsilon B_{t/\varepsilon},\varepsilon\langle B\rangle_{t/\varepsilon}\right)|_{t\in[0,T]}\in K_l^c\right)=-\infty. \quad \square$$

2.3. Large deviations for G-Brownian motion and its quadratic variation process

By Theorems 2.1 and 2.2, we obtain the following large deviation principle (LDP). We will give a representation of the rate function in next subsection.

Theorem 2.3. *Let $0\le\alpha<1/2$ and $0\le\beta<1$. Let B be G-Brownian motion and let $\langle B\rangle=\langle B,B\rangle=(\langle B^i,B^j\rangle)_{1\le i,j\le d}$ be its quadratic variation process. Then for any closed subset F in $(C_0^\alpha([0,T],\mathbb{R}^d),\|\cdot\|_\alpha)\times(C_0^\beta([0,T],\mathbb{R}^{d\times d}),\|\cdot\|_\beta)$*

$$\limsup_{\varepsilon\to 0}\varepsilon\log\bar{C}\left(\left(\varepsilon B_{t/\varepsilon},\varepsilon\langle B\rangle_{t/\varepsilon}\right)|_{t\in[0,T]}\in F\right)\le-\inf_{(f,g)\in F}J(f,g), \tag{2.6}$$

for any open subset O in $(C_0^\alpha([0,T],\mathbb{R}^d),\|\cdot\|_\alpha)\times(C_0^\beta([0,T],\mathbb{R}^{d\times d}),\|\cdot\|_\beta)$,

$$\liminf_{\varepsilon\to 0}\varepsilon\log\bar{C}\left(\left(\varepsilon B_{t/\varepsilon},\varepsilon\langle B\rangle_{t/\varepsilon}\right)|_{t\in[0,T]}\in O\right)\ge-\inf_{(f,g)\in O}J(f,g), \tag{2.7}$$

where

$$J(f,g)=\sup_{\substack{0=t_0<t_1<\cdots<t_m=T\\ m\ge 1}}\sum_{l=1}^m\lambda_{t_l-t_{l-1}}\left(f(t_l)-f(t_{l-1}),g(t_l)-g(t_{l-1})\right). \tag{2.8}$$

Corollary 2.2. *Let $0\le\alpha<1/2$. Then for any closed subset F in $\left(C_0^\alpha([0,T],\mathbb{R}^d),\|\cdot\|_\alpha\right)$,*

$$\limsup_{\varepsilon\to 0}\varepsilon\log\bar{C}\left((\varepsilon B_{t/\varepsilon})|_{t\in[0,T]}\in F\right)\le-\inf_{f\in F}I_B(f),$$

and for any open subset O in $\left(C_0^\alpha([0,T],\mathbb{R}^d),\|\cdot\|_\alpha\right)$,

$$\liminf_{\varepsilon\to 0}\varepsilon\log\bar{C}\left((\varepsilon B_{t/\varepsilon})|_{t\in[0,T]}\in O\right)\ge-\inf_{f\in O}I_B(f),$$

where

$$I_B(f)=\inf_{g\in C_0([0,T],\mathbb{R}^{d\times d})}J(f,g) \tag{2.9}$$

and

$$I_B(f)=\sup_{\substack{0=t_0<t_1<\cdots<t_m=T\\ m\ge 1}}\check{I}_{t_1,\ldots,t_m}(f(t_1),\ldots,f(t_m)). \tag{2.10}$$

2.4. Rate functions

Lemma 2.7. *For any $\mu=(\mu_1,\ldots,\mu_d)^\tau\in\mathbb{R}^d$, $\nu=(\nu_{ij})_{1\le i,j\le d}\in\mathbb{R}^{d\times d}$ and $0\le s<t$,*

$$\exp\{(\mu,B_t-B_s)+(\nu,\langle B\rangle_t-\langle B\rangle_s)\}\in L_G^1(\Omega),$$

where $(\nu,\langle B\rangle_t):=\sum_{i,j=1}^d\nu_{ij}\langle B^i,B^j\rangle_t$.

Proof. Since for any $\delta > 0$,

$$\bar{\mathbb{E}} \exp\{\delta|B_t|\} < +\infty, \qquad \bar{\mathbb{E}} \exp\left\{\delta|\langle B^i, B^j\rangle_t|\right\} \le \exp\{\delta\bar{\sigma}t\} < +\infty$$

by Proposition 25 in [6] and quasi-continuity of

$$\exp\{(\mu, B_t - B_s) + (\nu, \langle B\rangle_t - \langle B\rangle_s)\},$$

we obtain the conclusion of the lemma. □

Define

$$\mathbb{H}^d = \left\{ f \in C_0([0,T], \mathbb{R}^d);\ f \text{ is absolutely continuous and } \|f\|_H^2 \right.$$
$$\left. := \int_0^T |f'(s)|^2 \mathrm{d}s < +\infty \right\}, \tag{2.11}$$

and

$$\mathbb{A} = \left\{ g = (g_{ij})_{d\times d} \in C_0([0,T], \mathbb{R}^{d\times d}) : \frac{g(t)-g(s)}{t-s} \in \Sigma \text{ for all } t > s \right\}$$
$$= \left\{ g = \int_0^t g'(s)\mathrm{d}s;\ g' : [0,T] \mapsto \mathbb{R}^{d\times d} \text{ Borel measurable and} \right.$$
$$\left. g'(t) \in \Sigma \text{ for all } t \in [0,T] \right\}. \tag{2.12}$$

Then $\mathbb{A}$ is a closed subset in $(C_0^\beta([0,T], \mathbb{R}^{d\times d}), \|\cdot\|_\beta)$ for any $\beta \in [0,1)$ and by the condition (1.11), $\bar{C}(\varepsilon\langle B\rangle_{\cdot/\varepsilon} \notin \mathbb{A}) = 0$. Therefore, by the lower bound of large deviations, for any $g \notin \mathbb{A}$, $J(f,g) = \infty$. Since $I_B(f) = \inf_{g\in C_0([0,T],\mathbb{R}^{d\times d})} J(f,g)$ and $I_B(f) = \infty$ for all $f \notin \mathbb{H}^d$, we also have $J(f,g) = \infty$ for all $f \notin \mathbb{H}^d$. Thus

$$J(f,g) = +\infty, \quad \text{for all } (f,g) \notin \mathbb{H}^d \times \mathbb{A}. \tag{2.13}$$

Lemma 2.8. *For any $\mu = (\mu_1, \ldots, \mu_d)^\tau$, $\nu = (\nu_{ij})_{1\le i,j\le d} \in \mathbb{R}^{d\times d}$,*

$$\mathbb{E}\exp\{(\mu, B_t) + (\nu, \langle B\rangle_t)\} = \exp\left\{G\left(\mu\mu^\tau + 2\nu^s\right)t\right\} \tag{2.14}$$

where $\nu^s := (\nu^s_{ij})_{d\times d}$ is a $d\times d$ symmetric matrix with $\nu^s_{ij} = \frac{\nu_{ij}+\nu_{ji}}{2}$.

Proof. By the independence of increments of G-Brownian motion, we obtain that

$$\mathbb{E}\exp\{(\mu, B_t) + (\nu, \langle B\rangle_t)\} = \prod_{i=1}^n \mathbb{E}\exp\left\{\left(\mu, B_{\frac{i}{n}t - \frac{i-1}{n}t}\right) + \left(\nu, \langle B\rangle_{\frac{i}{n}t - \frac{i-1}{n}t}\right)\right\}$$
$$= \left(\mathbb{E}\exp\left\{\left(\mu, B_{\frac{t}{n}}\right) + \left(\nu, \langle B\rangle_{\frac{t}{n}}\right)\right\}\right)^n$$
$$= \left(\mathbb{E}\left(1 + \left(\mu, B_{\frac{t}{n}}\right) + \left(\nu, \langle B\rangle_{\frac{t}{n}}\right) + \frac{1}{2}\left(\mu, B_{\frac{t}{n}}\right)^2 + R_n(t)\right)\right)^n$$
$$= \left(\mathbb{E}\left(1 + \left(\nu, \langle B\rangle_{\frac{t}{n}}\right) + \frac{1}{2}\left(\mu, B_{\frac{t}{n}}\right)^2 + R_n(t)\right)\right)^n,$$

where $R_n(t) = \mathrm{e}^{\left(\mu, B_{\frac{t}{n}}\right)+\left(\nu,\langle B\rangle_{\frac{t}{n}}\right)} - \left(1+\left(\mu, B_{\frac{t}{n}}\right)+\left(\nu, \langle B\rangle_{\frac{t}{n}}\right)+\frac{1}{2}\left(\mu, B_{\frac{t}{n}}\right)^2\right)$. Applying Taylor formula to the function e^x at the point 0, we can obtain that

$$|R_n(t)| \le \left(\left|\left(\mu, B_{\frac{t}{n}}\right)\right|^3 + \left|\left(\mu, B_{\frac{t}{n}}\right)\left(\nu, \langle B\rangle_{\frac{t}{n}}\right)\right| + \left(\nu, \langle B\rangle_{\frac{t}{n}}\right)^2\right) \times \exp\left\{\left|\left(\mu, B_{\frac{t}{n}}\right)\right| + \left|\left(\nu, \langle B\rangle_{\frac{t}{n}}\right)\right|\right\}.$$

By Hölder inequality it is easy to get $\mathbb{E}|R_n(t)| = o(1/n)$. Then by

$$\mathbb{E}\left(\left(\nu, \langle B\rangle_{\frac{t}{n}}\right) + \frac{1}{2}\left(\mu, B_{\frac{t}{n}}\right)^2 + R_n(t)\right) \le \mathbb{E}\left(\left(\nu, \langle B\rangle_{\frac{t}{n}}\right) + \frac{1}{2}\left(\mu, B_{\frac{t}{n}}\right)^2\right) + \mathbb{E}|R_n(t)|$$

and

$$\mathbb{E}\left(\left(\nu, \langle B\rangle_{\frac{t}{n}}\right) + \frac{1}{2}\left(\mu, B_{\frac{t}{n}}\right)^2 + R_n(t)\right) \ge \mathbb{E}\left(\left(\nu, \langle B\rangle_{\frac{t}{n}}\right) + \frac{1}{2}\left(\mu, B_{\frac{t}{n}}\right)^2\right) - \mathbb{E}|R_n(t)|$$

we obtain

$$\left|\mathbb{E}\left(\left(\nu, \langle B\rangle_{\frac{t}{n}}\right) + \frac{1}{2}\left(\mu, B_{\frac{t}{n}}\right)^2 + R_n(t)\right) - \mathbb{E}\left(\left(\nu, \langle B\rangle_{\frac{t}{n}}\right) + \frac{1}{2}\left(\mu, B_{\frac{t}{n}}\right)^2\right)\right| = o(1/n).$$

Since for all $t \ge 0$ (cf. Example 5.3.23 in [12]),

$$\mathbb{E}\left((\mu, B_t)^2 - (\mu\mu^\tau, \langle B\rangle_t)\right) = 0, \qquad \mathbb{E}\left((\mu\mu^\tau, \langle B\rangle_t) - (\mu, B_t)^2\right) = 0,$$

and

$$\mathbb{E}\left((\nu, \langle B\rangle_t) + \frac{1}{2}(\mu, B_t)^2\right) = G\left(\mu\mu^\tau + 2\nu^s\right)t,$$

we have

$$\begin{aligned}\mathbb{E}\exp\{(\mu, B_t) + (\nu, \langle B\rangle_t)\} &= \lim_{n\to\infty}\left(1 + \mathbb{E}\left(\left(\nu, \langle B\rangle_{\frac{t}{n}}\right) + \frac{1}{2}\left(\mu, B_{\frac{t}{n}}\right)^2\right) + o(1/n)\right)^n \\ &= \lim_{n\to\infty}\left(1 + \frac{G\left(\mu\mu^\tau + 2\nu^s\right)t}{n} + o(1/n)\right)^n \\ &= \exp\left\{G\left(\mu\mu^\tau + 2\nu^s\right)t\right\}. \quad \square\end{aligned}$$

Define

$$\tilde{I}_B(x) = \sup_{\mu\in\mathbb{R}^d}\left\{(x, \mu) - G\left(\mu\mu^\tau\right)\right\}, \quad x \in \mathbb{R}^d,$$

and

$$\tilde{\lambda}(x, z) = \sup_{(\mu,\nu)\in\mathbb{R}^d\times\mathbb{R}^{d\times d}}\left\{(x, \mu) + (z, \nu) - G\left(\mu\mu^\tau + 2\nu^s\right)\right\}, \quad (x, z) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}.$$

Then by minmax theorem (cf. [16]),

$$\tilde{I}_B(x) = \inf_{\theta\in\Sigma}\sup_{\mu\in\mathbb{R}^d}\left\{(x, \mu) - \frac{1}{2}\mathrm{tr}(\mu\mu^\tau\theta)\right\} = \frac{1}{2}\inf_{\theta\in\Sigma}(x, \theta^{-1}x), \tag{2.15}$$

and

$$\tilde{\lambda}(x,z) = \inf_{\theta\in\Sigma} \sup_{(\mu,\nu)\in\mathbb{R}^d\times\mathbb{R}^{d\times d}} \left\{\langle x,\mu\rangle + \langle z,\nu\rangle - \frac{1}{2}\mathrm{tr}(\mu\mu^\tau\theta) - \mathrm{tr}(\nu^s\theta)\right\}$$
$$= \begin{cases} \frac{1}{2}(x, z^{-1}x), & \text{if } z\in\Sigma,\\ \infty, & \text{otherwise.}\end{cases} \tag{2.16}$$

By the condition (1.11),

$$\frac{1}{2\bar{\sigma}}|x|^2 \le \tilde{I}_B(x) \le \frac{1}{2\underline{\sigma}}|x|^2$$

and by Lemma 2.8 and the Varadhan integral lemma (Lemma A.3), we have

$$\check{I}_t(x) = \sup_{\mu\in\mathbb{R}^d}\left\{(x,\mu) - G\left(\mu\mu^\tau\right)t\right\} = t\tilde{I}_B(x/t), \quad x\in\mathbb{R}^d, \tag{2.17}$$

and for any $(x,z)\in\mathbb{R}^d\times\mathbb{R}^{d\times d}$,

$$\lambda_t(x,z) = \sup_{(\mu,\nu)\in\mathbb{R}^d\times\mathbb{R}^{d\times d}}\left\{(x,\mu)+(z,\nu) - G\left(\mu\mu^\tau + 2\nu^s\right)t\right\} = t\tilde{\lambda}(x/t, z/t). \tag{2.18}$$

The following theorem gives precise representations of the rate functions I_B and J.

Theorem 2.4. $\left(\bar{C}\left(\varepsilon B_{t/\varepsilon}|_{t\in[0,T]}\in\cdot\right), \varepsilon>0\right)$ *satisfies large deviation principle with speed* ε *and rate function*

$$I_B(f) = \begin{cases} \frac{1}{2}\int_0^T \inf_{\theta\in\Sigma}(f'(s), \theta^{-1}f'(s))\mathrm{d}s, & \text{if } f\in\mathbb{H}^d,\\ +\infty, & \text{otherwise,}\end{cases} \tag{2.19}$$

and $\left(\bar{C}\left(\left(\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}\right)|_{t\in[0,T]}\in\cdot\right), \varepsilon>0\right)$ *satisfies large deviation principle with speed* ε *and rate function*

$$J(f,g) = \begin{cases} \frac{1}{2}\int_0^T (f'(s), (g'(s))^{-1}f'(s))\mathrm{d}s, & \text{if } (f,g)\in\mathbb{H}^d\times\mathbb{A},\\ +\infty, & \text{otherwise.}\end{cases} \tag{2.20}$$

Proof. We only prove (2.19) and (2.20). The proof is standard. We only give the proof of (2.20). For any $(f,g)\in\mathbb{H}^d\times\mathbb{A}$, by the convexity of $\tilde{\lambda}$ and Jansen's inequality, for any $0=t_0<t_1<\cdots<t_m=T$,

$$\sum_{l=1}^m \lambda_{t_l-t_{l-1}}\left(f(t_l)-f(t_{l-1}), g(t_l)-g(t_{l-1})\right)$$
$$= \sum_{l=1}^m (t_l-t_{l-1})\tilde{\lambda}\left(\frac{\int_{t_{l-1}}^{t_l} f'(s)\mathrm{d}s}{t_l-t_{l-1}}, \frac{\int_{t_{l-1}}^{t_l} g'(s)\mathrm{d}s}{t_l-t_{l-1}}\right)$$
$$\le \sum_{l=1}^m \int_{t_{l-1}}^{t_l}\tilde{\lambda}\left(f'(s), g'(s)\right)\mathrm{d}s = \int_0^T \tilde{\lambda}(f'(s), g'(s))\mathrm{d}s$$

which yields that $J(f,g)\le\int_0^T\tilde{\lambda}(f'(s),g'(s))\mathrm{d}s$.

On the other hand, if $(f, g) \in \mathbb{H}^d \times \mathbb{A}$ and $J(f(s), g(s))\mathrm{d}s < \infty$, set $t_l^n = \frac{lT}{2^n}, 0 \le l \le 2^n$ and $\mathcal{G}_n = \sigma([t_l^n, t_{l+1}^n], 0 \le l \le 2^n)$ and

$$\psi^n(s) = \sum_{l=0}^{2^n-1} \left(\frac{f(t_{l+1}^n) - f(t_l^n)}{t_{l+1}^n - t_l^n}, \frac{g(t_{l+1}^n) - g(t_l^n)}{t_{l+1}^n - t_l^n} \right) I_{[t_l^n, t_{l+1}^n)}(s).$$

Then by the convergence theorem of martingales and Fatou's lemma,

$$\int_0^T \tilde{\lambda}(f'(s), g'(s))\mathrm{d}s \le \limsup_{n\to\infty} \int_0^T \tilde{\lambda}(\psi^n(s))\mathrm{d}s \le J(f, g) < +\infty.$$

Therefore, $J(f, g) = \int_0^T \tilde{\lambda}(f'(s), g'(s))\mathrm{d}s$. □

Remark 2.1. The rate function I_B is different from the classical one (cf. [15,2]). In $d \ge 2$ case, it is not a quadratic form. But in one-dimensional case, it is a quadratic form.

Example 2.1. If $d = 1$ and $B_1 \sim N(0, [\underline{\sigma}, 1])$ where $0 < \underline{\sigma} \le 1$, then

$$I_B(f) = \begin{cases} \frac{1}{2} \int_0^T |f'(s)|^2 \mathrm{d}s, & \text{if } f \in \mathbb{H}^1, \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.21}$$

and

$$J(f, g) = \begin{cases} \frac{1}{2} \int_0^T \frac{|f'(s)|^2}{g'(s)} \mathrm{d}s, & \text{if } (f, g) \in \mathbb{H}^1 \times \mathbb{A}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.22}$$

Remark 2.2. In one-dimensional case, the rate function I_B of G-Brownian motion is the same as the classical Brownian motion. The joint rate function J of B and $\langle B \rangle$ is a new form due to quadratic variation uncertainty.

3. Large deviations for a stochastic differential equation driven by G-Brownian motion

In this section we use discrete time approximation to study large deviations for stochastic differential equations (SDEs) driven by G-Brownian motion. The method of the discrete time approximation is a basic method in large deviations of SDEs (cf. [3,4,10]). Our proof avoids the stopping time technique and the Girsanov transformation. Our main tool is exponential moment estimates.

3.1. Statement of result

For any $\varepsilon \ge 0$, we consider the following random perturbation SDEs driven by d-dimensional G-Brownian motion B

$$X_t^{x,\varepsilon} = x + \int_0^t b^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}s + \varepsilon \int_0^t \sigma^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}B_{s/\varepsilon} + \varepsilon \int_0^t h^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}\langle B, B\rangle_{s/\varepsilon}, \tag{3.1}$$

where $\langle B, B\rangle$ is treated as a $d \times d$-dimensional vector,

$$b^\varepsilon = (b_1^\varepsilon, \ldots, b_n^\varepsilon)^\tau : \mathbb{R}^n \to \mathbb{R}^n, \qquad \sigma^\varepsilon = (\sigma_{i,j}^\varepsilon) : \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^d$$

and $h^\varepsilon : \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^{d\times d}$ has the following form

$$h^\varepsilon = \begin{pmatrix} h_{11}^{\varepsilon,(1)} & \cdots & h_{1d}^{\varepsilon,(1)} & h_{21}^{\varepsilon,(1)} & \cdots & h_{2d}^{\varepsilon,(1)} & \cdots & h_{d1}^{\varepsilon,(1)} & \cdots & h_{dd}^{\varepsilon,(1)} \\ h_{11}^{\varepsilon,(2)} & \cdots & h_{1d}^{\varepsilon,(2)} & h_{21}^{\varepsilon,(2)} & \cdots & h_{2d}^{\varepsilon,(2)} & \cdots & h_{d1}^{\varepsilon,(2)} & \cdots & h_{dd}^{\varepsilon,(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ h_{11}^{\varepsilon,(n)} & \cdots & h_{1d}^{\varepsilon,(n)} & h_{21}^{\varepsilon,(n)} & \cdots & h_{2d}^{\varepsilon,(n)} & \cdots & h_{d1}^{\varepsilon,(n)} & \cdots & h_{dd}^{\varepsilon,(n)} \end{pmatrix}.$$

We also introduce the following conditions:
$(\mathrm{H1})_u$. $b^\varepsilon, \sigma^\varepsilon$ and h^ε are uniformly bounded, i.e., there exists a constant $L > 0$ such that for all $\varepsilon \geq 0$,

$$\sup_{x\in\mathbb{R}^n} \max\left\{|b^\varepsilon(x)|, \|\sigma^\varepsilon(x)\|_{\mathrm{HS}}, \|h^\varepsilon(x)\|_{\mathrm{HS}}|\right\} \leq L,$$

where $\|A\|_{\mathrm{HS}} := \sqrt{\sum_{ij} a_{ij}^2}$ is the Hilbert–Schmidt norm of a matrix $A = (a_{ij})$.
$(\mathrm{H2})_u$. $b^\varepsilon, \sigma^\varepsilon$ and h^ε are uniformly Lipschitz continuous, i.e., there exists a constant $L > 0$ such that for any $x, y \in \mathbb{R}^n$,

$$\max\left\{|b^\varepsilon(x) - b^\varepsilon(y)|, \|\sigma^\varepsilon(x) - \sigma^\varepsilon(y)\|_{\mathrm{HS}}, \|h^\varepsilon(x) - h^\varepsilon(y)\|_{\mathrm{HS}}\right\} \leq L|x - y|.$$

$(\mathrm{H3})_u$. $b^\varepsilon, \sigma^\varepsilon$ and h^ε converge uniformly $b := b^0, \sigma := \sigma^0$ and $h = h^0$, respectively, i.e.,

$$\lim_{\varepsilon\to 0} \sup_{x\in\mathbb{R}^n} \max\left\{|b^\varepsilon(x) - b(x)|, \|\sigma^\varepsilon(x) - \sigma(x)\|_{\mathrm{HS}}, \|h^\varepsilon(x) - h(x)\|_{\mathrm{HS}}\right\} = 0.$$

Let the definition of $\mathbb{H}^d$ and $\mathbb{A}$ be the same as in Section 2. For any $(f, g) \in \mathbb{H}^d \times \mathbb{A}$, let $\Psi(f, g)(t) \in C([0, T], \mathbb{R}^n)$ be a unique solution of the following ordinary differential equation:

$$\Psi(f, g)(t) = x + \int_0^t b(\Psi(f, g)(s))\mathrm{d}s + \int_0^t \sigma(\Psi(f, g)(s)) f'(s)\mathrm{d}s + \int_0^t h(\Psi(f, g)(s)) g'(s)\mathrm{d}s. \tag{3.2}$$

Theorem 3.1. *Let* $0 \leq \alpha < 1/2$ *and let* $(\mathrm{H2})_u$ *and* $(\mathrm{H3})_u$ *hold. Let* $X = \{X_t^{x,\varepsilon}, t \geq 0\}$ *be a unique solution of the G-SDE (3.1). Then for any closed subset* F *in* $\left(C_0^\alpha([0, T], \mathbb{R}^n), \|\cdot\|_\alpha\right)$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \bar{C}\left((X_t^{x,\varepsilon} - x)|_{t\in[0,T]} \in F\right) \leq -\inf_{\psi\in F} I(\psi) \tag{3.3}$$

and for any open subset O *in* $\left(C_0^\alpha([0, T], \mathbb{R}^n), \|\cdot\|_\alpha\right)$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \bar{C}\left((X_t^{x,\varepsilon} - x)|_{t\in[0,T]} \in O\right) \geq -\inf_{\psi\in O} I(\psi), \tag{3.4}$$

where

$$I(\psi) = \inf\{J(f, g); \psi = \Psi(f, g) - x\}. \tag{3.5}$$

Remark 3.1. (1). If $h \equiv 0$, then

$$I(\psi) = \inf\{I_B(f); \psi = \Psi(f) - x\} \tag{3.6}$$

where $\Psi(f)(t) \in C_0([0,T],\mathbb{R}^n)$ is a unique solution of the following ordinary differential equation:

$$\Psi(f)(t) = x + \int_0^t b(\Psi(f)(s))\mathrm{d}s + \int_0^t \sigma(\Psi(f)(s)) f'(s)\mathrm{d}s. \tag{3.7}$$

(2). If $h \equiv 0$ and $\sigma^\tau\sigma > 0$, then the solution of (3.7) satisfies

$$\Psi'(f)(t) - b(\Psi(f)(t)) = \sigma(\Psi(f)(t)) f'(t),$$

which implies

$$f'(t) = \left(\sigma^\tau(\Psi(f)(t))\sigma(\Psi(f)(t))\right)^{-1} \sigma^\tau(\Psi(f)(t))(\Psi'(f)(t) - b(\Psi(f)(t))).$$

Therefore

$$I(\psi) = \begin{cases} I_B(\left(\sigma^\tau(\psi+x)\sigma(\psi+x)\right)^{-1}\sigma^\tau(\psi+x)(\psi' - b(\psi+x))), \\ \quad \text{if } \psi \text{ absolutely continuous}, \\ +\infty, \quad \text{otherwise}. \end{cases}$$

In particular, if $d = n = 1$, $B_1 \sim N(0, [\underline{\sigma}, 1])$, $h \equiv 0$ and $\sigma > 0$, then

$$I(\psi) = \begin{cases} \frac{1}{2}\int_0^T |\sigma^{-1}(\psi(t)+x)(\psi'(t) - b(\psi(t)+x))|^2, & \text{if } \psi \text{ absolutely continuous}, \\ +\infty, & \text{otherwise}. \end{cases}$$

That is, in one-dimensional case, if $h \equiv 0$, the rate function is the same as the classical case (cf. [5,7,8]).

Example 3.1. Let $d = n = 1$ and $B_1 \sim N(0, [\underline{\sigma}, 1])$. Consider a linear SDE:

$$X_t^\varepsilon = 1 + b\int_0^t X_s^\varepsilon \mathrm{d}t + \varepsilon h \int_0^t X_s^\varepsilon \mathrm{d}\langle B\rangle_{s/\varepsilon} + \varepsilon\sigma \int_0^t X_s^\varepsilon \mathrm{d}B_{s/\varepsilon}.$$

Then for any $(f, g) \in \mathbb{H}^1 \times \mathbb{A}$,

$$\Psi(f,g)(t) = \exp\{bt + hg(t) + \sigma f(t)\}.$$

Therefore, if ψ absolutely continuous, then

$$I(\psi) = \inf\left\{\frac{1}{2}\int_0^T \frac{|f'(t)|^2}{g'(t)}\mathrm{d}t, \psi(t) = \exp\{bt + hg(t) + \sigma f(t)\} - 1\right\}.$$

3.2. Proof of Theorem 3.1: SDE with bounded coefficients

If we define $B_t^\varepsilon = B_t - f_{\varepsilon t}/\varepsilon$,

$$c^\varepsilon(s,x) = \sigma^\varepsilon(x) f_s' + b^\varepsilon(x) + h^\varepsilon(x) g_s'$$

and

$$c(s,x) = \sigma(x) f_s' + b(x) + h(x) g_s'.$$

Then

$$\Psi(f,g)(t) = \int_0^t c(s, \Psi(f,g)(s))\mathrm{d}s$$

and

$$X_t^{x,\varepsilon} = x + \int_0^t c^\varepsilon(s, X_s^{x,\varepsilon})\mathrm{d}s + \varepsilon \int_0^t \sigma^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}B^\varepsilon_{s/\varepsilon} + \int_0^t h^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}(\varepsilon\langle B\rangle_{s/\varepsilon} - g(s)).$$

For $N \geq 1$, set $t_k = kT/N, k = 0, 1, \ldots, N$ and let $X_N^{x,\varepsilon} = \{X_N^{x,\varepsilon}(t) = X_{N,t}^{x,\varepsilon}, t \geq 0\}$ defined by

$$X_{N,t}^{x,\varepsilon} = X_{t_{k-1}}^{x,\varepsilon}, \quad t \in [t_{k-1}, t_k), k = 1, \ldots, N.$$

As usual, we denote by $\|\psi\| = \sup_{t\in[0,T]} |\psi(t)|$ for any function ψ on $[0, T]$.

Lemma 3.1. *Assume that* $(\mathrm{H1})_u$ *holds. Then for any* $\rho > 0$,

$$\limsup_{N\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \bar{C}\left(\|X^{x,\varepsilon} - X_N^{x,\varepsilon}\| > \rho\right) = -\infty \tag{3.8}$$

Proof. For any $\rho > 0$,

$$\begin{aligned}\bar{C}\left(\|X^{x,\varepsilon} - X_N^{x,\varepsilon}\| > \rho\right) \leq{}& \sum_{k=1}^N \bar{C}\left(\sup_{t\in[t_{k-1},t_k]} \int_{t_{k-1}}^t |b^\varepsilon(X_s^{x,\varepsilon})|\mathrm{d}s > \rho/3\right) \\ &+ \sum_{k=1}^N \bar{C}\left(\sup_{t\in[t_{k-1},t_k]} \left|\int_{t_{k-1}}^t \varepsilon\sigma^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}B_{s/\varepsilon}\right| > \rho/3\right) \\ &+ \sum_{k=1}^N \bar{C}\left(\sup_{t\in[t_{k-1},t_k]} \left|\int_{t_{k-1}}^t \varepsilon h^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}\langle B\rangle_{s/\varepsilon}\right| > \rho/3\right).\end{aligned}$$

Since there exists a constant M depending only on n, d, L such that

$$\sup_{t\in[t_{k-1},t_k]} \int_{t_{k-1}}^t |b^\varepsilon(X_s^{x,\varepsilon})|\mathrm{d}s \leq \frac{MT}{N} \quad \text{and} \quad \sup_{t\in[t_{k-1},t_k]} \left|\int_{t_{k-1}}^t \varepsilon h^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}\langle B\rangle_{s/\varepsilon}\right| \leq \frac{MT\bar{\sigma}}{N},$$

when $N \geq \frac{3MT\max\{1,\bar{\sigma}\}}{\rho}$, for all $k = 1, \ldots, N$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \bar{C}\left(\sup_{t\in[t_{k-1},t_k]} \int_{t_{k-1}}^t |b^\varepsilon(X_s^{x,\varepsilon})|\mathrm{d}s > \rho/3\right) = -\infty$$

and

$$\limsup_{\varepsilon\to 0} \varepsilon \log \bar{C}\left(\sup_{t\in[t_{k-1},t_k]} \left|\int_{t_{k-1}}^t \varepsilon h^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}\langle B\rangle_{s/\varepsilon}\right| > \rho/3\right) = -\infty.$$

On the other hand, for any $0 < \mu < 1/2$,

$$\sup_{t\in[t_{k-1},t_k]} \left|\int_{t_{k-1}}^t \varepsilon\sigma^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}B_{s/\varepsilon}\right| \leq \frac{T^\mu}{N^\mu} \sup_{s,t\in[t_{k-1},t_k]} \frac{\left|\int_s^t \varepsilon\sigma^\varepsilon(X_u^{x,\varepsilon})\mathrm{d}B_{u/\varepsilon}\right|}{|s-t|^\mu}$$

and by Lemma 3.3 in [9], there exists a constant $\delta = \delta(n, d, L) > 0$ such that

$$\sup_{\varepsilon>0} \max_{1\leq k\leq N} \sup_{s,t\in[t_{k-1},t_k]} \bar{\mathbb{E}}\left(\exp\left\{\delta \frac{\left|\int_s^t \varepsilon\sigma^\varepsilon(X_u^{x,\varepsilon})\mathrm{d}B_{u/\varepsilon}\right|^2}{|s-t|}\right\}\right)$$

$$\leq \sup_{\varepsilon>0} \max_{1\leq k\leq N} \sup_{s,t\in[0,T/(N\varepsilon)]} \bar{\mathbb{E}} \left(\exp\left\{ \delta \frac{\varepsilon \left| \int_s^t \sigma^\varepsilon(X_u^{x,\varepsilon}) \mathrm{d}B_{u/\varepsilon} \right|^2}{|s-t|} \right\} \right) < \infty.$$

Therefore by (3.1) in [9], there exists a constant $\delta > 0$ such that

$$\sup_{\varepsilon>0} \max_{1\leq k\leq N} \bar{\mathbb{E}} \left(\exp\left\{ \delta \sup_{s,t\in[t_{k-1},t_k]} \frac{\varepsilon \left| \int_s^t \sigma^\varepsilon(X_u^{x,\varepsilon}) \mathrm{d}B_{u/\varepsilon} \right|^2}{|s-t| \log(1+1/(2|s-t|))} \right\} \right) < \infty$$

which implies that for any $0 < \mu < 1/2$, there exists a constant $\delta > 0$ such that

$$\sup_{\varepsilon>0} \max_{1\leq k\leq N} \bar{\mathbb{E}} \left(\exp\left\{ \delta \sup_{s,t\in[t_{k-1},t_k]} \frac{\varepsilon \left| \int_s^t \sigma^\varepsilon(X_u^{x,\varepsilon}) \mathrm{d}B_{u/\varepsilon} \right|^2}{|s-t|^{2\mu}} \right\} \right) < \infty.$$

Now by Chebyshev's inequality, we have

$$\begin{aligned} &\bar{C} \left(\sup_{t\in[t_{k-1},t_k]} \left| \int_{t_{k-1}}^t \varepsilon \sigma^\varepsilon(X_s^{x,\varepsilon}) \mathrm{d}B_{s/\varepsilon} \right| > \rho/3 \right) \\ &\leq \bar{C} \left(\frac{T^\mu \sqrt{\varepsilon}}{N^\mu} \sup_{s,t\in[t_{k-1},t_k]} \frac{\left| \int_s^t \sqrt{\varepsilon} \sigma^\varepsilon(X_u^{x,\varepsilon}) \mathrm{d}B_{u/\varepsilon} \right|}{|s-t|^\mu} > \rho/3 \right) \\ &\leq \exp\left\{ -\frac{\rho^2 \delta N^{2\mu}}{9T^{2\mu}\varepsilon} \right\} \max_{1\leq k\leq N} \bar{\mathbb{E}} \left(\exp\left\{ \delta \sup_{s,t\in[t_{k-1},t_k]} \frac{\varepsilon \left| \int_s^t \sigma^\varepsilon(X_u^{x,\varepsilon}) \mathrm{d}B_{u/\varepsilon} \right|^2}{|s-t|^{2\mu}} \right\} \right), \end{aligned}$$

and so

$$\limsup_{N\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \left(N \max_{1\leq k\leq N} \bar{C} \left(\sup_{t\in[t_{k-1},t_k]} \left| \int_{t_{k-1}}^t \varepsilon \sigma^\varepsilon(X_s^{x,\varepsilon}) \mathrm{d}B_{s/\varepsilon} \right| > \rho/3 \right) \right) = -\infty.$$

Therefore, (3.8) holds. □

Lemma 3.2. *Assume that* $(\mathrm{H1})_u$, $(\mathrm{H2})_u$ *and* $(\mathrm{H3})_u$ *hold. Then for any* $a \in (0, +\infty)$ *and* $\rho > 0$,

$$\begin{aligned} &\limsup_{\mu\to 0} \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{\|f\|_H^2 \leq a} \bar{C} \left(\left\| \varepsilon \int_0^\cdot \sigma^\varepsilon(X_s^{x,\varepsilon}) \mathrm{d}B_{s/\varepsilon}^\varepsilon \right\| > \rho, \|\varepsilon B_{\cdot/\varepsilon} - f\| < \mu \right) \\ &= -\infty. \end{aligned} \tag{3.9}$$

Proof. Set

$$\begin{aligned} A_1 &= \left\{ \|X^{x,\varepsilon} - X_N^{x,\varepsilon}\| > \tau \right\}, \\ A_2 &= \left\{ \left\| \varepsilon \int_0^\cdot \left(\sigma^\varepsilon(X_s^{x,\varepsilon}) - \sigma^\varepsilon(X_{N,s}^{x,\varepsilon}) \right) \mathrm{d}B_{s/\varepsilon}^\varepsilon \right\| > \rho/2, \|X^{x,\varepsilon} - X_N^{x,\varepsilon}\| \leq \tau \right\}, \\ A_3 &= \left\{ \left\| \varepsilon \int_0^\cdot \sigma^\varepsilon(X_{N,s}^{x,\varepsilon}) \mathrm{d}B_{s/\varepsilon}^\varepsilon \right\| > \rho/2, \left\| \varepsilon B_{\cdot/\varepsilon} - f \right\| < \mu, \|X^{x,\varepsilon} - X_N^{x,\varepsilon}\| \leq \tau \right\}. \end{aligned}$$

Then

$$\left\{\left\|\varepsilon\int_0^{\cdot}\sigma^{\varepsilon}(X_s^{x,\varepsilon})\mathrm{d}B^{\varepsilon}_{s/\varepsilon}\right\|>\rho,\ \|\varepsilon B_{\cdot/\varepsilon}-f\|<\mu\right\}\subset A_1\cup A_2\cup A_3$$

and

$$A_2\subset A_{21}\cup A_{22}\cup A_{23}$$

where

$$\begin{aligned}
A_{21}&=\left\{\left\|\varepsilon\int_0^{\cdot}\left(\sigma^{\varepsilon}(X_s^{x,\varepsilon})-\sigma(X_s^{x,\varepsilon})\right)\mathrm{d}B^{\varepsilon}_{s/\varepsilon}\right\|>\rho/6\right\},\\
A_{22}&=\left\{\left\|\varepsilon\int_0^{\cdot}\left(\sigma(X_s^{x,\varepsilon})-\sigma(X_{N,s}^{x,\varepsilon})\right)\mathrm{d}B^{\varepsilon}_{s/\varepsilon}\right\|>\rho/6,\ \|X^{x,\varepsilon}-X_N^{x,\varepsilon}\|\le\tau\right\},\\
A_{23}&=\left\{\left\|\varepsilon\int_0^{\cdot}\left(\sigma(X_{N,s}^{x,\varepsilon})-\sigma^{\varepsilon}(X_{N,s}^{x,\varepsilon})\right)\mathrm{d}B^{\varepsilon}_{s/\varepsilon}\right\|>\rho/6\right\}.
\end{aligned}$$

Set $\kappa(\varepsilon)=\sup_{x\in\mathbb{R}^n}\|\sigma^{\varepsilon}(x)-\sigma(x)\|_{\mathrm{HS}}$, $\iota=\sup_{x\in\mathbb{R}^n}\|\sigma(x)\|_{\mathrm{HS}}$ and take a function $\phi_\tau(x)\in L_{ip}(\mathbb{R})$ such that $0\le\phi_\tau\le 1$, $\phi_\tau(x)=1$ for all $|x|\le\tau$ and $\phi_\tau(x)=0$ for all $|x|\ge 2\tau$. Choose $\varepsilon_0>0$ and $\tau_0>0$ such that for all $0<\varepsilon\le\varepsilon_0$ and all $0<\tau\le\tau_0$,

$$\kappa(\varepsilon)\sqrt{Ta}<\rho/12,\qquad 2\iota\tau\sqrt{Ta}<\rho/12.$$

Then for $\|f\|_H^2\le a$, by

$$\int_0^t|f'(s)|\mathrm{d}s\le\sqrt{T}\left(\int_0^T|f'(s)|^2\mathrm{d}s\right)^{1/2}\le\sqrt{Ta},\quad t\in[0,T],$$

we have that for all $0<\varepsilon\le\varepsilon_0$ and all $0<\tau\le\tau_0$,

$$\begin{aligned}
\bar{C}(A_{21})&\le\bar{C}\left(\left\|\varepsilon\int_0^{\cdot}\left(\sigma^{\varepsilon}(X_s^{x,\varepsilon})-\sigma(X_s^{x,\varepsilon})\right)\mathrm{d}B_{s/\varepsilon}\right\|>\rho/12\right),\\
\bar{C}(A_{23})&\le\bar{C}\left(\left\|\varepsilon\int_0^{\cdot}\left(\sigma^{\varepsilon}(X_{N,s}^{x,\varepsilon})-\sigma(X_{N,s}^{x,\varepsilon})\right)\mathrm{d}B_{s/\varepsilon}\right\|>\rho/12\right),
\end{aligned}$$

and by Lemma 2.1 in [9],

$$\bar{C}(A_{22})\le\bar{C}\left(\left\|\varepsilon\int_0^{\cdot}\phi_\tau(X_s^{x,\varepsilon}-X_{N,s}^{x,\varepsilon})\left(\sigma(X_s^{x,\varepsilon})-\sigma(X_{N,s}^{x,\varepsilon})\right)\mathrm{d}B_{s/\varepsilon}\right\|>\rho/12\right).$$

By Lemma 3.3 in [9], there exist constants $M_1>0$ and $M_2>0$ such that for any $\lambda>0$ with $\lambda\kappa(\varepsilon)^2M_1<1$,

$$\sup_{s,t\in[0,T]}\bar{\mathbb{E}}\left(\exp\left\{\frac{\lambda\left|\sqrt{\varepsilon}\int_s^t\left(\sigma^{\varepsilon}(X_{N,u}^{x,\varepsilon})-\sigma(X_{N,u}^{x,\varepsilon})\right)\mathrm{d}B_{u/\varepsilon}\right|^2}{|t-s|}\right\}\right)\le\frac{M_2}{1-\lambda\kappa(\varepsilon)^2M_1}.$$

Therefore, by Chebyshev's inequality, we have

$$\limsup_{\varepsilon\to 0}\varepsilon\log\bar{C}\left(\left\|\varepsilon\int_0^{\cdot}\left(\sigma^{\varepsilon}(X_s^{x,\varepsilon})-\sigma(X_s^{x,\varepsilon})\right)\mathrm{d}B_{s/\varepsilon}\right\|>\frac{\rho}{12}\right)=-\infty.$$

Similarly, one can obtain

$$\limsup_{\tau\to 0}\sup_{N\geq 1}\limsup_{\varepsilon\to 0}\varepsilon\log\bar{C}\left(\left\|\varepsilon\int_0^{\cdot}\phi_\tau(X_s^{x,\varepsilon}-X_{N,s}^{x,\varepsilon})\left(\sigma(X_s^{x,\varepsilon})\right.\right.\right.$$
$$\left.\left.\left.-\ \sigma(X_{N,s}^{x,\varepsilon})\right)\mathrm{d}B_{s/\varepsilon}\right\|>\frac{\rho}{12}\right)=-\infty$$

and

$$\limsup_{N\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log\bar{C}\left(\left\|\varepsilon\int_0^{\cdot}\left(\sigma^\varepsilon(X_{N,s}^{x,\varepsilon})-\sigma(X_{N,s}^{x,\varepsilon})\right)\mathrm{d}B_{s/\varepsilon}\right\|>\frac{\rho}{12}\right)=-\infty.$$

Thus

$$\limsup_{\tau\to 0}\limsup_{N\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log\sup_{\|f\|_H^2\leq a}\bar{C}(A_2)=-\infty. \tag{3.10}$$

Next let us consider A_3. On $\{\|\varepsilon B_{\cdot/\varepsilon}-f\|<\mu\}$

$$\begin{aligned}\left|\varepsilon\int_0^{\cdot}\sigma^\varepsilon(X_{N,s}^{x,\varepsilon})\mathrm{d}B_{s/\varepsilon}^\varepsilon\right| &= \left|\sum_{k=1}^N\sigma^\varepsilon(X_{t_k}^{x,\varepsilon})\left((\varepsilon B_{t_k/\varepsilon}-f_{t_k})-(\varepsilon B_{t_{k-1}}-f_{t_{k-1}})\right)\right|\\ &\leq 2N\mu\sup_{x\in\mathbb{R}^n}|\sigma^\varepsilon(x)|\end{aligned}$$

which yields

$$\limsup_{\mu\to 0}\limsup_{\varepsilon\to 0}\varepsilon\log\sup_{\|f\|_H^2\leq a}\bar{C}(A_3)=-\infty. \tag{3.11}$$

Finally, combining (3.10), (3.11) and Lemma 3.1, we obtain (3.9). □

Lemma 3.3. *Assume that* (H1)$_u$, (H2)$_u$ *and* (H3)$_u$ *hold. Then for any* $a\geq 0$ *and* $\rho>0$*,*

$$\limsup_{\substack{\mu\to 0\\ \nu\to 0}}\limsup_{\varepsilon\to 0}\varepsilon\log\sup_{\|f\|_H^2\leq a,g\in\mathbb{A}}\bar{C}\left(\|X^{x,\varepsilon}-\Psi(f,g)\|>\rho,\right.$$
$$\left.\|\varepsilon B_{\cdot/\varepsilon}-f\|<\mu,\|\varepsilon\langle B\rangle_{\cdot/\varepsilon}-g\|<\nu\right)=-\infty. \tag{3.12}$$

Proof. For any $(f,g)\in\mathbb{H}^d\times\mathbb{A}$ with $\|f\|_H^2\leq a$,

$$\begin{aligned}X_t^{x,\varepsilon}-\Psi(f,g)_t &= \int_0^t\left(c^\varepsilon(s,X_s^{x,\varepsilon})-c(s,X_s^{x,\varepsilon})\right)\mathrm{d}s\\ &\quad+\int_0^t\left(c(s,X_s^{x,\varepsilon})-c(s,\Psi(f,g)_s)\right)\mathrm{d}s\\ &\quad+\varepsilon\int_0^t\sigma^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}B_{s/\varepsilon}^\varepsilon+\int_0^t h^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}(\varepsilon\langle B\rangle_{s/\varepsilon}-g_s).\end{aligned}$$

Set

$$\Lambda(\varepsilon)=\sup_{x\in\mathbb{R}^n}\left(\|\sigma^\varepsilon(x)-\sigma(x)\|_{\mathrm{HS}}+|b^\varepsilon(x)-b(x)|+\|h^\varepsilon(x)-h(x)\|_{\mathrm{HS}}\right).$$

Then by conditions $(\mathrm{H1})_u$, $(\mathrm{H2})_u$ and $\int_0^T(1+|f'(s)|+|g'(s)|)\mathrm{d}s \le ((1+\bar{\sigma})T+\sqrt{Ta})$, there exists a constant $M \in (0,\infty)$ such that for all $t \in [0,T]$ and all $(f,g) \in \mathbb{H}^d \times \mathbb{A}$ with $\|f\|_H^2 \le a$,

$$\left|\int_0^t \left(c^\varepsilon(s, X_s^{x,\varepsilon}) - c(s, X_s^{x,\varepsilon})\right)\mathrm{d}s\right| \le M\Lambda(\varepsilon),$$

$$\left|\int_0^t \left(c(s, X_s^{x,\varepsilon}) - c(s, \Psi(f,g)_s)\right)\mathrm{d}s\right| \le M\int_0^t (1+|f'(s)|+|g'(s)|)|X_s^{x,\varepsilon} - \Psi(f,g)_s|\mathrm{d}s$$

and

$$\left|\int_0^t h^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}(\varepsilon\langle B\rangle_{s/\varepsilon} - g(s))\right| \le M\|\varepsilon\langle B\rangle_{\cdot/\varepsilon} - g\|.$$

Therefore,

$$|X_t^{x,\varepsilon} - \Psi(f,g)_t| \le M\|\varepsilon\langle B\rangle_{\cdot/\varepsilon} - g\| + \left\|\varepsilon\int_0^\cdot \sigma^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}B^\varepsilon_{s/\varepsilon}\right\| + M\Lambda(\varepsilon) + M\int_0^t (1+|f'(s)|+|g'(s)|)|X_s^{x,\varepsilon} - \Psi(f,g)_s|\mathrm{d}s$$

which implies from Gronwall's inequality

$$\|X^{x,\varepsilon} - \Psi(f,g)\| \le \left(M\|\varepsilon\langle B\rangle_{\cdot/\varepsilon} - g\| + \left\|\varepsilon\int_0^\cdot \sigma^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}B^\varepsilon_{s/\varepsilon}\right\| + M\Lambda(\varepsilon)\right) \times \mathrm{e}^{((1+\bar{\sigma})T+\sqrt{Ta})M}.$$

Choose $\varepsilon_0 > 0$ and $\nu_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ and $0 < \nu \le \nu_0$, $M\Lambda(\varepsilon)\mathrm{e}^{((1+\bar{\sigma})T+\sqrt{Ta})M} < \rho/3$ and $M\nu\mathrm{e}^{((1+\bar{\sigma})T+\sqrt{Ta})M} < \rho/3$. Then

$$\left\{\|X^{x,\varepsilon} - \Psi(f,g)\| > \rho, \|\varepsilon B_{\cdot/\varepsilon} - f\| < \mu, \|\varepsilon\langle B\rangle_{\cdot/\varepsilon} - g\| < \nu\right\} \subset \left\{\left\|\varepsilon\int_0^\cdot \sigma^\varepsilon(X_s^{x,\varepsilon})\mathrm{d}B^\varepsilon_{s/\varepsilon}\right\| > \mathrm{e}^{-((1+\bar{\sigma})T+\sqrt{Ta})M}\rho/3, \|\varepsilon B_{\cdot/\varepsilon} - f\| < \mu\right\},$$

and so (3.12) holds by Lemma 3.2. □

Theorem 3.2. *Let* $(\mathrm{H1})_u$, $(\mathrm{H2})_u$ *and* $(\mathrm{H3})_u$ *hold. Then for any closed subset* F *and any open subset* O *in* $(C_0([0,T],\mathbb{R}^d), \|\cdot\|) \times (C_0([0,T],\mathbb{R}^{d\times d}), \|\cdot\|) \times (C_0([0,T],\mathbb{R}^n), \|\cdot\|)$

$$\limsup_{\varepsilon\to 0} \varepsilon\log\bar{C}\left(\left(\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}, X_t^{x,\varepsilon} - x\right)|_{t\in[0,T]} \in F\right) \le -\inf_{(f,g,\psi)\in F}\hat{I}(f,g,\psi), \quad (3.13)$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon\log\bar{C}\left(\left(\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}, X_t^{x,\varepsilon} - x\right)|_{t\in[0,T]} \in O\right) \ge -\inf_{(f,g,\psi)\in O}\hat{I}(f,g,\psi), \quad (3.14)$$

where

$$\hat{I}(f,g,\psi) = \begin{cases} J(f,g), & if\, (f,g) \in \mathbb{H}^d \times \mathbb{A},\ x+\psi = \Psi(f,g) \\ +\infty, & otherwise. \end{cases}$$

Proof. Let us first prove the lower bound. For any open set $G \subset (C_0([0,T],\mathbb{R}^d), \|\cdot\|) \times (C_0([0,T],\mathbb{R}^{d\times d}), \|\cdot\|) \times (C_0([0,T],\mathbb{R}^n), \|\cdot\|)$, without loss of generality we assume

$\inf_{(f,g,\psi)\in G} \hat{I}(f,g,\psi) < \infty$. For any $\delta > 0$, choose $(f_0, g_0, \psi_0) \in G$ such that $\hat{I}(f_0, g_0, \psi_0) \leq \inf_{(f,g,\psi)\in G} \hat{I}(f,g,\psi) + \delta$ and $x + \psi_0 = \Psi(f_0, g_0)$. Choose $\rho > 0$ such that

$$U_\rho(f_0, g_0, \psi_0) := \{(f,g,\psi); \|f - f_0\| \leq \rho, \|g - g_0\| \leq \rho, \|\psi - \psi_0\| \leq \rho\} \subset G.$$

Then for any $\mu > 0$ and $\nu > 0$ small enough,

$$\begin{aligned}
&\bar{C}((\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}, X_t^{x,\varepsilon} - x) \in G) \\
&\quad \geq \bar{C}((\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}, X_t^{x,\varepsilon} - x) \in U_\rho(f_0, g_0, \psi_0)) \\
&\quad \geq \bar{C}(\|\varepsilon B_{\cdot/\varepsilon} - f_0\| < \mu, \|\varepsilon\langle B\rangle_\cdot - g_0\| < \nu) \\
&\qquad - \bar{C}(\|X^{x,\varepsilon} - (x + \psi_0)\| > \rho, \|\varepsilon B_{\cdot/\varepsilon} - f_0\| < \mu, \|\varepsilon\langle B\rangle_\cdot - g_0\| < \nu).
\end{aligned}$$

Therefore, by Lemma 3.3 and Theorem 2.3, we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \varepsilon \log \bar{C}\left((\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}, X_t^{x,\varepsilon} - x) \in G\right) \\
&\quad \geq -J(f_0, g_0) \geq -\inf_{(f,g,\psi)\in G} \hat{I}(f,g,\psi) - \delta
\end{aligned}$$

which yields the lower bound.

Next let us show that the rate function $\hat{I}$ is a good rate function. For each $a < \inf_{(f,g,\psi)\in F} \hat{I}(f,g,\psi)$, set $K_a = \{(f,g); J(f,g) \leq a\}$ and $\tilde{K}_a = \left\{(f,g,\psi); \hat{I}(f,g,\psi) \leq a\right\}$. Then K_a is compact and $\tilde{K}_a = \{(f,g,\Psi(f,g)); (f,g) \in K_a\}$. It is easy to check that $\Psi|_{K_a}$ is continuous. Therefore $\tilde{K}_a$ is also compact.

Finally, we show the upper bound. Let F be a closed subset in $(C_0([0,T],\mathbb{R}^d), \|\cdot\|) \times (C_0([0,T],\mathbb{R}^{d\times d}), \|\cdot\|) \times (C_0([0,T],\mathbb{R}^n), \|\cdot\|)$. For each $a < \inf_{(f,g,\psi)\in F} \hat{I}(f,g,\psi)$, set $K_a = \{(f,g); J(f,g) \leq a\}$ and $\tilde{K}_a = \left\{(f,g,\psi); \hat{I}(f,g,\psi) \leq a\right\}$. Then K_a is compact, $\tilde{K}_a \cap F = \emptyset$ and for any $(f,g,\psi) \in \tilde{K}_a$, there exists $\rho = \rho_{f,g} > 0$ such that $U_\rho(f,g,\psi)$ and F are disjoint. For each $(f,g) \in \mathbb{H}^d \times \mathbb{A}$ and $x + \psi = \Psi(f,g)$, by Lemma 3.3, for each $R > 0$, there exist $\mu = \mu_{f,g} > 0$, $\nu = \nu_{f,g} > 0$ and $\varepsilon_{f,g} > 0$ such that for all $0 < \varepsilon < \varepsilon_{f,g}$,

$$\begin{aligned}
&\bar{C}\left(\|X^{x,\varepsilon} - (x + \psi)\| > \rho_{f,g}, \|\varepsilon B_{\cdot/\varepsilon} - f\| < \mu_{f,g}, \|\varepsilon\langle B\rangle_{\cdot/\varepsilon} - g\| < \nu_{f,g}\right) \\
&\quad \leq \exp\left\{-\frac{R}{\varepsilon}\right\}.
\end{aligned}$$

Since K_a is compact, there exists a finite subset $\{(f_1, g_1), \ldots, (f_l, g_l)\}$ of K_a such that $K_a \subset U := \cup_{i=1}^l \{\|f - f_i\| < \mu_{f_i,g_i}, \|g - g_i\| < \nu_{f_i,g_i}\}$. Then for ε small enough,

$$\begin{aligned}
&\bar{C}((\varepsilon B_{\cdot/\varepsilon}, \varepsilon\langle B\rangle_{\cdot/\varepsilon}, X^{x,\varepsilon} - x) \in F) \\
&\quad \leq \bar{C}\left((\varepsilon B_{\cdot/\varepsilon}, \varepsilon\langle B\rangle_{\cdot/\varepsilon}, X^{x,\varepsilon} - x) \in F, (\varepsilon B_{\cdot/\varepsilon}, \varepsilon\langle B\rangle_{\cdot/\varepsilon}) \in U\right) \\
&\qquad + \bar{C}\left((\varepsilon B_{\cdot/\varepsilon}, \varepsilon\langle B\rangle_{\cdot/\varepsilon}) \in U^c\right) \\
&\quad \leq \sum_{i=1}^l \bar{C}\left(\|X^{x,\varepsilon} - \Psi(f_i, g_i)\| > \rho_{f_i,g_i}, \|\varepsilon B_{\cdot/\varepsilon} - f_i\| < \mu_{f_i,g_i},\right. \\
&\qquad \left.\|\varepsilon\langle B\rangle_{\cdot/\varepsilon} - g_i\| < \nu_{f_i,g_i}\right) + \bar{C}((\varepsilon B_{\cdot/\varepsilon}, \varepsilon\langle B\rangle_{\cdot/\varepsilon}) \in U^c) \\
&\quad \leq l \exp\left\{-\frac{R}{\varepsilon}\right\} + \exp\left\{-\frac{a}{\varepsilon}\right\}.
\end{aligned}$$

Therefore

$$\limsup_{\varepsilon\to 0} \varepsilon \log \bar{C}((\varepsilon B_{\cdot/\varepsilon}, \varepsilon\langle B\rangle_{\cdot/\varepsilon}, X^{x,\varepsilon} - x) \in F) \leq \max\{-R, -a\}.$$

First letting $R \to +\infty$, then letting $a \to \hat{I}(F)$, the upper bound is proved. □

We know that the LDP for a diffusion process also holds (cf. [4,10]). Next we extend the above LDP to the Hölder norm.

Lemma 3.4. *Let* $0 \leq \alpha < 1/2$ *and* $0 \leq \beta < 1$. *Assume that* $(\mathrm{H1})_u$, $(\mathrm{H2})_u$ *and* $(\mathrm{H3})_u$ *hold. Then*

$$\left\{\bar{C}\left(\left(\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}, X_t^{x,\varepsilon} - x\right)|_{t\in[0,T]} \in \cdot\right), \varepsilon > 0\right\}$$

is exponentially tight in $(C_0^\alpha([0,T],\mathbb{R}^d), \|\cdot\|_\alpha) \times (C_0^\beta([0,T],\mathbb{R}^{d\times d}), \|\cdot\|_\beta) \times (C_0^\alpha([0,T],\mathbb{R}^n), \|\cdot\|_\alpha)$.

Proof. By Lemma 3.3 in [9], there exists a constant $\delta = \delta(n,d,L) > 0$ such that

$$\sup_{\varepsilon>0} \sup_{s,t\in[0,T]} \bar{\mathbb{E}}\left(\exp\left\{\delta \frac{\left|\int_s^t \sqrt{\varepsilon}\sigma^\varepsilon(X_u^\varepsilon)\mathrm{d}B_{u/\varepsilon}\right|^2}{|s-t|}\right\}\right)$$
$$\leq \sup_{\varepsilon>0} \sup_{s,t\in[0,T/\varepsilon]} \bar{\mathbb{E}}\left(\exp\left\{\delta \frac{\left|\int_s^t \sigma^\varepsilon(X_u^\varepsilon)\mathrm{d}B_{u/\varepsilon}\right|^2}{|s-t|}\right\}\right) < \infty$$

which implies that for any $0 \leq \alpha < \gamma < 1/2$ there exists a constant $\delta = \delta(n,d,L) > 0$ such that

$$\sup_{\varepsilon>0} \bar{\mathbb{E}}\left(\exp\left\{\delta \sup_{s,t\in[0,T]} \frac{\left|\int_s^t \sqrt{\varepsilon}\sigma^\varepsilon(X_u^\varepsilon)\mathrm{d}B_{u/\varepsilon}\right|^2}{|s-t|^{2\gamma}}\right\}\right) < \infty.$$

Since b^ε and h^ε are bounded, we have that for R large enough,

$$\bar{C}\left(\|X^\varepsilon\|_\gamma \geq R\right) \leq \bar{C}\left(\sqrt{\varepsilon} \sup_{s,t\in[0,T]} \frac{\left|\int_s^t \sqrt{\varepsilon}\sigma^\varepsilon(X_u^\varepsilon)\mathrm{d}B_{u/\varepsilon}\right|}{|s-t|^\gamma} \geq R/2\right).$$

Therefore

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \bar{C}\left(\|X^\varepsilon\|_\gamma \geq R\right) = -\infty,$$

and the conclusion of the lemma is proved by Lemma 4.2 in [9]. □

By Theorem 3.2 and Lemma 3.4, we obtain the following result.

Theorem 3.3. *Let* $0 \leq \alpha < 1/2$ *and* $0 \leq \beta < 1$. *Assume that* $(\mathrm{H1})_u$, $(\mathrm{H2})_u$ *and* $(\mathrm{H3})_u$ *hold. Then for any closed subset* F *and any open subset* O *in* $(C_0^\alpha([0,T],\mathbb{R}^d), \|\cdot\|_\alpha) \times (C_0^\beta([0,T],\mathbb{R}^{d\times d}), \|\cdot\|_\beta) \times (C_0^\alpha([0,T],\mathbb{R}^n), \|\cdot\|_\alpha)$

$$\limsup_{\varepsilon\to 0} \varepsilon \log \bar{C}\left(\left(\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}, X_t^{x,\varepsilon} - x\right)|_{t\in[0,T]} \in F\right) \leq -\inf_{(f,g,\psi)\in F} \hat{I}(f,g,\psi), \quad (3.15)$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon \log \bar{C}\left(\left(\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}, X_t^{x,\varepsilon} - x\right)|_{t\in[0,T]} \in O\right) \geq -\inf_{(f,g,\psi)\in O} \hat{I}(f,g,\psi). \quad (3.16)$$

3.3. *Proof of Theorem 3.1: general case*

Lemma 3.5. *Assume that* $(\mathrm{H2})_u$ *and* $(\mathrm{H3})_u$ *hold. Then*

$$\limsup_{r\to\infty} \lim_{\varepsilon\to 0} \varepsilon \log \bar{C}\left(\sup_{0\leq s\leq T} |X^{x,\varepsilon}(s)| \geq r\right) = -\infty. \quad (3.17)$$

Proof. By the proof of Lemma 5.1 in [9], there exist constants $c_1(T), c_2(T) > 0$ such that for any $0 < \varepsilon < 1/2, x \in \mathbb{R}^n, t \in [0,T]$,

$$\bar{\mathbb{E}}\left(\sup_{0\leq t\leq T}(1+|X^{x,\varepsilon}(t)|^2)^{1/\varepsilon}\right) \leq \frac{c_1(T)(1+|x|^2)^{1/\varepsilon}}{\varepsilon} \exp\{c_2(T)/\varepsilon\}. \quad (3.18)$$

Now by Chebyshev's inequality, we have

$$\bar{C}\left(\sup_{0\leq s\leq T} |X^{x,\varepsilon}(s)| \geq r\right) \leq \frac{c_1(T)(1+|r|^2)^{-1/\varepsilon}(1+|x|^2)^{1/\varepsilon}}{\varepsilon} \exp\{c_2(T)/\varepsilon\}$$

which implies (3.17). □

By Lemma 3.5 and the proofs of Lemmas 3.2 and 3.3, we can get also the following estimate.

Lemma 3.6. *Assume that* $(\mathrm{H2})_u$ *and* $(\mathrm{H3})_u$ *hold. Then for any* $a \geq 0$ *and* $\rho > 0$,

$$\limsup_{\substack{\mu\to 0\\ \nu\to 0}} \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{\|f\|_H^2\leq a, g\in\mathbb{A}} \bar{C}\Big(\|X^{x,\varepsilon} - \Psi(f,g)\| > \rho,$$
$$\|\varepsilon B_{\cdot/\varepsilon} - f\| < \mu, \|\varepsilon\langle B\rangle_{\cdot/\varepsilon} - g\| < \nu\Big) = -\infty. \quad (3.19)$$

By Lemma 3.5 and the proofs of Lemma 3.4, the following tightness holds also.

Lemma 3.7. *Let* $0 \leq \alpha < 1/2$ *and* $0 \leq \beta < 1$. *Assume that* $(\mathrm{H2})_u$ *and* $(\mathrm{H3})_u$ *hold. Then*

$$\left\{\bar{C}\left(\left(\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}, X_t^{x,\varepsilon} - x\right)|_{t\in[0,T]} \in \cdot\right), \varepsilon > 0\right\}$$

is exponentially tight in $(C_0^\alpha([0,T],\mathbb{R}^d), \|\cdot\|_\alpha) \times (C_0^\beta([0,T],\mathbb{R}^{d\times d}), \|\cdot\|_\beta) \times (C_0^\alpha([0,T],\mathbb{R}^n), \|\cdot\|_\alpha)$.

Now by Lemma 3.7 and the proofs of Theorem 3.2, we obtain the following LDP which implies Theorem 3.1 by the contraction principle.

Theorem 3.4. *Let* $0 \leq \alpha < 1/2$ *and* $0 \leq \beta < 1$. *Assume that* $(\mathrm{H2})_u$ *and* $(\mathrm{H3})_u$ *hold. Then for any closed subset* F *and any open subset* O *in* $(C_0^\alpha([0,T],\mathbb{R}^d), \|\cdot\|_\alpha) \times (C_0^\beta([0,T],\mathbb{R}^{d\times d}), \|\cdot\|_\beta) \times (C_0^\alpha([0,T],\mathbb{R}^n), \|\cdot\|_\alpha)$

$$\limsup_{\varepsilon\to 0} \varepsilon \log \bar{C}\left(\left(\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}, X_t^{x,\varepsilon} - x\right)|_{t\in[0,T]} \in F\right) \leq -\inf_{(f,g,\psi)\in F} \hat{I}(f,g,\psi), \quad (3.20)$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon \log \bar{C}\left(\left(\varepsilon B_{t/\varepsilon}, \varepsilon\langle B\rangle_{t/\varepsilon}, X_t^{x,\varepsilon} - x\right)|_{t\in[0,T]} \in O\right) \geq - \inf_{(f,g,\psi)\in O} \hat{I}(f,g,\psi). \quad (3.21)$$

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Appendix. Some general results on large deviations for the capacity

In this Appendix, we present some general results on large deviations for the $\bar{C}$-capacity. The proofs of these results are the same as probability case (cf. [5,7]). Here, we only give a proof of Varadhan's integral theorem.

Definition A.1. Let (S, ρ) be a Polish space. Let $(V^\varepsilon, \varepsilon > 0)$ be a family of measurable maps from Ω into (S, ρ) and let $\lambda(\varepsilon), \varepsilon > 0$ be a positive function satisfying $\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$. A nonnegative function I on (S, ρ) is called to be (good) rate function if $\{I \leq l\}$ is (compact) closed for all $0 \leq l < \infty$.

(1) $(\bar{C}(V^\varepsilon \in \cdot), \varepsilon > 0)$ is said to satisfy large deviation principle (LDP) with speed $\lambda(\varepsilon)$ and rate function $I(x)$ if for any closed subset $F \subset S$,

$$\limsup_{\varepsilon\to 0} \lambda(\varepsilon) \log \bar{C}(V^\varepsilon \in F) \leq - \inf_{x\in F} I(x); \quad (A.1)$$

and for any open subset $O \subset S$,

$$\liminf_{\varepsilon\to 0} \lambda(\varepsilon) \log \bar{C}(V^\varepsilon \in O) \geq - \inf_{x\in O} I(x). \quad (A.2)$$

(A.1) is referred to as upper bound of large deviations with speed $\lambda(\varepsilon)$ and rate function $I(x)$ (ULD) and (A.2) is lower bound of large deviations with speed $\lambda(\varepsilon)$ and rate function $I(x)$ (LLD).

(2) $(\bar{C}(V^\varepsilon \in \cdot), \varepsilon > 0)$ is said to satisfy w-upper bound of large deviations with speed $\lambda(\varepsilon)$ and rate function $I(x)$ if for any compact subset $K \subset S$,

$$\limsup_{\varepsilon\to 0} \lambda(\varepsilon) \log \bar{C}(V^\varepsilon \in K) \leq - \inf_{x\in K} I(x). \quad (A.3)$$

If $(\bar{C}(V^\varepsilon \in \cdot), \varepsilon > 0)$ satisfies w-upper bound of large deviations with speed $\lambda(\varepsilon)$ and rate function $I(x)$ and lower bound of large deviations with speed $\lambda(\varepsilon)$ and rate function $I(x)$, then $(\bar{C}(V^\varepsilon \in \cdot), \varepsilon > 0)$ is called to satisfy w-large deviation principle.

(3) $(\bar{C}(V^\varepsilon \in \cdot), \varepsilon > 0)$ is said to be exponentially tight if for any $L > 0$, there exists a compact set $K_L \subset S$ such that

$$\limsup_{\varepsilon\to 0} \lambda(\varepsilon) \log \bar{C}(V^\varepsilon \in K_L^c) \leq -L. \quad (A.4)$$

Lemma A.1. *Let $(\bar{C}(V^\varepsilon \in \cdot), \varepsilon > 0)$ satisfy w-large deviation principle with speed $\lambda(\varepsilon)$ and rate function I. Then it satisfies large deviation principle with the rate function I if $(\bar{C}(V^\varepsilon \in \cdot), \varepsilon > 0)$ is exponentially tight.*

Lemma A.2. *Suppose $S = \mathbb{R}^d$. If for any $y \in \mathbb{R}^d$, there exists a $\delta > 0$ such that*

$$\Lambda(\delta y) := \limsup_{\varepsilon \to 0} \lambda(\varepsilon) \log \bar{\mathbb{E}} \exp \left\{ \frac{\langle V^\varepsilon, \delta y \rangle}{\lambda(\varepsilon)} \right\} \in \mathbb{R}, \tag{A.5}$$

then $(\bar{C}(V^\varepsilon \in \cdot), \varepsilon > 0)$ satisfies upper bound of large deviations with speed $\lambda(\varepsilon)$ and good rate function Λ^* defined by

$$\Lambda^*(x) = \sup_{y \in \mathbb{R}^d} \{\langle x, y \rangle - \Lambda(y)\}, \quad x \in \mathbb{R}^d.$$

Remark A.1. Since $\bar{\mathbb{E}}$ is not linear, Cramér's method is not useful for lower bound of large deviations.

Lemma A.3 (*Varadhan Integral Theorem*)**.** *Let S be a Polish space.*

(1) Let $(\bar{C}(V^\varepsilon \in \cdot), \varepsilon > 0)$ satisfy the LLD with speed $\lambda(\varepsilon)$ and rate function I. If $\Phi : S \to [-\infty, +\infty]$ is lower semicontinuous (l.s.c.), then

$$\liminf_{\varepsilon \to 0} \lambda(\varepsilon) \log \bar{\mathbb{E}} \left(\exp \left(\frac{\Phi(V^\varepsilon)}{\lambda(\varepsilon)} \right) \right) \geq \sup \{\Phi(x) - I(x);\ \Phi(x) \wedge I(x) < +\infty\}. \tag{A.6}$$

(2) Let $(\bar{C}(V^\varepsilon \in \cdot), \varepsilon > 0)$ satisfy the ULD with speed $\lambda(\varepsilon)$ and good rate function I. If $\Phi : S \to [-\infty, +\infty]$ is upper semicontinuous (u.s.c.) and there exists some $\delta > 0$ such that

$$\limsup_{\varepsilon \to 0} \lambda(\varepsilon) \log \bar{\mathbb{E}} \left(\exp \left((1+\delta) \frac{\Phi(V^\varepsilon)}{\lambda(\varepsilon)} \right) \right) < \infty, \tag{A.7}$$

then

$$\limsup_{\varepsilon \to 0} \lambda(\varepsilon) \log \bar{\mathbb{E}} \left(\exp \left(\frac{\Phi(V^\varepsilon)}{\lambda(\varepsilon)} \right) \right) \leq \sup \{\Phi(x) - I(x) : x \in S\}. \tag{A.8}$$

Proof. (1) Let $x \in S$ with $\Phi(x) \wedge I(x) < +\infty$. Then for any neighborhood N_x of x,

$$\begin{aligned} \liminf_{\varepsilon \to 0} \lambda(\varepsilon) \log \bar{\mathbb{E}} \left(\exp \left(\frac{\Phi(V^\varepsilon)}{\lambda(\varepsilon)} \right) \right) &\geq \liminf_{\varepsilon \to 0} \lambda(\varepsilon) \log \bar{\mathbb{E}} \left(\exp \left(\frac{\Phi(V^\varepsilon)}{\lambda(\varepsilon)} \right) I_{N_x}(V^\varepsilon) \right) \\ &\geq \inf_{y \in N_x} \Phi(y) + \liminf_{\varepsilon \to 0} \lambda(\varepsilon) \log \bar{C}(V^\varepsilon \in N_x) \\ &\geq \inf_{y \in N_x} \Phi(y) - I(x). \end{aligned}$$

Therefore, (A.6) holds by the lower semicontinuity of Φ.

(2) First, assume $\sup_{x \in E} \Phi(x) \leq M < \infty$ for some $M > 0$. Given $L > 0$, set $K_L := \{I \leq L\}$. Choose $x_1, \ldots, x_n \in K_L$ and their neighborhood $B_{x_1}, \ldots, B_{x_n}$ such that $K_L \subset \bigcup_{i=1}^n B_{x_i} := G$, $\sup_{x \in \bar{B}_{x_i}} \Phi(x) \leq \Phi(x_i) + \delta$ and $\inf_{x \in \bar{B}_{x_i}} I(x) \geq I(x_i) - \delta$. Then

$$\begin{aligned} \bar{\mathbb{E}} \left(\exp \left(\frac{\Phi(V^\varepsilon)}{\lambda(\varepsilon)} \right) \right) &\leq \exp \left\{ \frac{1}{\lambda(\varepsilon)} (M + \lambda(\varepsilon) \log \bar{C}(V^\varepsilon \in G^c)) \right\} \\ &\quad + \sum_{i=1}^n \exp \left\{ \frac{1}{\lambda(\varepsilon)} \left(\Phi(x_i) + \delta + \lambda(\varepsilon) \log \bar{C}(V^\varepsilon \in \bar{B}_{x_i}) \right) \right\}. \end{aligned}$$

Thus

$$\begin{aligned}\limsup_{\varepsilon\to 0}\lambda(\varepsilon)\log\bar{\mathbb{E}}\left(\exp\left(\frac{\Phi(V^\varepsilon)}{\lambda(\varepsilon)}\right)\right) &\le (M-L)\vee\max_{1\le i\le n}\{\Phi(x_i)-I(x_i)+2\delta\}\\ &\le (M-L)\vee\sup_{x\in S}\{\Phi(x)-I(x)\}+2\delta.\end{aligned}$$

First letting $\delta\downarrow 0$, and then letting $L\uparrow\infty$, we obtain (A.8).

For general case, set $\Phi_M=\Phi\wedge M$. Then

$$\begin{aligned}\limsup_{\varepsilon\to 0}\lambda(\varepsilon)\log\bar{\mathbb{E}}\left(\exp\left\{\frac{\Phi(V^\varepsilon)}{\lambda(\varepsilon)}\right\}\right) &\le A_M\vee\sup_{x\in S}\{\Phi_M(x)-I(x)\}\\ &\le A_M\vee\sup_{x\in S}\{\Phi(x)-I(x)\},\end{aligned}$$

where

$$A_M:=\limsup_{\varepsilon\to 0}\lambda(\varepsilon)\log\bar{\mathbb{E}}\left(\exp\left\{\frac{\Phi(V^\varepsilon)}{\lambda(\varepsilon)}\right\}I_{\{\Phi\ge M\}}(V^\varepsilon)\right)\to-\infty$$

as $M\to\infty$. Therefore, (2) holds. □

If $S=C([0,T],\mathbb{R}^d)$, let $\mathcal{A}:=\{\{t_1,t_2,\ldots,t_n\}\subset[0,T];n\ge 1\}$ and $\Phi:X\to(\mathbb{R}^d)^{[0,T]}$, $x\to(x(t),t\in[0,T])$. For any $\alpha\in\mathcal{A}$, let p_α be the canonical projection of $(\mathbb{R}^d)^{[0,T]}$ to $(\mathbb{R}^d)^\alpha$. For any $x\in C([0,T],\mathbb{R}^d)$, set $\|x\|=\sup_{t\in[0,T]}|x(t)|$.

Lemma A.4. *Let $(\bar{C}(V^\varepsilon\in\cdot),\varepsilon>0)$ be exponentially tight. If $(\bar{C}(p_\alpha(\Phi(V^\varepsilon))\in\cdot),\varepsilon>0)$ satisfies the LDP with speed $\lambda(\varepsilon)$ and rate function I_α for any $\alpha\in\mathcal{A}$, then $(\bar{C}(V^\varepsilon\in\cdot),\varepsilon>0)$ satisfies the LDP with speed $\lambda(\varepsilon)$ and rate function I defined by*

$$I(x)=\sup_{\alpha\in\mathcal{A}}I_\alpha\left(p_\alpha(\Phi(x))\right),\quad x\in C([0,T],\mathbb{R}^d).$$

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