Results on homomorphic realization of automata by $\alpha_0$-products

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Abstract

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The notion of an irreducible semigroup has been fundamental to the Krohn–Rhodes decomposition. In this paper we study a similar concept and point out its equivalence with the Krohn–Rhodes irreducibility. We then use the new aspect of irreducible semigroups to provide cascade decompositions of automata in a situation when a strict letter-to-letter replacement is essential. The results are stated in terms of completeness theorems. Our terminology follows Gécseg (1986), so that the cascade composition is referred to as the $\alpha_0$-product.

Introduction

There are two different approaches in the structural theory of finite automata. When no sharp distinction is made between transformations induced by input letters and those induced by input words, semigroup theoretical methods come into effect. The celebrated Prime Decomposition Theorem of Krohn and Rhodes is a beautiful result in the semigroup theoretical approach. In the opposite case, when one accepts the distinction as that between operations and derived operations, methods of universal algebra become applicable. Even though the semigroup theoretical methods have been much more successful in the field. In the recent papers [5, 12], we looked for conditions that together with those implied by the Krohn–Rhodes decomposition ensure completeness with respect to the cascade composition also when letters have to be assigned to letters. We developed a construction using

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counters and definite automata to attain the group-like automaton corresponding
to a subgroup $G$ of the semigroup of an automaton, provided that the transformations
belonging to $G$ are induced by words of some constant length. On the basis of this
construction we gave a definition of "divisibility in equal lengths" between a
semigroup and the semigroup of an automaton in [4]. The aim of the present paper
is to explore this new notion of divisibility. The Krohn–Rhodes decomposition
readily implies that only irreducible semigroups can be irreducible with respect to
divisibility in equal lengths. So the best that can be hoped for is that the converse
is also true, which is one of the results here. The new aspect of irreducible semigroups
is then used to obtain (relative) completeness criteria for some classes of automata
that arise in connection with the Krohn–Rhodes decomposition. The results have
been announced in [6, 11]. For some consequences not treated here see [7, 8].

1. Basic notions and notation

Given a finite nonempty set $X$, let $X^*$ denote the free monoid of all words over $X,$ including the empty word $\lambda$. We set $X^+ = X^* - \{\lambda\}$ and $X^\lambda = X \cup \{\lambda\}$. The length
of a word $u$ is denoted $|u|$. An automaton $A = (A, X, \delta)$ consists of the finite nonempty
sets $A$ (states), $X$ (input letters) and transition $\delta : A \times X \rightarrow A$ that extends to words
as usual. Given a word $u \in X^*$ we define the transformation $uA : A^+ \rightarrow A$ by
$auA = \delta((a, u), \cdots)$, for all $a \in A$. Set $S_i(A) = \{u^\lambda : u \in X^*\}$ and $S(A) = \{u^\lambda : u \in X^+\}$. $S_i(A)$ is
called the characteristic monoid of $A$, while $S(A)$ is the characteristic semigroup of $A$.

Our fundamental notion is the $\alpha_0$-product (cascade composition) of automata, see
[1, 15, 16]. Let $A_t = (A_t, X_t, \delta_t), t = 1, \ldots, n, n \geq 0$, be automata. For each $t$, let
$\varphi_t : A_1 \times \cdots \times A_{t-1} \times X \rightarrow X^+_t$
be a (feedback) function, where $X$ is a new finite nonempty set. The $\alpha_0^+$-product
$A = A_1 \times \cdots \times A_n(X, \varphi)$ is defined to be the automaton $(A, X, \delta)$, where $A = A_1 \times \cdots \times A_n$ and
\[
\delta((a_1, \ldots, a_n), x) = (\delta_1(a_1, u_1), \ldots, \delta_n(a_n, u_n)),
\]
$u_t = \varphi_t(a_1, \ldots, a_{t-1}, x), t = 1, \ldots, n,$
for all $(a_1, \ldots, a_n) \in A$ and $x \in X$. In the special case that each feedback function
$\varphi_t$ maps into $X^+_t$ ($X^+_t$, $X_t$) we obtain the notion of the $\alpha_0^+$-product ($\alpha_0^+$-product,
$\alpha_0$-product). A quasi-direct product is an $\alpha_0$-product with the additional property
that no feedback function depends on any state variable. Thus if $A = A_1 \times \cdots \times A_n(X, \varphi)$ is a quasi-direct product with components $A_t = (A_t, X_t, \delta_t)$, then
each $\varphi_t$ can be viewed as a mapping $X \rightarrow X_t$. When $X_1 = \cdots = X_n = X$ and each $\varphi_t$ is the identical mapping $X \rightarrow X$, the quasi-direct product reduces to the direct product.

Let $\mathcal{H}$ be any class of automata. We define
- $\text{P}_\alpha(\mathcal{H}) :=$ all $\alpha_0$-products of automata from $\mathcal{H}$;
- $\text{P}_q(\mathcal{H}) :=$ all quasi-direct products of automata from $\mathcal{H}$;
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- $P(\mathcal{K}) := \text{all direct products of automata from } \mathcal{K}$;
- $H(\mathcal{K}) := \text{all homomorphic images of automata from } \mathcal{K}$;
- $S(\mathcal{K}) := \text{all subautomata of automata from } \mathcal{K}$.

The operators $P_{\alpha_0}, P_{\alpha_0}^+, P_{\alpha_0}^\lambda$ are defined likewise and correspond to the formations of $\alpha_0^+$-products, $\alpha_0^\lambda$-products and $\alpha_0^\lambda$-products. We write

$$P_{\alpha_0}(\mathcal{K}) = \{A(X, \varphi) : A \in \mathcal{K}, A(X, \varphi) \text{ is quasi-direct product}\}$$

for the class of all first quasi-direct powers of automata from $\mathcal{K}$.

The main object of study in this paper is the combination $HSP_{\alpha_0}(\mathcal{K})$. It is known that $HSP_{\alpha_0}(\mathcal{K})$ is the $\alpha_0$-variety generated by $\mathcal{K}$, i.e., the smallest class that contains $\mathcal{K}$ and is closed under the operators $H, S$ and $P_{\alpha_0}$. Similar facts are true for the rest of our product notions. Any class closed under $H, S$ and $P_{\alpha_0}$ for $\alpha \in \{*, +, \lambda\}$ will be referred to an $\alpha_0$-variety. Every $\alpha_0^\lambda$-variety is both an $\alpha_0^+$-variety and an $\alpha_0^\lambda$-variety. Every $\alpha_0^+$-variety or $\alpha_0^\lambda$-variety is an $\alpha_0$-variety.

As defined here, the $\alpha_0$-product is obtained as a special case of each of the following: $\alpha_0^*$-product, $\alpha_0^+$-product and $\alpha_0^\lambda$-product. It is however important to observe that the converse also holds. For an automaton $A = (A, X, \delta)$ define $A^* = (A, S_1(A), \delta^*)$ with $\delta^*(a, u^A) = \delta(a, u)$, for all $a \in A$ and $u \in X^*$. Similarly, let $A^+ = (A, S(A), \delta^+)$ and $A^\lambda = (A, \{x^A : x \in X^\lambda\}, \delta^\lambda)$, where $\delta^\lambda(a, u^A) = \delta(a, u)$ and $\delta^\lambda(a, x^A) = \delta(a, x)$, for all $a \in A$, $u \in X^+$ and $x \in X^\lambda$. Notice that $S_1(A^*) = S_1(A^+) = S_1(A^\lambda) = S(A^*) = S(A^+) = S(A^\lambda) = S_1(A)$. If $\mathcal{K}$ is a class of automata and $\alpha$ is any modifier $*, +$ or $\lambda$, then we have $P_{\alpha_0}(\mathcal{K}) = P_{\alpha_0}(\mathcal{K}^\alpha)$, so that the $\alpha_0^\lambda$-product can be defined in terms of the $\alpha_0$-product. The automaton $A^*$ corresponds to the transformation monoid of the automaton $A$ and $A^+$ is just the transformation semigroup of $A$. The operators $P_{\alpha_0}^+, P_{\alpha_0}^\lambda$ thus correspond to the wreath product of transformation semigroups and/or monoids, see [1, 9, 19]. To be more explicit, there is a 1-1 correspondence between $\alpha_0^+$-varieties ($\alpha_0^\lambda$-varieties) and closed classes of transformation semigroups (transformation monoids) as defined in [9]. In the sequel we will use some elementary concepts of semigroup and group theory as well as universal algebra. For the latter we refer to [17]. Except for free semigroups, a semigroup is always assumed to be finite. If $S$ is a semigroup then $S^1$ is the smallest monoid containing $S$ as a subsemigroup, i.e., $S^1 = S$ if $S$ is a semigroup and $S^1$ is obtained from $S$ by adjoining an identity element if $S$ is not a monoid. By $\text{Aut}(S)$ we denote the automaton $(S^1, S, \delta)$ with $\delta(s, t) = st$ for all $s \in S^1$ and $t \in T$. We set $\text{Aut}(\mathcal{F}) = \{\text{Aut}(S) : S \in \mathcal{F}\}$ for a class $\mathcal{F}$ of semigroups (or groups).

2. The Krohn–Rhodes decomposition

The Krohn–Rhodes Decomposition Theorem, that we recall below, provides the basis for studying the $\alpha_0$-product. The theorem itself has a number of equivalent formalizations in terms of automata, transformation semigroups and even semigroups, see [1, 9, 15, 16, 19]. Our formalization is most closely related to those in [15, 16]. We start with some more definitions.
Definition 2.1. Let $S$ and $T$ be semigroups. We say that $S$ divides $T$, written $S < T$, if $S$ is a homomorphic image of a subsemigroup of $T$.

It is known that if $S$ is a monoid (group) then $S < T$ if and only if $S$ is a homomorphic image of a submonoid (subgroup) of $T$.

Definition 2.2. A semigroup $S$ is irreducible if for every nonempty class $K$ of automata and $A \in \text{HSP}_0(K)$, the condition $S < S(A)$ implies the existence of an automaton $B \in K$ with $S < S(B)$.

It is immediately seen that we can replace $P_0$ by the operator $P_0^*$ in the above definition. Replacing $P_0$ by $P_0^*$ or $P_0^\lambda$ we still get the same class of irreducible semigroups if we use characteristic monoids instead of characteristic semigroups, although now somewhat more argument is needed. Nevertheless we will make use of these observations. Following [1], we denote by $U_3$ a monoid consisting of the identity and two right zero elements. The divisors of $U_3$ are the trivial semigroup $U_0$, the two-element right zero semigroup $U_1$ and the two element monoid with a right zero $U_2$. The semigroups $U_i$, $i = 0, 1, 2, 3$, are called units. For later use we define $U_i - \text{Aut}(U_i)$. We note that in each of our considerations, the automata $U_3$ and $U_1$, that have three states, can be replaced by two-state automata: the identity reset automaton and the reset automaton.

Theorem 2.3 (Krohn–Rhodes Decomposition Theorem, Part I). The irreducible semigroups are the simple groups and the units.

The second part of the Krohn–Rhodes Decomposition Theorem is a strong kind of completeness result. Recall that a permutation automaton is an automaton $A = (A, X, \delta)$ such that each transformation $x^A$, $x \in X$, is a permutation of the state set. An equivalent condition is that $S_1(A)$ is a group. A discrete automaton is an automaton as above with $x^A$ the identical mapping $A \to A$, for all $x \in X$.

Theorem 2.4 (Krohn–Rhodes Decomposition Theorem, Part II). Let $A$ be an automaton and $\mathcal{G}$ a set of simple groups containing an isomorphic copy of each simple group $G$ with $G < S(A)$. Then $A \in \text{HSP}_0(\text{Aut}(\mathcal{G}) \cup \{U_3\})$. If $A$ is a permutation automaton which is not discrete then $A \in \text{HSP}_0^\lambda(\text{Aut}(\mathcal{G}))$.

Let $\mathcal{G}$ be a nonempty class of simple groups. We define $\mathcal{K}_i(\mathcal{G}) = \text{HSP}_0(\text{Aut}(\mathcal{G}) \cup \{U_i\})$ for $i = 0, 1, 2, 3$. Furthermore, we define $\mathcal{K}_{1,2}(\mathcal{G}) = \text{HSP}_0(\text{Aut}(\mathcal{G}) \cup \{U_1, U_2\})$. Further combinations need not be treated for they coincide with one of the classes $\mathcal{K}_i(\mathcal{G})$ of $\mathcal{K}_{1,2}(\mathcal{G})$. The classes $\mathcal{K}_0(\mathcal{G})$, $\mathcal{K}_1(\mathcal{G})$ and $\mathcal{K}_3(\mathcal{G})$ are $\alpha_0^\lambda$-varieties, for instance $\mathcal{K}_3(\mathcal{G}) = \text{HSP}_0^\lambda(\text{Aut}(\mathcal{G}) \cup \{U_3\}) = \text{HSP}_0((\text{Aut}(\mathcal{G}) \cup \{U_3\})^*) = \text{HSP}_0^\lambda(\text{Aut}(\mathcal{G}) \cup \{U_3\})$. Similarly the classes $\mathcal{K}_1(\mathcal{G})$ and $\mathcal{K}_{1,2}(\mathcal{G})$ are $\alpha_0^\lambda$-varieties. Each class $\mathcal{K}_i(\mathcal{G})$ or $\mathcal{K}_{1,2}(\mathcal{G})$ is well-known and has some
kind of characterization. When $\mathcal{G}$ is the class of all simple groups, by the Krohn-Rhodes Decomposition Theorem, $\mathcal{H}_i(\mathcal{G})$ is the class of all automata and $\mathcal{H}_0(\mathcal{G})$ is the class of all permutation automata. The class $\mathcal{H}_i(\mathcal{G})$, where $i = 1, 2$, consists of those automata $A$ with $U_{i-1} \triangleleft S(A)$, and an automaton $A$ belongs to $\mathcal{H}_1(\mathcal{G})$ if and only if $U_1 \triangleleft S(A)$. Now let $\mathcal{G}$ solely consist of trivial groups. Then $\mathcal{H}_1(\mathcal{G})$ is the class of aperiodic automata, i.e. of automata whose characteristic semigroups only contain trivial subgroups. The classes $\mathcal{H}_i(\mathcal{G})$ and $\mathcal{H}_0(\mathcal{G})$ can be identified as the definite automata and the monotone (or partially ordered) automata. $\mathcal{H}_0(\mathcal{G})$ is the class of all trivial automata. The class $\mathcal{H}_1(\mathcal{G})$ can be called the class of locally monotone automata. For the above discussion as well as other characterizations see [9, 21, 22] and the references contained therein. For connections to language varieties we refer to [9, 20].

Next we treat some consequences of the Krohn-Rhodes Decomposition Theorem in terms of completeness. Let $\mathcal{K}$ and $\mathcal{H}_0$ be two classes of automata and take any variant of the $\alpha_0$-product. Let $Q$ be the corresponding product operator. We say that $\mathcal{H}_i$ is $\alpha_i$-complete ($\alpha_i^*$-complete, . . . ) for $\mathcal{H}_0$ if $\mathcal{H}_0 \subseteq HSQ(\mathcal{H}_i)$. Thus $\mathcal{H}_i$ is $\alpha_i$-complete for $\mathcal{H}_0$ if and only if $\mathcal{H}$ is included in the $\alpha_i$-variety generated by $\mathcal{H}_0$. In particular, an $\alpha_0$-complete ($\alpha_i^*$-complete, . . . ) class for the class of all automata is referred to an $\alpha_0$-complete ($\alpha_i^*$-complete, . . . ) class. To avoid trivial situations we assume that $\mathcal{G}$ contains a nontrivial simple group when writing $\mathcal{H}_0(\mathcal{G})$ below.

Corollary 2.5. Let $\mathcal{G}$ be a nonempty class of simple groups. A class $\mathcal{K}$ is $\alpha_0^*$-complete ($\alpha_i^*$-complete) for $\mathcal{H}_i(\mathcal{G})$, $i = 0, 1, 2, 3$, if and only if the following hold:

(i) For every $G \in \mathcal{G}$ there is $A \in \mathcal{K}$ with $G \triangleleft S(A)$ ($G < S(A)$).

(ii) There is an automaton $A \in \mathcal{K}$ with $U_i \triangleleft S(A)$ ($U_i < S(A)$).

$\mathcal{K}$ is $\alpha_0^*$-complete ($\alpha_i^*$-complete) for $\mathcal{H}_1(\mathcal{G})$ if and only if it satisfies (i) and (ii) with $i = 1, 2$.

Note that in (ii) above we could have written just $G \triangleleft S(A)$ even for $\alpha_i^*$-completeness.

Corollary 2.6. A class $\mathcal{K}$ is $\alpha_0^*$-complete ($\alpha_i^*$-complete) if and only if the following conditions hold.

(i) For every (simple) group there is $A \in \mathcal{K}$ with $G \triangleleft S(A)$ ($G < S(A)$).

(ii) There is an automaton $A \in \mathcal{K}$ with $U_i \triangleleft S(A)$ ($U_i < S(A)$).

In Section 5 we give necessary and sufficient conditions regarding $\alpha_0$-completeness for some of the classes $\mathcal{H}_i(\mathcal{G})$ and $\mathcal{H}_1(\mathcal{G})$.

3. Divisibility in equal lengths

Recall from Section 2 that $S \triangleleft S(A)$ for a semigroup $S$ and an automaton $A$ if and only if there is a subsemigroup $T$ of $S(A)$ which is mapped homomorphically
onto \( S \). Although this definition implicitly implies some structural properties of those words inducing the transformations contained in \( T \), it does not provide anything explicit whatsoever. From a semigroup theoretical standpoint any explicit requirement could be an irrelevant matter in most investigations, e.g. when no sharp distinction is made between transformations induced by letters and words. In a more automata theoretical view one is however also interested in exactly which words induce which transformations. Thus a corresponding notion of divisibility should reveal something explicit about this relation between words and induced transformations. Although a very detailed definition of divisibility could be given, taking into account every aspect of the words, such a concept would not provide a broad scope for achieving deep results in the theory of compositions of automata, mainly because of bad irreducibility properties. In other words, care should be taken in finding a right balance between semigroup theoretical and automata theoretical concepts. Below we define two slightly different notions of "divisibility in equal lengths" between a semigroup and an automaton that only take into account the lengths of the words. One of these notions already appears in [4] and can be traced back to [10], or in more preliminary form to [3, 5, 12]. Both divisibility relations will be used for providing cascade decompositions of automata by making use of counters and definite automata in a situation when a strict letter-to-letter replacement is essential. If one is intended to use also other primitive automata, such as reverse definite automata or commutative permutation automata, then certainly different divisibility concepts are needed. The references [4, 10] contain some indications.

**Definition 3.1.** Let \( S \) be a semigroup and \( A = (A, X, \delta) \) an automaton.

(i) \( S \mid^{(n)} S(A) \) for an integer \( n \geq 1 \) if and only if there are a subsemigroup \( T \) of \( S(A) \) and an onto homomorphism \( \psi: T \to S \) such that

\[
\psi^{-1}(s) \cap \{u^k : u \in X^+, |u| = n\} \neq \emptyset, \quad \text{for all } s \in S.
\]

(ii) \( S \mid S(A) \) if and only if \( S \mid^{(n)} S(A) \) for some \( n \).

(iii) \( S \mid||^{(n)} S(A) \) for an integer \( n \geq 1 \) if and only if there is a subsemigroup \( T \) of \( S(A) \) which can be mapped homomorphically onto \( S \) and which satisfies

\[
T \subseteq \{u^k : u \in X^+, |u| = n\},
\]

i.e., each member of \( T \) is induced by some word of length \( n \).

(iv) \( S \parallel S(A) \) if and only if \( S \parallel^{(n)} S(A) \) for some \( n \geq 1 \).

**Remark 3.2.** If \( S \) is a monoid (group), then in (iii) above we can require that \( T \) be a submonoid (subgroup) of \( S \). It is obvious that \( S \parallel^{(n)} S(A) \) implies \( S \parallel^{(n)} S(A) \), which in turn yields \( S < S(A) \). Consequently, if \( S \parallel S(A) \) then also \( S \mid S(A) \), and if \( S \mid S(A) \) then \( S < S(A) \). None of the converse implications holds, see the examples below. Although the two relations \(|\) and \(\parallel\) are different, they coincide in the important special case that \( S^2 = S \), see Lemma 3.3. Note that a semigroup \( S \) satisfies the condition \( S^2 = S \) if and only if \( S = SES \) with \( E \) denoting the set of idempotent
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It is proved in [7] that for any semigroup $S$ with $|S| = k$ we have $S^k = SES$, a subsemigroup $T$ of $S$ satisfying $T^2 = T$. We think that each of $|n|$ and $||n||$ has some advantage on the other. E.g., $|n|$ has better irreducibility properties with respect to the $a_0$-product, while $||n||$ gives stronger decomposition results (see Propositions 4.3, 4.4 and Theorem 5.1).

Examples. Let $Z_2$ denote the additive group of integers modulo 2 and let $C_2 = (\{0, 1\}, x, \delta)$ be a counter with length 2, i.e., $\delta(i, x) = i + 1 \mod 2$, $i = 0, 1$. It is clear that $S(C_2) \cong Z_2$, so that $Z_2 < S(C_2)$. On the other hand the relation $Z_2 \not| S(C_2)$ does not hold. Let $S$ be a nilpotent semigroup with $|S| = 2$. We show that there is an automaton $A$ with $S \not| S(A)$ and such that $S || S(A)$ does not hold. Let $A = (\{0, 1, 2, 3\}, \{x, y\}, \delta)$ with $\delta(0, x) = 1$, $\delta(0, y) = 2$ and $\delta(i, z) = 3$ for $i \geq 1$ and $z = x, y$. Thus $A$ is a commutative automaton generated by a single state (and henceforth a free automaton ). $S(A)$ consists of the transformations $x^A, y^A$ and $xy^A$.

Let $\theta$ be the congruence with blocks $\{x, xy\}$ and $\{y\}$. Since $S(A)/\theta \cong S$ we see that $S \not| (1) S(A)$. If $T$ is a subsemigroup of $S(A)$ contained in the set $\{w^A : w \in \{x, y\}^*, |w| = n\}$ for an integer $n \geq 1$, then $T$ is trivial. Therefore $S || S(A)$ does not hold.

Lemma 3.3. Let $A = (A, X, \delta)$ be an automaton and $S$ a semigroup with $S^2 = S$. If $S \not| (n) S(A)$ then $S \not| (m) S(A)$ for a multiple $m$ of $n$. Thus $S \not| S(A)$ if and only if $S || S(A)$.

Proof. Since $S \not| (n) S(A)$, there are a subsemigroup $T$ of $S(A)$ and a homomorphism $\psi$ of $T$ onto $S$ such that

$$\psi^{-1}(s) \cap \{u^A : u \in X^+, |u| = n\} \neq \emptyset,$$

for all $s \in S$.

Set

$$T_0 = T \cap \{u^A : u \in X^+, |u| = n\}.$$

We have $\psi(T_0) = S$. Successively compute the powers $T_0^l$ ($l \geq 1$) until a repetition occurs, say $T_0^{k+d} = T_0^k$ for some $k, d \geq 1$. It follows that $T_0^{k+l} = T_0^{k+l}$ for all $l \geq 0$. Hence we may assume that $d$ divides $k$. But then, since $T_0^k = T_0^{k+ld} = T_0^{k+ld} = \cdots$, we obtain $T_0^k = T_0^{2k}$, showing that $T_0^k$ is a subsemigroup of $T$. Since $S^2 = S$, the restriction of $\psi$ onto $T_0^k$ maps $T_0^k$ homomorphically onto $S$. On the other hand, each transformation in $T_0^k$ is induced by a word of length $m = kn$. □

The easy proof of the next statement is omitted.

Proposition 3.4. Let $A$ be an automaton and $S$ a semigroup with $S^2 = S$. Suppose $m$ and $n$ are positive integers such that $m$ is a multiple of $n$.

(i) If $S \not| (n) S(A)$ then $S \not| (m) S(A)$.

(ii) If $S || (n) S(A)$ then $S || (m) S(A)$.
The semigroups $S$ under consideration here will be mainly monoids or groups, anyway they will satisfy $S^2 = S$. If this is the case, on the basis of Lemma 3.3, we feel free to write $S \parallel S(A)$ even if only $S \mid S(A)$ has been established.

It would be interesting to characterize those semigroups $S$ (with $S^2 = S$) for which $S < S(A)$ always implies $S \parallel S(A)$. Some partial results including the group case are given below. Recall that a semigroup $S$ is (ideal) simple if $SsS = S$ for all $s \in S$. Simple semigroups are described up to isomorphism by the Rees–Suschkewitsch Theorem, see [1, 17]. In our proof of Proposition 3.5 we shall not however use the Rees–Suschkewitsch representation. In fact, besides the definition, we shall only use that every simple semigroup is the disjoint union of pairwise isomorphic maximal subgroups. Moreover, if $\psi$ is a homomorphism of a simple semigroup $S$ onto a group $G$ and $H$ is a maximal subgroup of $S$, then the restriction of $\psi$ to $H$ is a homomorphism of $H$ onto $G$.

Since abelian groups are closed under subdirect product each semigroup $S$ has a unique minimum congruence relation $\rho$ such that $S/\rho$ is an abelian group. Let $S'$ be the block containing the idempotents under the partition induced by $\rho$. Clearly then $S'$ is a subsemigroup of $S$. If $S$ is a group then $S'$ is the commutator subgroup of $S$.

**Proposition 3.5.** Let $S$ be a simple semigroup.

(i) If $S < S(A)$ for an automaton $A$ then $S' \parallel S(A)$.

(ii) The following two conditions are equivalent:

1. For every automaton $A$, $S < S(A)$ implies $S/\rho S(A)$.

2. $S$ has no nontrivial abelian group quotient.

**Proof.** To prove our first statement assume that $S < S(A)$, where $A = (A, X, \delta)$ is any automaton. Let $T$ be a subsemigroup of $S(A)$ that is mapped homomorphically onto $S$ under a homomorphism $\psi$. Put $\rho = \text{Ker } \psi$, so that $\rho$ is the congruence relation induced by $\psi$ and $T/\rho \equiv S$. Let $e$ be any fixed idempotent of $S$ and $T_e = \psi^{-1}(e)$, a subsemigroup of $T$. Let $m \equiv 1$ be the greatest integer such that $\lvert u \rvert = 0 \mod m$ for all $u \in X^+$ with $u^k \in T_e$. We define a function $\# : T \to \mathbb{Z}_m$ by $\#(g) = i$ if and only if there is a word $u \in X^+$ with $u^i = g$ and $\lvert u \rvert = i \mod m$. This definition not only makes sense but has the following property: If $g_1, g_2, u_1, u_2$ are words in $X^+$ with $u^i_1 = g_1, i = 1, 2$, then $\lvert u_1 \rvert = \lvert u_2 \rvert \mod m$, so that $\#(g_1) = \#(g_2)$. To see this, let $\psi(g_1) = \psi(g_2) = s$. Since $e \in S = SsS$ and $\psi$ maps $T$ onto $S$, there are $h_1, h_2 \in T$ with $h_1 g_1 h_2, h_1 g_2 h_2 \in T_e$. If $v_1, v_2 \in X^+$ are words for which $v_i^k = h_i, i = 1, 2$, then $\lvert v_1 u_1 v_2 \rvert = \lvert v_1 \rvert + \lvert u_1 \rvert + \lvert v_2 \rvert = 0 \mod m$ and $\lvert v_1 u_2 v_2 \rvert = \lvert v_1 \rvert + \lvert u_2 \rvert + \lvert v_2 \rvert = 0 \mod m$. Thus $\lvert u_1 \rvert = \lvert u_2 \rvert \mod m$. On the other hand, the function $\#$ is clearly a homomorphism and $\rho \subseteq \theta = \text{Ker } \#$. Since $S \equiv T/\rho$, we see that $S$ maps homomorphically onto $\text{Im } \#$ under the assignment $r : s \mapsto \#(s)$ if and only if there exists $g \in T$ with $\psi(g) = \theta$ and $\#(g) = i$. Let $\theta_0$ be the smallest congruence relation such that $S/\theta_0$ is an abelian group. We have $\theta_0 \subseteq \text{Ker } \tau$, so that $S' = \theta_0(e) \subseteq \tau^{-1}(\theta)$. Let $T_0 = \psi^{-1}(S')$. It follows that $\#(g) = 0$ for all $g \in T_0$. 

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Now let \( u_1, \ldots, u_k \in X^+ (k \geq 1) \) be words with \( u^k \in T_e \) and \( m = \gcd(|u_1|, \ldots, |u_k|) \).

If \( \lambda_i (i \in [k]) \) are nonnegative integers one of which is not zero, then let

\[
u(\lambda_1, \ldots, \lambda_k) = u_1^{\lambda_1} \cdots u_k^{\lambda_k}.
\]

We have \( u^k \in T_e \). Since

\[
|u(\lambda_1, \ldots, \lambda_k)| = \lambda_1|u_1| + \cdots + \lambda_k|u_k|
\]

and \( m = \gcd(|u_1|, \ldots, |u_k|) \), there exists an integer \( t_0 \) such that for every \( t \geq t_0 \) there is a system \( (\lambda_1, \ldots, \lambda_k) \) with \( |u(\lambda_1, \ldots, \lambda_k)| = tm \). (As is well known, \( t_0 = \lcm(|u_1|, \ldots, |u_k|)/m \) suffices.) Let \( g_1, \ldots, g_r \) be a full set of representatives of \( T/\rho \), i.e., \( T/\rho = \{\rho(g_1), \ldots, \rho(g_r)\} \), where \( \rho(g_i) \) denotes the block containing \( g_i \) under the partition induced by \( \rho \). For each \( i \in [r] \) let \( v_i \in X^+ \) be a word with \( v_i^k - g_i \). Set

\[
I = \{(i,j): 1 \leq i, j \leq r, g_i T g_j \in T_0\}.
\]

By \( \#(g, e g_j) = \#(g_i) + \#(g_j) = 0 \) we have \( |v_i| + |v_j| \equiv 0 \mod m \) for all \( (i, j) \in I \), so that \( |v_i| + |v_j| = k_0 m \) for an integer \( k_0 \geq 1 \). Let

\[
t = \max\{k_0 + t_0: (i, j) \in I\}.
\]

Define \( t_{ij} = t - k_0 \) for all \( (i, j) \in I \). Since \( t_{ij} \geq t_0 \), a word \( v_{ij} \) can be found with \( v_{ij}^k = g_i T g_j \) and \( |v_{ij}| = t_{ij} m \). Consider now the elements of \( T_o \) of the form \( h_{ij} = g_i g_j g_i \) with \( (i, j) \in I \). Since \( S' \subseteq S \subseteq S \) and \( \psi(h_{ij}) = \psi(g_i) e \psi(g_j) \), they form a full set of representatives of \( T_o/\rho \). On the other hand, \( h_{ij} = (v_i v_j v_i)^\delta \) and \( |v_i v_j v_i| = |v_i| + |v_j| + |v_i| = k_0 m + t_{ij} m = tm \). It follows that \( S^f(S(A)) \), where \( n = tm \). The proof of the first statement is complete. To see that (2) implies (1) just notice that \( S' = S \) whenever \( S \) has no nontrivial abelian group quotient and then use part (i).

Now for the converse implication. Supposing \( S \) has nontrivial abelian group quotient, there is a homomorphism \( \psi \) of \( S \) onto \( Z_p \), where \( p > 1 \) is a prime number. Let \( G_1, \ldots, G_k \) be any collection of maximal subgroups of \( S \) that together generate \( S \), e.g., take all maximal subgroups. For each \( i \) \((1 \leq i \leq k)\), the restriction of \( \psi \) to \( G_i \) is an onto homomorphism \( \psi_i: G_i \to Z_p \). Set \( X_i = \psi_i^{-1}(\overline{1}) \), so that each \( X_i \) generates the subgroup \( G_i \). Since \( \bigcup \{G_i: 1 \leq i \leq k\} \) generates the semigroup \( S \), it follows that \( S \) is also generated by \( X = \bigcup \{X_i: 1 \leq i \leq k\} \). Define the automaton \( A = (S', X, S) \) by \( \delta(s, x) = sx \), for all \( s \in S^1 \) and \( x \in X \). We have \( S \cong S(A) \), one isomorphism \( \psi_i : S(A) \to S \) is the map taking \( u^k \in S(A), u \in X^+ \), to \( \psi_i(u^k) = 1 u^k \). Let \( \bar{\psi} \) be the composite of \( \psi \) by \( \psi \), so that \( \bar{\psi} : S(A) \to Z_p \) is an onto homomorphism. Given \( u \in X^+ \), we have \( \bar{\psi}(u) = \bar{i} \) if and only if \( |u| = i \mod p \). This follows at once, because \( \bar{\psi}(x^k) = 1 \), for all \( x \in X \). Consequently, \( S \parallel S(A) \) is not possible. \( \Box \)

**Corollary 3.6.** (i) Let \( S \) be a simple semigroup with maximal subgroup \( G \). If \( G = G' \) then \( S < S(A) \) implies \( S \parallel S(A) \).

(ii) A group \( G \) satisfies \( G = G' \) if and only if for every automaton \( A, G < S(A) \) implies \( G \parallel S(A) \).
We note that the converse of (i) of Corollary 3.6 does not hold, though \( G \neq G' \), \( S \) may not have nontrivial abelian group quotients. (See [1] for the description of the maximal group homomorphic image of a simple semigroup). Besides (ii) above, other examples when \( G = G' \) for the maximal subgroup of a simple semigroup \( S \) is necessary and sufficient for having \( S \parallel S(A) \) whenever \( S < S(A) \) include right groups, or any direct product of a group with a rectangular band.

It is worthwhile to express the previous results in different terms. Let \( S \) be a semigroup and \( X \subseteq S \) a set of generators. For a word \( u \in X^+ \) denote by \( \tilde{u} \) the image of \( u \) under the homomorphism \( X^+ \rightarrow S \) that embeds \( X \) into \( S \). Let \( \alpha \) be any homomorphism of \( X^+ \) into \( N \), the additive semigroup of positive integers. We call the pair \( (X, \alpha) \) a weighted set of generators.

**Corollary 3.7.** The following conditions are equivalent for a simple semigroup \( S \):

(i) For every weighted set of generators \( (X, \alpha) \) there is a positive integer \( n \) with \( S = \{ \tilde{u} : u \in X^+, \alpha(u) = n \} \).

(ii) For every set \( X \) of generators there is \( n \geq 1 \) with \( S = \{ \tilde{u} : u \in X^+, |u| = n \} \).

(iii) \( S \) has no nontrivial abelian group quotient.

**Remark.** Proposition 3.5 and Corollary 3.7 easily derive from an interesting result in [2] for the group case. Let \( G \) be a group of order \( n \), say \( G = \{ g_1, \ldots, g_n \} \). Denote by \( PG \) the set of all products

\[ g_\pi = g_{\pi(1)} \cdots g_{\pi(n)}, \]

where \( \pi \) is any permutation of \( [n] = \{1, \ldots, n\} \). It is proved in [2] that \( PG \) is always a coset of \( G' \), hence \( PG = G \) whenever \( G = G' \). Now, using the notations of the proof of Proposition 3.5, if \( G < S(A) \) then for each \( g_i \in G \) take a word \( u_i \in X^+ \) with \( u_i^\alpha \in T \) and \( \psi(u_i^\alpha) = g_i \). Then, for each permutation \( \pi \) of \( [n] \), define

\[ v_\pi = u_{\pi(1)} \cdots u_{\pi(n)}. \]

We have \( v_\pi \in T \) and \( \psi(v_\pi) = g_\pi \). Since the words \( v_\pi \) have equal lengths, it follows that \( G \parallel S(A) \). The reason why we have presented an elementary proof—in fact our original proof—of Proposition 3.5 is partly because the result in [2] uses the Feit-Thompson Theorem. It should be also mentioned that the equivalence of (ii) and (iii) of Corollary 3.7 also derives from facts proved in [18].

**Proposition 3.8.** Suppose that \( S \) is generated by idempotents. Then, for every automaton \( A \), \( S < S(A) \) implies \( S \parallel S(A) \).

We omit the easy proof but note that if \( S \) is a homomorphic image of a subsemigroup \( T \) of \( S(A) \), then \( T \) contains a subsemigroup \( T' \) that can be mapped homomorphically onto \( S \) and such that each member of \( T' \) is induced by some word of constant length \( n \). In addition to the semigroups appearing in Propositions
3.5 and 3.8, there are other interesting cases when $S < S(A)$ implies $S \parallel S(A)$. E.g., any union of simple semigroups (henceforth also union of groups) is a good example, provided that each simple semigroup itself has this property. Further examples are obtained by taking direct products.

We end this section by providing a pure automata theoretic characterization of the relations $\parallel$, $\|$, and $<$ between a semigroup and the characteristic semigroup of an automaton. The results also reflect the difference between these relations.

**Proposition 3.9.** $S \parallel S(A)$ if and only if $\text{Aut}(S) \in \text{HSP}(\{B\})$ for some $B \in \mathcal{P}_{\text{id}}(\{A\})$.

**Proof.** If $\mathcal{K}$ is a finite class of automata with the same input set, then $\text{HSP}(\mathcal{K})$ is the class of all finite automata in the equational class generated by $\mathcal{K}$. This observation allows us to use some elementary facts about equational classes in our proof, see [15]. Let $A = (A, X, \delta)$ and suppose $S \parallel S(A)$. Let $T$ be a subsemigroup of $S(A)$ and $\psi$ a homomorphism of $T$ onto $S$ such that each $s \in S$ has an inverse image $x^A_s$ in the set $\{x^A_x: x \in X\}$. Define $B = (A, S, \delta')$ with $\delta'(a, s) = \delta(a, x_s)$, for all $a \in A$ and $s \in S$, i.e., $s^B = x^A_s$. For a word $u \in S^*$, let $\bar{u}$ denote the corresponding product in $S^1$, so that $\lambda = 1$, the identity in $S^1$. A straightforward inductive argument yields $u^u \in T$ and $\psi(u^B) = \bar{u}$, for all $u \in S^+$. Since $\text{Aut}(S)$ is generated by the identity in $S^1$, in order to show that $\text{Aut}(S) \in \text{HSP}(\{B\})$, it suffices to prove that every nontrivial equation in a single variable that holds in $B$ also holds in $\text{Aut}(S)$. In other words this means that if $u, v \in S^*$, $u \neq v$, induce equal transformations in $B$, then they induce equal transformations in $\text{Aut}(S)$, i.e., $u^B = v^B$ implies $u^\text{Aut}(S) = v^\text{Aut}(S)$. Notice that the latter equality can be rephrased as $\bar{u} = \bar{v}$. Supposing $u, v \in S^+$ we obtain $\bar{u} = \psi(u^B) = \psi(v^B) = \bar{v}$. Assume now that $u \neq \lambda$ and $v = \lambda$. Then $T$ is a monoid and $u^B$ is the identity in $T$. Therefore $\bar{u} = \psi(u^B) = 1 = \bar{v}$.

Conversely, suppose that $\text{Aut}(S) \in \text{HSP}(\{B\})$ for an automaton $B = (A, S, \delta') \in \mathcal{P}_{\text{id}}(\{A\})$. Let $F(B) = (S_1(B), S, \delta'')$ with $\delta''(u^B, s) = (us)^B$, for all $u \in S^*$ and $s \in S$. The definition is unique. The automaton $F(B)$ is just the free automaton on one generator (namely $1_A$, the identity mapping $A \to A$) in the equational class generated by $B$. Since $\text{Aut}(S) \in \text{HSP}(\{B\})$, it follows that $\text{Aut}(S)$ is the homomorphic image of $F(B)$ under the unique homomorphism $\psi: S_1(B) \to S^1$ satisfying $\psi(1_A) = 1$. Define $\psi': S(B) \to S$ by $\psi'(u^B) = \psi(u^B)$, for all $u \in S^+$. The definition is correct, for $\psi(u^B) = 1$, $u \in S^+$, implies $1 \in S$. The mapping $\psi'$ is also onto, because

$$s = \tilde{\delta}(1, s) = \psi(\delta''(1, A, s)) = \psi(s^B) = \psi'(s^B), \quad \text{for all } s \in S,$$

where $\tilde{\delta}$ denotes the transition in $\text{Aut}(S)$. It is easy to check that $\psi$ is a semigroup homomorphism. Since $\psi(s^B) = s$ for all $s \in S$, we see that $S \parallel S(B)$. The proof is now easily completed. Since $S \parallel S(B)$ and also $S(B) \parallel S(A)$, we have $S \parallel S(A)$.

**Definition 3.10.** Let $A = (A, X, \delta)$ be an automaton and $n \geq 1$ an integer. Let $S^{(n)}(A)$ denote the collection of all transformations induced by words of length $n$, i.e.,

$$S^{(n)}(A) = \{u^\lambda: u \in X^+, |u| = n\}.$$
We define an automaton $A^{(n)} = (A, S^{(n)}(A), \delta^{(n)})$ by $\delta^{(n)}(a, u^A) = au^A$, for all $a \in A$ and $u \in X^+$ with $|u| = n$. Let $S$ be any subsemigroup of $S(A)$ contained in $S^{(n)}(A)$. We obtain an automaton

$$B = (A, S, \delta')$$

by defining $\delta'(a, s) = as$, for all $a \in A$ and $s \in S$. We let $\mathcal{H}^{(n)}(A)$ denote the set of all these automata $B$. Note that $\mathcal{H}^{(n)}(A)$ is possibly empty.

**Fact 3.11.** Let $A$ be an automaton, $n \geq 1$ an integer and let $S$ be a semigroup.

(i) $S \parallel^{(n)} S(A)$ if and only if $S \parallel^{(1)} S(A^{(n)})$.

(ii) $S \parallel^{(n)} S(A)$ if and only if $S \parallel^{(1)} S(B)$ for some $B \in \mathcal{H}^{(n)}(A)$.

(iii) $S \preceq S(A)$ if and only if $S \parallel^{(1)} S(A^+)$.

Combining Proposition 3.9 with the above fact we get the following statement.

**Proposition 3.12.** Let $A$ be an automaton and $S$ a semigroup.

(i) $S \parallel^{(n)} S(A)$ if and only if $\text{Aut}(S) \in \text{HSP}_{nq}(\{A^{(n)}\})$ for some $B \in \mathcal{P}_{1q}(\{A^{(n)}\})$.

(ii) $S \parallel^{(n)} S(A)$ if and only if $\text{Aut}(S) \in \text{HSP}(\{B\})$ for an automaton $B \in \mathcal{P}_{1q}(\mathcal{H}^{(n)}(A))$.

(iii) $S \preceq S(A)$ if and only if $\text{Aut}(S) \in \text{HSP}(\{B\})$ for an automaton $B \in \mathcal{P}_{1q}(\{A^+\})$.

The above proposition, in particular (iii), should be thought of as an analogon of known results.

### 4. Irreducibility

The Krohn-Rhodes Decomposition Theorem determines the irreducible semigroups with respect to the relation $\prec$. Our principal aim in this section is to show that exactly the same semigroups are irreducible with respect to the relations $\mid$ and $\parallel$.

**Definition 4.1.** A semigroup $S$ is called $\mid$-irreducible ($\parallel$-irreducible) if and only if for every nonempty class $\mathcal{H}$ and automaton $A \in \text{HSP}_{nq}(\mathcal{H})$, the condition $S \mid S(A)$ ($S \parallel S(A)$) implies that $S \mid S(B)$ ($S \parallel S(B)$) for some $B \in \mathcal{H}$.

Given a semigroup $S$, let $\mathcal{H}$ consist of the automaton $U_3$ and the automata $\text{Aut}(G)$ for each simple group $G$ with $G < S$. Since the characteristic semigroup of $\text{Aut}(S)$ is isomorphic to $S$, we have $\text{Aut}(S) \in \text{HSP}_{nq}(\mathcal{H})$ by the second part of the Krohn-Rhodes Decomposition Theorem. If $S$ is not a unit or a simple group, then $S \parallel^{(1)} S(\text{Aut}(S))$ and henceforth also $S \parallel^{(1)} S(\text{Aut}(S))$, but for no automaton $A \in \mathcal{H}$ do we have $S \prec S(A)$. Since for any automaton $B$, $S \parallel S(B)$ implies $S \mid S(B)$ which in turn yields $S \prec S(B)$, we see that $S$ is neither $\mid$-irreducible nor $\parallel$-irreducible. Thus the best we can hope is that the irreducible semigroups are both $\mid$-irreducible and $\parallel$-irreducible.
Theorem 4.2. The following conditions are equivalent for a semigroup $S$:

(i) $S$ is irreducible,
(ii) $S$ is $\|-$irreducible,
(iii) $S$ is $\|-\|-$irreducible.

Proof. We have already seen that every $|-\|-$irreducible or $\|\|-$irreducible semigroup is irreducible. Since irreducible semigroups satisfy $S^2 = S$, $|-\|-$irreducibility coincides with $\|\|-$irreducibility by Lemma 3.3. If $G$ is a nonabelian simple group then $G$ is $\|\|-$irreducible by the first part of the Krohn–Rhodes Decomposition Theorem and Corollary 3.6. Similarly, the Krohn–Rhodes Decomposition Theorem and Proposition 3.8 imply that the units are $\|\|-$irreducible. The $\|\|-$irreducibility of the abelian simple groups can be proved directly by using Proposition 4.7 below. The fact that (i) implies (ii) also derives from the stronger statements formulated in Propositions 4.3, 4.4 and 4.6.

Let $B \times C(X, \varphi)$ be an $\alpha_0$-product of automata $B = (B, Y, \delta_1)$ and $C = (C, Z, \delta_2)$. The feedback function $\varphi_1$ extends to a monoid homomorphism $X^* \to Y^*$. Similarly, $\varphi_2$ extends to a map $\varphi_2^*: B \times X^* \to Z^*$ by defining

$$\varphi_2^*(b, xu) = \varphi_2(b, xu)$$

for all $b \in B$, $x \in X$ and $u \in X^*$.

Proposition 4.3. Let $A = B \times C(X, \varphi)$ be an $\alpha_0$-product. If $G$ is a simple group with $G \|\|^{(n)} S(A)$ then either $G \|\|^{(n)} S(B)$ or $G \|\|^{(n)} S(C)$.

Proof. Although the argument below follows the proof of the analogous result for $< \text{divisibility}$ as appearing e.g. in [16], it contains some simplifications. Without these simplifications only a weaker statement could be proved.

Let $A = (A, X, \delta), B = (B, Y, \delta_1)$ and $C = (C, Z, \delta_2)$. If $u \in X^*$ then denote $\bar{u} = \varphi_1(u)$ and $\bar{u}_b = \varphi_2(b, u)$, for all $b \in B$. For every $(b, c) \in A$ we have $(b, c)u^A = (b\bar{u}^B, c\bar{u}_C)$. The map $u^A \mapsto \bar{u}^B (u \in X^*)$ is a well-defined homomorphism of $S(A)$ into $S(B)$. For each $b \in B$ let $S_b \subseteq S(A)$ be the collection of those transformations $u^A (u \in X^*)$ with $b\bar{u}^B = b$. If $S_b$ is nonempty then it is a subsemigroup of $S(A)$ and the mapping

$$u^A \mapsto \bar{u}_b^C (u \in X^*, u^A \in S_b)$$

is a homomorphism of $S_b$ into $S(C)$.

Since $G \|\|^{(n)} S(A)$ there is a subgroup $H$ of $S(A)$ such that $G$ is a homomorphic image of $H$ and $H \subseteq \{u^A: u \in X^+, |u| = n\}$. Since $H$ is a subgroup of $S(A)$ there is a nonempty set $W \subset B \times C$ with the following properties:

(a) The restriction $u^A|_W$ of each $u^A \in H$ ($u \in X^+$) to $W$ is a permutation of $W$.

(b) The map $u^A \mapsto u^A|_W (u^A \in H, u \in X^+)$ is an isomorphism of $H$ onto a permutation group over $W$. (For this and some other facts to be used below, see [14]).

Thus for $u^A, v^A \in H$ ($u, v \in X^+$) we have $u^A = v^A$ if and only if $(b, c)u^A = (b, c)v^A$ for all $(b, c) \in W$. Let $B_1$ be the set of the first components of the elements of $W$. For each $u^A \in H$ with $u \in X^+$ define $\psi(u^A)$ to be the permutation of $B_1$ obtained by taking the restriction of $\bar{u}^B$ to $B_1$, i.e., $\psi(u^A) = \bar{u}^B|_{B_1}$. The assignment $u^A \mapsto \psi(u^A)$ is
a well-defined homomorphism of $H$ onto a permutation group over $B_1$. Set $N = \text{Ker } \psi$, so that $N$ is a normal subgroup of $H$ and $H/N = \text{Im } \psi$. Let $\psi': H \to S(B)$ be the homomorphism taking $u^A \in H$ ($u \in X^+$) to $\tilde{u}^B$. $\text{Im } \psi'$ is a subgroup of $S(B)$, denoted $H_1$. We can consider $\psi'$ as a homomorphism of $H$ onto $H_1$. If $\tilde{u}^B = \tilde{v}^B$ for some $u^A, v^A \in H$ ($u, v \in X^+$), then also $\tilde{u}^B|_{B_1} = \tilde{v}^B|_{B_1}$. Therefore $\text{Ker } \psi' \subseteq N$ and $\psi$ factors through $\psi'$. It follows that $H/N$ is a homomorphic image of $H_1$. From $H \leq \{u^A: u \in X^+, |u| = n\}$ we also have $H_1 \leq \{v^B: v \in Y^+, |v| = n\}$.

Since the simple group $G$ is a homomorphic image of $H$ and $N$ is a normal subgroup of $H$, either $G$ is a homomorphic image of $H/N$ or $N$ maps homomorphically onto $G$. In the former case $G$ is a homomorphic image of $H$ and therefore $G ||^{(n)} S(B)$. From now on we assume that $G$ is a homomorphic image of $N$. Let $b \in B_1$ be any fixed state. If $u^A \in N$ for a word $u \in X^+$ then $bu^B = b$, i.e., $N \subseteq S_b$. Define $\psi_b: N \to S(C)$ by $\psi_b(u^A) = \tilde{u}_C^A$, for all $u^A \in N$ with $u \in X^+$. We already know that $\psi_b$ is a well-defined homomorphism of $N$ into $S(C)$. Therefore $H_b = \text{Im } \psi_b$ must be a group, a subgroup of $S(C)$. We can also view $\psi_b$ as a homomorphism of $N$ onto $H_b$. If $u, v \in X^+$ with $u^A, v^A \in N$ and $u^A \neq v^A$, then there is a pair $(b, c) \in W$ with $(b, c)u^A \neq (b, c)v^A$. But $(b, c)u^A = (b, cu^B_C)$ and $(b, c)v^A = (b, cv^B_C)$, so that $\psi_b(u^A) = \tilde{u}_C^A \neq \tilde{v}_C^A = \psi_b(v^A)$. Thus $\bigcap \{\text{Ker } \psi_b : b \in B_1\}$ is the trivial normal subgroup consisting of the identity of $N$. Since $\bigcap \{\text{Ker } \psi_b : b \in B_1\}$ is trivial, $N$ is isomorphic to a subdirect product of the groups $H_b, b \in B_1$. Since the simple group $G$ is a homomorphic image of $N$, it is also a homomorphic image of a subgroup of some $H_b$, i.e., $G < H_b$. The group $H_b$ consists of the transformations of the form $\tilde{u}_C^A$, where $u \in X^+$ and $u^A \in N$. Since each member of $N$ is induced by some word of length $n$, the same holds for $H_b$, i.e., $H_b \subseteq \{v^C : v \in Z^+, |v| = n\}$. From $G < H_b$ we obtain $G ||^{(n)} S(C)$. □

**Proposition 4.4.** Let $U_i$ be one of the units. If $U_i ||^{(n)} S(A)$ for an $\alpha_0$-product $A = B \times C(X, \varphi)$, then $U_i ||^{(n)} S(B)$ or $U_i ||^{(n)} S(C)$.

**Proof.** The proof makes use of the known fact, see [1, 9, 16], that if $U_i$ is a homomorphic image of a semigroup $S$, then $S$ contains an isomorphic copy of $U_i$. Thus if $U_i ||^{(n)} S(A)$, then there is a subsemigroup $T$ of $S(A)$ isomorphic to $U_i$ and such that $T \subseteq \{u^A : u \in X^+, |u| = n\}$. The case $i = 0$ is trivial (and is also handled by Proposition 4.3). So we assume $i \neq 0$. Let $u^A$ and $v^A$ be two distinct elements of $T$ such that none of them is the identity in $T$ if $i = 3$. There is a pair $(b_0, c_0)$ in the state set of $A$ with $(b_1, c_1) = (b_0, c_0)u^A \neq (b_0, c_0)v^A = (b_2, c_2)$. It is now easy to see that the elements of $T$ map $W = \{(b_1, c_1), (b_2, c_2)\}$ into itself. Moreover, the assignment $u^A \mapsto u^A|_W$ $(u \in X^+, u^A \in T)$ is an isomorphism of $T$ onto a mapping semigroup over $W$. If $b_1 \neq b_2$, then, in the notations of the proof of Proposition 4.3, $T_i = \{\tilde{u}^B : u \in X^+, u^A \in T\}$ is a subsemigroup of $S(B)$ isomorphic to $U_i$. Thus $U_i ||^{(n)} S(B)$. If $b_1 = b_2 = b$ then $T_2 = \{\tilde{u}_C^A : u \in X^+, u^A \in T\}$ is isomorphic to $U_i$ and $U_i ||^{(n)} S(C)$. □

Neither Proposition 4.3 nor Proposition 4.4 holds if $||^{(n)}$ is replaced by $|^{(n)}$. Nevertheless the following corollary is true by Lemma 3.3.
Corollary 4.5. Let $S$ be one of the units or a simple group. If $S \mid^n S(A)$ for an $\alpha_0$-product of automata $B$ and $C$, then there is a multiple $m$ of $n$ such that $S \mid^m S(B)$ or $S \mid^m S(C)$.

Next we turn to subautomata and homomorphic images.

Proposition 4.6. Let $A$ and $B$ be automata with $B \in H(\{A\})$ or $B \in S(\{A\})$. If $S \mid^n S(B)$ for a semigroup $S$, then $S \mid^n S(A)$. If $S \mid^n S(B)$ and $S^2 = S$, then $S \mid^m S(A)$ for a multiple $m$ of $n$.

Proof. Let $X$ denote the common input set of the automata $A$ and $B$. The map $u^A\mapsto u^B$ ($u \in X^*$) is a homomorphism of $S(A)$ onto $S(B)$. Therefore if $S \mid^n S(B)$ then $S \mid^n S(A)$. Supposing $S \mid^n S(B)$ and $S^2 = S$ we have $S \mid^n S(B)$ and thus $S \mid^n S(A)$. Apply Lemma 3.3. $\square$

Except for the abelian simple groups, each irreducible semigroup $S$ has the property that $S < S(A)$ is equivalent to $S \mid S(A)$. For a prime number $p$ and an automaton $A$, $Z_p < S(A)$ if and only if $S(A)$ has a subgroup isomorphic of $Z_p$. This is further equivalent to the condition that for some distinct states $a_0, \ldots, a_{p-1}$ and a nonempty input word $u$ we have $a_0 u^A = a_{i+1 \mod p}$. We end this section by pointing out a similar fact for the relation $\mid$.

Proposition 4.7. Let $A = (A, X, \delta)$ be an automaton and $p$ a prime number. $Z_p \mid S(A)$ if and only if there are an integer $n \geq 1$, words $u, v \in X^*$ with $|u| = |v| = n$ and pairwise distinct states $a_0, \ldots, a_{p-1} \in A$ such that $a_i u^A = a_i$ and $a_i v^A = a_{i+1 \mod p}$ hold for all $i$.

5. Completeness

In this section we present relative completeness results with respect to the $\alpha_0$-product and some of the classes defined in Section 2. If $S < S(A)$ for a semigroup $S$ and an automaton $A$, then $\text{Aut}(S) \in \text{HSP}_p(\{A\}) = \text{HSP}_p(\{A^\infty\})$ by Proposition 3.12. A similar fact is not true if $P_p^n$ is replaced by $P_q^n$, even $\text{Aut}(S) \in \text{HSP}_q(\{A\})$ may not hold. Our basic tool, Theorem 5.1 below, provides a way of constructing $\text{Aut}(S)$ from $A$ itself and certain primitive automata, provided that $S \mid^n S(A)$ holds. The present form of this result is drawn from [4], see however [3, 10, 12]. A counter of length $n \geq 1$ is an automaton $C_n = (C, \{x\}, \delta)$ with $C = \{c_0, \ldots, c_{n-1}\}$ and $\delta(c_i, x) = c_{i+1 \mod n}$, so that $x$ induces a cyclic permutation of the state set.

Theorem 5.1 (Dömösi and Ésik [4]). If $S \mid^n S(A)$ for a semigroup $S$ and an automaton $A$, then $\text{Aut}(S) \in \text{HSP}_m(\{C_n, U_1, A\})$.

By Corollary 2.5 to the Krohn-Rhodes Decomposition Theorem, a class $\mathcal{X}$ is $\alpha_0^+$-complete for the class of definite automata if and only if $U_1 < S(A)$ holds for
some $A \in \mathcal{H}$. For $\alpha_0$-completeness, no necessary and sufficient condition is known (besides $U_1 \in \text{HSP}_{\alpha_0}(\mathcal{H})$). The situation is similar concerning aperiodic automata.

A partial result is formulated below. An automaton $A = (A, X, \delta)$ is **strongly connected** if for each pair of states $a, b \in A$ there is a word $u \in X^\ast$ with $\delta(a, u) = b$. Moreover, $A$ is **ambiguous** if there are $a \in A$ and $x, y \in X$ with $\delta(a, x) \neq \delta(a, y)$.

**Theorem 5.2** (Dömösi and Ésik [5]). Let $\mathcal{H}$ be $\alpha_0$-complete for the class of counters, i.e., $C_n \in \text{HSP}_{\alpha_0}(\mathcal{H})$ for all $n \geq 1$. Then $\mathcal{H}$ is $\alpha_0$-complete for the class of all definite (resp. aperiodic) automata if and only if (i) and (ii) hold:

(i) $\text{HSP}_{\alpha_0}(\mathcal{H})$ contains a strongly connected ambiguous automaton:
(ii) There is an automaton $A \in \mathcal{H}$ with $U_1 < S(A)$ (resp. $U_1 < S(A)$).

The proof of Theorem 5.2 uses the fact that $U_i < S(A)$ implies $U_i | S(A)$ for the unit semigroups and the fact, seen in [5], that if $K$ is $\alpha_0$-complete for the class of counters, then (i) is equivalent to requiring that $U_1 \in \text{HSP}_{\alpha_0}(\mathcal{H})$ or $\text{Aut}(C_n^m) \in \text{HSP}_{\alpha_0}(\mathcal{H})$ for some $m > 1$. It is then shown that $U_1 \in \text{HSP}_{\alpha_0}((C_n^m, C_n, A))$ if $A$ is an automaton with $U_1 < S(A)$. For $U_1$, one applies also Theorem 5.1. We are now ready to state our completeness results.

**Theorem 5.3.** Let $\mathcal{G}$ be a nonempty class of simple groups and $\mathcal{H}$ a class of automata $\alpha_0$-complete for the class of counters. Then $\mathcal{H}$ is $\alpha_0$-complete for $\mathcal{H}(\mathcal{G})$, where $i = 1$ or $i = 3$, if and only if the following hold:

(i-1) For every nonabelian $G \in \mathcal{G}$ there is $A \in \mathcal{H}$ with $G < S(A)$.
(i-2) For every abelian $G \in \mathcal{G}$ there is $A \in \mathcal{H}$ with $G \parallel S(A)$.
(ii) There is an automaton $A \in \mathcal{H}$ with $U_i < S(A)$.
(iii) $\text{HSP}_{\alpha_0}(\mathcal{H})$ contains a strongly connected ambiguous automaton.

Moreover, $\mathcal{H}$ is $\alpha_0$-complete for $\mathcal{H}_{1,2}(\mathcal{G})$ if and only if the above conditions hold with (ii) both for $i = 1$ and $i = 2$.

**Proof.** We note that, in view of the results of Section 3, the relation $<$ can be replaced by $\parallel$ or $|$ in (i-1) and (ii). We only prove the first statement. The necessity of (iii) is obvious and the necessity of the rest of the conditions derives either from the first part of the Krohn–Rhodes Decomposition Theorem or from Theorem 4.2. Conversely, we must show that if each of the conditions (i-1), (i-2), (ii) and (iii) holds then $U_i \in \text{HSP}_{\alpha_0}(\mathcal{H})$ and $\text{Aut}(\mathcal{G}) \in \text{HSP}_{\alpha_0}(\mathcal{H})$, for all $G \in \mathcal{G}$. As regards $U_i \in \text{HSP}_{\alpha_0}(\mathcal{H})$, it is already done by Theorem 5.2. Let $\mathcal{G}$ be a simple group in $\mathcal{G}$. From our assumptions and Corollary 3.6 we obtain $G \parallel S(A)$ for an automaton $A \in \mathcal{H}$, so that also $G \parallel S(A)$ for an integer $n \geq 1$. Since $C_n \in \text{HSP}_{\alpha_0}(\mathcal{H})$ by assumption and $U_1 \in \text{HSP}_{\alpha_0}(\mathcal{H})$ (even if $i = 3$), Theorem 5.1 gives $\text{Aut}(G) \in \text{HSP}_{\alpha_0}(\mathcal{H})$.

If $\mathcal{G}$ contains all of the abelian simple groups, then the presence of the counters in $\text{HSP}_{\alpha_0}(\mathcal{H})$ is already necessary for $\mathcal{H}$ to be $\alpha_0$-complete for $\mathcal{H}(\mathcal{G})$. Thus we can turn Theorem 5.3 to a full necessary and sufficient condition. □
Theorem 5.4. Let \( \mathcal{G} \) be a class of simple groups containing the abelian simple groups. A class \( \mathcal{K} \) of automata is \( \alpha_0 \)-complete for \( \mathcal{K}(\mathcal{G}) \), where \( i = 1 \) or \( i = 3 \), if and only if the following hold:

1. (i-1) For every nonabelian \( G \in \mathcal{G} \) there is \( A \in \mathcal{K} \) with \( G \triangleleft S(A) \).
2. (i-2) For every abelian \( G \in \mathcal{G} \) there is \( A \in \mathcal{K} \) with \( G \parallel S(A) \).
3. (ii) There is an automaton \( A \in \mathcal{K} \) with \( U_i \triangleleft S(A) \).
4. (iii) \( \text{HSP}_{\alpha_0}(\mathcal{K}) \) contains the counters and at least one strongly connected ambiguous automaton.

\( \mathcal{K} \) is \( \alpha_0 \)-complete for \( \mathcal{K}_{1,2}(\mathcal{G}) \) if and only if the above conditions hold with (ii) both for \( i = 1 \) and \( i = 2 \).

By letting \( \mathcal{G} \) to be the class of all simple groups, \( \mathcal{K}(\mathcal{G}) \) becomes the class of all automata. We obtained a new proof of the following corollary, which is the main result of [5].

Corollary 5.5 (Dömösi and Ésik [5]). A class \( \mathcal{K} \) of automata is \( \alpha_0 \)-complete if and only if the following conditions hold:

1. (i) For every (simple) group \( G \) there is \( A \in \mathcal{K} \) with \( G \triangleleft S(A) \).
2. (ii) There is an automaton \( A \in \mathcal{K} \) with \( U_i \triangleleft S(A) \).
3. (iii) \( \text{HSP}_{\alpha_0}(\mathcal{K}) \) contains the counters and at least one strongly connected ambiguous automaton.

Proof. By Theorem 5.4 and the fact that every group is embedded in a nonabelian simple group.  \( \Box \)

Corollary 5.6 (Ésik and Viragh [13]). There exists an \( \alpha_0 \)-complete class of automata with 2 input letters.

It should be noted that Dömösi [3] proves similar result for automata with 3 input letters. One thing that could be asked at this point is whether Theorem 5.4 and Corollary 5.5 are in a sense the best possible results. As regards Corollary 5.5, it has been shown in [5]. We slightly extend the ideas in [5] to show that the same holds for Theorem 5.4.

Definition 5.7. Let \( \mathcal{G} \) be a nonempty class of simple groups and \( \mathcal{K}_0 \) a class of automata. \( \mathcal{K}_0 \) is called critical for \( \mathcal{K}_i(\mathcal{G}) \), where \( i = 1 \) or \( i = 3 \), if for every \( \mathcal{K} \) satisfying the conditions (i-1), (i-2) and (ii) of Theorem 5.4 as well as the inclusion \( \mathcal{K}_0 \subseteq \text{HSP}_{\alpha_0}(\mathcal{K}) \) it follows that \( \mathcal{K} \) is \( \alpha_0 \)-complete for \( \mathcal{K}_i(\mathcal{G}) \). Similarly, \( \mathcal{K}_0 \) is critical for \( \mathcal{K}_{1,2}(\mathcal{G}) \) if for every class \( \mathcal{K} \), the conditions (i-1), (i-2), (ii) with \( i = 1, 2 \) and \( \mathcal{K}_0 \subseteq \text{HSP}_{\alpha_0}(\mathcal{K}) \) jointly imply that \( \mathcal{K} \) is \( \alpha_0 \)-complete for \( \mathcal{K}_{1,2}(\mathcal{G}) \). Let \( \mathcal{G} \) be the class of all simple groups. A class critical for \( \mathcal{K}_3(\mathcal{G}) \) is termed a critical class. Thus \( \mathcal{K}_0 \) is critical if and only if for every class \( \mathcal{K} \) satisfying the conditions (i) and (ii) of Corollary 5.5 as well as the inclusion \( \mathcal{K}_0 \subseteq \text{HSP}_{\alpha_0}(\mathcal{K}) \) we have that \( \mathcal{K} \) is \( \alpha_0 \)-complete.
It is immediately seen that $X_0$ is critical for $\mathcal{H}_i(\mathcal{G})$, $i = 1, 3$, if and only if $\text{HSP}_{\alpha_0}(\mathcal{H}_0)$ is critical for $\mathcal{H}_i(\mathcal{G})$. Similarly for $\mathcal{H}_{1,2}(\mathcal{G})$. Therefore we are interested in critical $\alpha_0$-varieties rather than just critical classes. By (iii) of Theorem 5.4, every $\alpha_0$-variety containing the counters and at least one strongly connected ambiguous automaton is critical for any class $\mathcal{H}_i(\mathcal{G})$ with $i = 1, 3$ and also for $\mathcal{H}_{1,2}(\mathcal{G})$. (Of course $\mathcal{G}$ is assumed to contain the simple abelian groups). The following facts are drawn from [12, p. 141] and [5, p. 7]. The class of counters with length $n \geq 1$ is denoted $\mathcal{E}_n$. We set $\mathcal{E} = \bigcup (\mathcal{E}_n : n > 1)$.

**Fact 5.8** (Ésik and Dömösi [12]). There is an $\alpha_0$-variety $\mathcal{K}$, which satisfies (i) and (ii) of Corollary 5.5, contains the counters, but does not contain any strongly connected ambiguous automaton.

**Fact 5.9** (Dömösi and Ésik [5]). Let $q = p^r$ for a prime number $p$ and integer $r \geq 1$.

There is an $\alpha_0$-variety $\mathcal{K}_q$ with the following properties:

(i) $\mathcal{K}_q$ satisfies both (i) and (ii) of Corollary 5.5;

(ii) For every $\alpha_0$-variety $\mathcal{K}$ we have $\mathcal{E}_q \not\subseteq \text{HSP}_{\alpha_0}(\mathcal{K} \cup \mathcal{K}_q)$ unless $\mathcal{E}_q \subseteq \mathcal{K}$.

We now turn back to Theorem 5.4. It is clear that none of the conditions (i-1), (i-2) and (ii) can be removed. (We can however discard (i-2) if for every abelian $G_1 \in \mathcal{G}$ there is a nonabelian $G_2 \in \mathcal{G}$ with $G_1 \leq G_2$, in which case $G_1$ is embedded in $G_2$). Below we point out that not only (iii) cannot be removed either, but it is exactly the condition that describes critical classes for $\mathcal{H}_i(\mathcal{G})$, or $\mathcal{H}_{1,2}(\mathcal{G})$.

**Theorem 5.10.** Let $\mathcal{G}$ be a class of simple groups containing the abelian ones. An $\alpha_0$-variety $\mathcal{H}_0$ is critical for $\mathcal{H}_i(\mathcal{G})$, $i = 1, 3$, if and only if it contains the counters and at least one strongly connected ambiguous automaton. The same holds true for $\mathcal{H}_{1,2}(\mathcal{G})$.

**Proof.** We spell out the proof only for classes of the form $\mathcal{H}_i(\mathcal{G})$, $i = 1, 3$. The sufficiency is a restatement of one part of Theorem 5.4. Suppose $\mathcal{H}_0$ is critical for $\mathcal{H}_i(\mathcal{G})$.

Claim 1: $\mathcal{C} \subseteq \mathcal{H}_0$. Indeed, if $\mathcal{C} \not\subseteq \mathcal{H}_0$ then there is a prime power $q = p^r$ ($r \geq 1$) with $\mathcal{C}_q \not\subseteq \mathcal{H}_0$. Let $\mathcal{K} = \mathcal{H}_0 \cup \mathcal{K}_q$, where $\mathcal{K}_q$ is the $\alpha_0$-variety provided by Fact 5.9. We see that the conditions (i-1), (i-2) and (ii) of Theorem 5.4 hold, for (i-2) one also uses Corollary 3.6 and the fact that each group is embedded in a nonabelian simple group. The inclusion $\mathcal{H}_0 \subseteq \text{HSP}_{\alpha_0}(\mathcal{K})$ is obvious. Furthermore, $\mathcal{C}_q \not\subseteq \text{HSP}_{\alpha_0}(\mathcal{K})$ by Fact 5.9. On the other hand, since $\mathcal{G}$ contains the abelian simple groups, we have $\mathcal{C}_q \not\subseteq \mathcal{H}_0(\mathcal{G}) \subseteq \mathcal{H}_i(\mathcal{G})$ by the second part of the Krohn–Rhodes Decomposition Theorem. Thus $\mathcal{H}_i(\mathcal{G}) \not\subseteq \text{HSP}_{\alpha_0}(\mathcal{K})$, contradicting our assumption that $\mathcal{H}_0$ is critical.

Claim 2: $\mathcal{H}_0$ contains a strongly connected ambiguous automaton. The claim is established by taking $\mathcal{K} = \mathcal{H}_0 \cup \mathcal{K}_i$, where $\mathcal{K}_i$ is a class as in Fact 5.8. We again see that (i-1), (i-2) and (ii) of Theorem 5.4 are satisfied by $\mathcal{K}$ and that $\mathcal{H}_0 \subseteq \text{HSP}_{\alpha_0}(\mathcal{K})$. Since $\mathcal{H}_0$ is critical we obtain $\mathcal{H}_i(\mathcal{G}) \subseteq \text{HSP}_{\alpha_0}(\mathcal{K})$, so that $\text{HSP}_{\alpha_0}(\mathcal{K})$, and even
SP_{\alpha_0}(\mathcal{H})$, must contain a strongly connected ambiguous automaton. It follows that an $\alpha_0$-product $B$ of a counter $C_n$ with an automaton $A \in \mathcal{H}$ contains a strongly connected ambiguous automaton. By Claim 1 and Fact 5.8 we have $C_n \in \mathcal{H}_0 \cap \mathcal{H}_1$, and again by Fact 5.8, $A \in \mathcal{H}_1$. Thus $C_n, A \in \mathcal{H}_0$, and since $\mathcal{H}_0$ is an $\alpha_0$-variety, also $B \in \mathcal{H}_0$. 

**Corollary 5.11.** Let $\mathcal{G}$ be a class of simple groups containing the abelian simple groups. The following are equivalent for an $\alpha_0$-variety $\mathcal{H}_0$:

(i) $\mathcal{H}_0$ is critical,

(ii) $\mathcal{H}_0$ is critical for $\mathcal{H}_1(\mathcal{G})$,

(iii) $\mathcal{H}_0$ is critical for $\mathcal{H}_2(\mathcal{G})$,

(iv) $\mathcal{H}_0$ is critical for $\mathcal{H}_{1,2}(\mathcal{G})$.

Let $\mathcal{H}$ be an $\alpha_0$-variety with $\mathcal{G} \subseteq \mathcal{H}$. By [5, Lemma 4], $\mathcal{H}$ contains a strongly connected ambiguous automaton if and only if either $U_1 \in \mathcal{H}$ or $C^p \in \mathcal{H}$ for a prime number $p$. (It is easy to see that $C^p \in \mathcal{H}$ if and only if only if Aut($Z_p$) $\in \mathcal{H}$, see [14]). We obtain that each critical $\alpha_0$-variety includes a minimal critical $\alpha_0$-variety and each critical $\alpha_0$-variety is of the form $HSP_{\alpha_0}(\mathcal{G} \cup \{U_1\})$ or $HSP_{\alpha_0}(\mathcal{G} \cup \{C^p\})$, where $p$ is any prime. We now turn to the $\alpha_0$-product. Notice that a class $\mathcal{H}$ is $\alpha_0$-complete for $\mathcal{H}_1(\mathcal{G})$ if and only if it is $\alpha_0$-complete for $\mathcal{H}_2(\mathcal{G})$. The same holds if $\mathcal{H}_1(\mathcal{G})$ is replaced by $\mathcal{H}_{1,2}(\mathcal{G})$. This is the reason why we shall only treat the classes $\mathcal{H}_2(\mathcal{G})$.

**Proposition 5.12.** If $S < S(A)$ for a semigroup $S$ and an automaton $A$ then there is an integer $n_0$ such that $S \upharpoonright (n)$ $S(A^*)$ for $n \geq n_0$.

The easy proof makes use of the existence of an input letter of $A^*$ that induces the identity state transformation. We call an automaton $A = (A, X, \delta)$ counter-free if $\delta(a, x^*) = a$ implies $\delta(a, x) = a$ for all $a \in A, x \in X$ and $n \geq 1$. Note that an equivalent condition is as follows: there exists an integer $n \geq 0$ such that for all $x \in X$ we have $(x^n)^A = (x^{n+1})^A$. Henceforth also $(x^m)^A = (x^{m+1})^A$ whenever $m \geq n$. It is easily seen that this property is preserved by $\alpha_0$-products, subautomata and homomorphic images, so that the counter-free automata form an $\alpha_0$-variety. This observation implies the following fact (a class $\mathcal{H}$ of automata is counter-free if each member of $\mathcal{H}$ is counter-free).

**Proposition 5.13.** The three conditions below are equivalent for a class $\mathcal{H}$.

(i) $HSP_{\alpha_0}(\mathcal{H})$ contains a counter $C_n$ with $n > 1$.

(ii) $HSP_{\alpha_0}(\mathcal{H})$ contains an automaton $C_n^A$ with $n > 1$.

(iii) $\mathcal{H}$ is not counter-free.

The previous two propositions are now utilized in describing $\alpha_0$-complete classes for $\mathcal{H}_2(\mathcal{G})$. First we treat the case that $\mathcal{G}$ contains trivial groups only.
Theorem 5.14. Suppose that \( \mathcal{K} \) is not counter-free. Then \( \mathcal{K} \) is \( \alpha_0^\lambda \)-complete for the class of aperiodic automata if and only if there is an automaton \( A \in \mathcal{K} \) with \( U_3 < S_1(A) \).

Proof. The necessity follows from the first part of the Krohn–Rhodes Decomposition Theorem. For the converse direction, suppose \( U_3 < S_1(A) \) for an automaton \( A \in \mathcal{K} \). Since \( S_1(A) - S(A^k) \) we have \( U_3 < S(A^k) \) and, by Proposition 5.12, \( U_3 \perp S(A^k) \) for all but a finite number of positive integers. Since \( \mathcal{K} \) is not counter-free, \( C_n^\alpha \in \text{HSP}_{\alpha_0}^\lambda(\mathcal{K}) \) holds for an integer \( q > 1 \). By a result in [14], also \( \text{Aut}(Z,q) \in \text{HSP}_{\alpha_0}^\lambda(\mathcal{K}) \). The second part of the Krohn–Rhodes Decomposition Theorem gives \( C_n \in \text{HSP}_{\alpha_0}^\lambda(\mathcal{K}) \) whenever every prime divisor of \( n \) divides \( q \). It follows that there is an integer \( n \geq 1 \) with \( C_n \in \text{HSP}_{\alpha_0}^\lambda(\mathcal{K}) \) and \( U_3 \not\perp S(A^k) \). As part of the proof of [5, Theorem 2], it is shown that \( U_1 \in \text{HSP}_{\alpha_0}^\lambda(\{C_n^\lambda, C_n, A\}) \). Theorem 5.1 now implies \( U_3 \in \text{HSP}_{\alpha_0}^\lambda(\{U_1, C_n, A\}) \). The proof is completed by using the Krohn–Rhodes Decomposition Theorem. \( \square \)

Theorem 5.15. Let \( \mathcal{G} \) be a class of simple groups that contains a nontrivial group. A class \( \mathcal{K} \) is \( \alpha_0^\lambda \)-complete for \( \mathcal{K}_3(\mathcal{G}) \) if and only if the following hold:

(i) For every \( G \in \mathcal{G} \) there is \( A \in \mathcal{K} \) with \( G < S_1(A) \).

(ii) There is an automaton \( A \in \mathcal{K} \) with \( U_3 < S_1(A) \).

(iii) \( \mathcal{K} \) is not counter-free.

Proof. The necessity of (i) and (ii) is again from the first part of the Krohn–Rhodes Decomposition Theorem. Since every nontrivial group contains an element with order at least 2, we obtain \( \text{Aut}(Z_n) \in \mathcal{K}_3(\mathcal{G}) \) for some \( n \geq 2 \). Therefore also \( C_n \in \mathcal{K}_3(\mathcal{G}) \). Assuming \( \mathcal{K}_3(\mathcal{G}) \in \text{HSP}_{\alpha_0}^\lambda(\mathcal{K}) \) we obtain from Proposition 5.13 that \( \mathcal{K} \) is not counter-free. Suppose that \( \mathcal{K} \) satisfies the conditions (i), (ii) and (iii). By Theorem 5.14 we have \( U_1 \in \text{HSP}_{\alpha_0}^\lambda(\mathcal{K}) \), so that also \( U_1 \in \text{HSP}_{\alpha_0}^\lambda(\mathcal{K}) \). Let \( G \in \mathcal{G} \) and \( A \in \mathcal{K} \) with \( G < S_1(A) \). Since \( G \) is a group we also have \( G < S(A) \). As in the proof of Theorem 5.14 we obtain an integer \( n \geq 1 \) with \( C_n \in \text{HSP}_{\alpha_0}^\lambda(\mathcal{K}) \) and \( G \not\perp S(A^k) \). Theorem 5.1 now implies \( \text{Aut}(G) \in \text{HSP}_{\alpha_0}^\lambda(\mathcal{K}) \). \( \square \)

Corollary 5.16. A class \( \mathcal{K} \) is \( \alpha_0^\lambda \)-complete if and only if the following hold:

(i) For every (simple) group there is \( A \in \mathcal{K} \) with \( G < S_1(A) \).

(ii) There is an automaton \( A \in \mathcal{K} \) with \( U_3 < S_1(A) \).

(iii) \( \mathcal{K} \) is not counter-free.

This result is stronger than a similar theorem in [13], where \( U_3 \in \text{HSP}_{\alpha_0}^\lambda(\mathcal{K}) \) was required instead of (iii).

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