Review of *Singular Systems of Differential Equations*, by S. L. Campbell*

N. J. Rose
Department of Mathematics
North Carolina State University
Raleigh, North Carolina 27650

This monograph is concerned with various problems associated with singular systems of linear differential or difference equations. Even though for the most part the coefficient matrices are assumed to be constant, there is a lot of interesting mathematics involved and a variety of applications to which the results can be applied.

Before describing the contents of this work, let us look at the simplest problem considered—that of solving the homogeneous system:

$$A\dot{x}+Bx=0,\qquad x(0)=x_0,\qquad t\geqslant 0, \tag{1}$$

where A and B are given constant $n\times n$ matrices, both of which may be singular. If A is singular, solutions, for a given initial vector x_0, may not exist or be unique. The condition that at most one solution exists is that $(\lambda A+B)^{-1}$ exists for some λ; this assumption is used throughout the book. Multiplying Eq. (1) by $(\lambda A+B)^{-1}$ we obtain the equivalent system

$$\hat{A}\dot{x}+\hat{B}x=0, \tag{2}$$

where $\hat{A}=(\lambda A+B)^{-1}A$ and $\hat{B}=(\lambda A+B)^{-1}B$. Letting $\hat{A}^D$ denote the Drazin inverse of $\hat{A}$, solutions exist if and only if

$$\hat{A}\hat{A}^D x_0=x_0, \tag{3}$$

and, if so, the solution is given by

$$x=e^{-\hat{A}^D\hat{B}t}x_0. \tag{4}$$

Note that $\hat{B}=I-\lambda\hat{A}$, so that Eqs. (3) and (4) depend only on the matrix $\hat{A}$.

*Pitman, Marshfield, MA.

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Similar, but more complicated results are obtained for the nonhomogeneous version of Eq. (1).

It is interesting to note that the role of the Drazin inverse in the solution of Eq. (1) is quite analogous to the role of the Moore–Penrose inverse (or any "1-inverse") in the solution of the algebraic system

$$Ax = b. \tag{5}$$

Recall that solutions exist for Eq. (5) if and only if $AA^+b = b$ and, if so, the solutions are given by

$$x = A^+b + (I - A^+A)h, \tag{6}$$

where A^+ is the Moore–Penrose inverse of A and h is an arbitrary vector.

Linear algebra and matrix theory are used extensively throughout the book. The relevant material is summarized in Chapter 1, which includes brief, but self-contained treatments of the Drazin inverse, functions of a matrix, and representation of matrix functions by contour integrals, the so-called "functional calculus" of operator theory. The second chapter provides a host of examples of singular problems from control theory, electrical circuits, population growth, economics, and singular perturbation problems.

In Chapter 3 closed form solutions for the nonhomogeneous version of Eq. (1) are developed in the case when unique solutions exist for appropriate initial conditions. The case when A and B are rectangular matrices is solved when $\lambda A + B$ has maximum rank. These results are applied to a control problem. Discrete systems analogous to Eq. (1) are discussed and some interesting results are obtained for the delay equation

$$A\dot{x}(t) + Bx(t) = Cx(t-1) + f(t)$$

when A is singular.

In Chapter 4, some of the results of Chapter 3 are obtained using the Laplace Transform. This is accomplished by finding an explicit representation for the coefficients in the Laurent expansion of $(sA + B)^{-1}$.

Chapter 5 is perhaps the most interesting chapter; it is devoted to singular perturbation problems. The simplest problem is

$$\begin{aligned} \dot{x} &= A_1x + A_2y, \\ \varepsilon\dot{y} &= B_1x + B_2y, \qquad x(0) = x_0, \qquad y(0) = y_0, \end{aligned} \tag{7}$$

where ε is a "small" parameter. When $\varepsilon > 0$, Eq. (7) can be written

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = (\tilde{A} + \tilde{B}/\varepsilon)\begin{bmatrix} x \\ y \end{bmatrix}, \tag{8}$$

where

$$\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 0 & 0 \\ B_1 & B_2 \end{bmatrix}. \tag{9}$$

The unique solution of Eq. (8) is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{(\tilde{A} + \tilde{B}/\varepsilon)t}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \tag{10}$$

The fundamental questions are (i) under what conditions does the solution of Eq. (10) approach a solution of the reduced (singular) system

$$\dot{x} = A_1 x + A_2 y,$$

$$0 = B_1 x + B_2 y,$$

as $\varepsilon \to 0^+$, (ii) to find an asymptotic expansion for the solution. To answer (i), it is necessary to consider the limit of the exponential term in Eq. (10). The author obtains an explicit representation for this limit using the notion of a semistable matrix; a (square) matrix is semistable if it has index 0 or 1 and all nonzero eigenvalues have negative real part. The result on (i) is

$$\operatorname*{Lim}_{\varepsilon \to 0^+} e^{(\tilde{A} + \tilde{B}/\varepsilon)t} = e^{(I - \tilde{B}\tilde{B}^D)At}(I - \tilde{B}\tilde{B}^D),$$

where the limit exists for all $t \geqslant 0$, if and only if B is semistable. In obtaining this result, and the asymptotic expansion, extensive use is made of the functional calculus. More general perturbation problems of the form

$$A(\varepsilon)\dot{x} + B(\varepsilon)x = f$$

are also considered, with appropriate conditions on the matrices.

In Chapter 6, systems with variable coefficients are considered with emphasis on conditions when a change of variables will convert the system to one with constant coefficients. The concluding chapter is a brief, but perceptive, discussion of algorithms suitable for numerical solution of singular systems.

The book is an outgrowth of the research papers of the author and some of his co-workers during the period 1976–1979, thus these relatively recent results are now available in book form. As a whole the writing is clear, if somewhat concise. There are perhaps 20 or so typographical errors, none of which should deter a careful reader for long. The methods and results should be interesting to mathematicians, engineers, and scientists who are interested in singular linear problems.

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