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The Cauchy problem for the BGK equation with an external force ☆

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ABSTRACT

In this paper, we study the Cauchy problem for the BGK equation with an external force. Firstly, we establish an L^∞ existence result for this equation, and obtain some weighted L^∞ estimates. Then, by means of the regularizing effects to the initial datum, we construct the approximate solutions and obtain some uniform estimates of the approximate solutions. Finally by using compactness method and passing to the limit, we prove the existence theorems of the L^1 and L^p solutions and establish the propagation properties of the L^p moments.

1. Introduction and main results

The BGK model is a relaxation model of the Boltzmann equation describing the evolution of a gas through a kinetic theory, this means the state of the gas is represented by the density $f(t,x,v)$ of particles which at time $t \geqslant 0$, at position $x \in \mathbb{R}^3$ move with the velocity $v \in \mathbb{R}^3$. In the presence of an external force $E(t,x,v)$, f is governed by the BGK equation [4,25]

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f + E(t,x,v) \cdot \nabla_v f = M[f] - f, \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{1.1}$$

where $f_0(x,v)$ is the initial state of gas. The nonlinear term $M[f]$, called a local Maxwellian, is implicitly defined through the moments of f

$$\begin{pmatrix} \rho \\ \rho u \\ \rho|u|^2 + 3\rho\theta \end{pmatrix}(t,x) = \int_{\mathbb{R}^3_v} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(t,x,v)\,dv \tag{1.2}$$

by the formula

$$M[f](t,x,v) = \frac{\rho(t,x)}{(2\pi\theta(t,x))^{3/2}} \exp\left(-\frac{|v-u(t,x)|^2}{2\theta(t,x)}\right). \tag{1.3}$$

It is well known that the BGK equation (1.1) is an important relaxation model of the Boltzmann equation. In the absence of the external force, the Cauchy problem and the initial–boundary value problem for the BGK equation have been

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extensively studied, see [3,6,18–22,26–28]. We only mention some works related to the problems considered in this paper. Assuming that the total mass, inertia, kinetic energy and entropy of the initial datum are finite, Perthame [20] built an L^1 theory and proved that the classical BGK equation has a positive solution in the distributional sense in 1989. Moreover this solution was proved to propagate the initial higher order moments in [27]. On the other hand, Perthame and Pulvirenti [21] also developed an L^∞ method and showed that there exists a unique polynomially decaying solution for x in a periodic domain; later this result was generalized to the Cauchy problem of BGK equation in [19]. The main ingredient of this method is the L^∞ estimates of the macroscopic quantities and the local Maxwellians obtained in [21]. Recently, by establishing weighted L^p estimates of the hydrodynamical quantities and local Maxwellians, an L^p existence theorem and certain regularity results were developed in [28] by means of weakly compact argument.

Recently, some authors have paid their attention to investigating BGK equations with given force terms as well as self induced electrostatic fields. Rejeb [23] gave an existence result of a classical solution to the Vlasov–Poisson–BGK equation in one dimension. Zhang [29] proved the global existence of weak solutions in L^p $(p > 9)$ space to the Cauchy problem of the three dimensional Vlasov–Poisson–BGK system. For the BGK equation with a given confining potential $\Phi(x)$ (namely, $E = -\nabla_x \Phi$), Bosi and Cáceres [5] studied the global existence in L^1 space and the long time behavior by the method of Perthame [20].

Here, we should mention that the Boltzmann equations with external forces were also extensively studied. For example, local existence theorems were given by Asano [1] and Glikson [12,13]. Bellomo, Lachowicz, Palczewski and Toscani [2] gave the global existence of mild solutions, see also the recent result [8,9]. Global existence of classical solutions with small amplitude was obtained by Guo [15] for the soft potential and by Duan, Yang and Zhu [10] for the general potential. For solutions near a global Maxwellian, Ukai, Yang and Zhao [24] proved the stability of stationary Maxwellian solutions to the Boltzmann equation with external forces through the energy method (see also related results in [16,17]).

In this paper, we study the Cauchy problem for the BGK equation with a general external force which depends on t, x and v. Specially, we assume throughout this paper that the external force $E(t,x,v)$ satisfies the following condition:

(A) $\nabla_v \cdot E = 0$ on $\mathbb{R}^+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v$ in distributional sense. Furthermore, $E(\cdot,\cdot,\cdot) \in L^1_{loc}((0,\infty), L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v))$, i.e., for any $T < \infty$, there exists a constant $E_T > 0$ such that

$$\int_0^T \|E(t)\|_\infty \, dt \leqslant E_T. \tag{1.4}$$

Remark 1.1. (A) is technical assumption. For understanding the hypotheses, we list two special examples: (1) Constant field: $E(t,x,v) = E_0$, where E_0 is constant vector; (2) $E(t,x,v) = P(t) \equiv (p_1(t), p_2(t), p_3(t))$, where $p_i(t)$ $(i = 1,2,3)$ could be a power function t^α $(\alpha > 0)$, exponential function e^t and trigonometric function $\sin(t)$ or $\cos(t)$. These examples are obviously suitable for the assumption (A). Even so, some important examples are not included. For example, gravitational field or repulsive field: $E(t,x,v) = \pm \frac{x}{|x|^3}$; Lorentz field: $E(t,x,v) = E_0(t,x) + v \times B_0(t,x)$, where $E_0(t,x)$ and $B_0(t,x)$ are given electric intensity and magnetic intensity respectively. These are still unknown problem for the BGK equation. We will consider these problems in the future.

Remark 1.2. The assumption that $\nabla_v \cdot E = 0$ is standard, which is to ensure the mapping $(x,v) \to (X(s), V(s))$ and $(x,v) \to (X^t(x,v), V^t(x,v))$ for any $s, t \geqslant 0$ preserves the measure. Under this condition, for any $1 \leqslant p \leqslant \infty$ and $f(t,x,v)$, we have

$$\big\| f\big(t, X^t(x,v), V^t(x,v)\big)\big\|_{L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)} = \big\| f(t,x,v)\big\|_{L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)},$$
$$\big\| f\big(t, X(s), V(s)\big)\big\|_{L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)} = \big\| f(t,\cdot,\cdot)\big\|_{L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)}.$$

For the sake of simplicity, we denote positive constants by C, C_1 and $C(\alpha_1, \alpha_2, \ldots, \alpha_n)$ depending on $\alpha_1, \alpha_2, \ldots, \alpha_n$. And for any function $f(t,x,v)$, we use notation:

$$\big\| f(t)\big\|_p = \big\| f(t,\cdot,\cdot)\big\|_{L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)}.$$

Then the main results of this paper can be described by the following theorems.

Theorem 1.1. *Suppose that the external force satisfies* (A), *and the initial datum* $f_0(x,v)$ *is a nonnegative function such that*

$$\int_{\mathbb{R}^3_x \times \mathbb{R}^3_v} \big(1 + |x|^2 + |v|^2 + \big|\log f_0(x,v)\big|\big) f_0(x,v)\, dx\, dv = C_0 < \infty. \tag{1.5}$$

Then there exists a distributional solution $f(t,x,v) \in C([0,\infty), L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)_+)$ *to the BGK equation* (1.1), (1.2) *and* (1.3) *such that*

$$\int_{\mathbb{R}^3_x \times \mathbb{R}^3_v} \big(1 + |x|^2 + |v|^2 + \big|\log f(t,x,v)\big|\big) f(t,x,v)\, dx\, dv \leqslant C(T, f_0, E_T), \quad \forall t \leqslant T. \tag{1.6}$$

Remark 1.3. Although Bosi and Cáceres [5] also gave the existence of L^1 solutions to the BGK equation, the external force considered by them is induced by a confining potential which depends only on the position variable x. Here, we consider external forces which not only depend on x but also on t and v. Our strategy of proving Theorem 1.1 relies on the L^∞ result established in Theorem 3.1 (which follows from the method introduced in [21]), while the result in [5] was obtained by the method established in [20]. Furthermore, the force term generated by the confining potential $\Phi(x)$ in [5] and the one considered in this paper belong to different classes.

Next, we describe our second result which do not require the assumption of finite initial entropy, but some L^p regularity is assumed.

Theorem 1.2. *Suppose that the external force satisfies* (A)*, and the initial datum $f_0(x,v)$ is a nonnegative function verifying*

$$f_0 \in L^1 \cap L^p\big(\mathbb{R}^3_x \times \mathbb{R}^3_v\big) \quad (1 < p \leqslant \infty), \qquad |v|^2 f_0 \in L^1\big(\mathbb{R}^3_x \times \mathbb{R}^3_v\big). \tag{1.7}$$

Then there exists a global solution $f(t,x,v) \in L^\infty([0,\infty), L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)_+)$ in the distributional sense such that

$$\sup_{0\leqslant t\leqslant T} \int\limits_{\mathbb{R}^3_x\times\mathbb{R}^3_v} \big(1+|v|^2\big) f(t,x,v)\,dx\,dv \leqslant C(T, f_0, E_T), \tag{1.8}$$

$$\sup_{0\leqslant t\leqslant T} \big\| f(t)\big\|_p \leqslant C(T, p, f_0, E_T). \tag{1.9}$$

Furthermore, if

$$\big(|x|^q + |v|^q\big) f_0 \in L^p\big(\mathbb{R}^3_x \times \mathbb{R}^3_v\big) \tag{1.10}$$

for some $q \in (1, 3/p') \cup (3/p' + 2, \infty)$, then we have the following propagation:

$$\sup_{0\leqslant t\leqslant T} \big\| \big(|x|^q + |v|^q\big) f(t,\cdot,\cdot)\big\|_p \leqslant C(p, q, f_0, T, E_T). \tag{1.11}$$

Remark 1.4. Zhang [29] considered self-consistent force $E(t,x)$ given by a potential equation and proved the existence of L^p solutions to a coupled system for $p > 9$. For the force term satisfying assumption (A), we obtain in Theorem 1.2 similar results under much weaker condition $p > 1$ for the initial datum f_0. Moreover, the total energy is conservative in [29] (at least for approximate solutions), which can be used to establish uniform estimate of kinetic energy. Nevertheless, the model considered in this paper does not have conservation law for energy, so we need to prove some uniform estimates for kinetic energy with different methods (see Lemma 4.3).

The rest of this paper is organized as follows. In Section 2, some preliminary lemmas are given for later use. Section 3 is devoted to establishing existence and uniqueness of solutions in weighted L^∞ space, which is the starting point for proving our main theorems. The main tools in this section include the Banach fixed point theorem and some weighted L^∞ estimates which will result in uniform estimates for the second velocity moment as well as uniform estimates for entropy and L^p norm. In Section 4, by means of the regularizing effects to the initial datum and the theorem of Section 3, we construct the approximate solutions and obtain some uniform estimates of the approximate solutions. Then we prove the existence theorem of the L^1 solutions by using compactness method and passing to the limit. Finally, in Section 5, combining the idea in Section 4 and the asymptotic method in [28], we prove the existence theorem of the L^p solutions and establish the propagation properties of some L^p moments.

2. Preliminaries

As usual, we will rewrite the BGK equation along the characteristics. Suppose that $E(t,x,v) \in C([0,\infty), C^1_b(\mathbb{R}^3_x \times \mathbb{R}^3_v))$, then for any fixed point (x,v) in $\mathbb{R}^3_x \times \mathbb{R}^3_v$, the forward bi-characteristics $[X^t(x,v), V^t(x,v)]$ generated by the external force $E(t,x,v)$ is defined by

$$\begin{cases} \dfrac{dX^t(x,v)}{dt} = V^t(x,v), \quad \dfrac{dV^t(x,v)}{dt} = E\big(t, X^t(x,v), V^t(,x,v)\big), \\ \big(X^t(x,v), V^t(x,v)\big)\big|_{t=0} = (x,v). \end{cases} \tag{2.1}$$

Notice that the Cauchy–Lipschitz theorem ensures the global existence and uniqueness of solutions to the above ODE system. Then the mild form of the BGK equation (1.1) becomes

$$f^\sharp(t,x,v) = f_0(x,v)e^{-t} + \int\limits_0^t e^{(s-t)} M[f]^\sharp(s,x,v)\,ds, \tag{2.2}$$

where $f^\sharp(t,x,v) = f(t, X^t(x,v), V^t(x,v))$, $M[f]^\sharp(t,x,v) = M[f](t, X^t(x,v), V^t(x,v))$.

On the other hand, for any fixed point (t,x,v) in $\mathbb{R}^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$, we define the backward bi-characteristics $[X(s;t,x,v), V(s;t,x,v)]$ by solutions to the ODE system

$$\begin{cases} \dfrac{dX(s;t,x,v)}{ds} = V(s;t,x,v), \quad \dfrac{dV(s;t,x,v)}{ds} = E\big(s, X(s;t,x,v), V(s;t,x,v)\big), \\ \big(X(s;t,x,v), V(s;t,x,v)\big)\big|_{s=t} = (x,v). \end{cases} \tag{2.3}$$

Due to the backward bi-characteristics, we obtain another representation of the BGK equation (1.1), namely

$$f(t,x,v) = f_0\big(X(0;t,x,v), V(0;t,x,v)\big)e^{-t} + \int_0^t e^{(s-t)} M[f]\big(s, X(s;t,x,v), V(s;t,x,v)\big)\, ds. \tag{2.4}$$

For the sake of simplicity, we will use the short hands $X(s) = X(s;t,x,v)$, $V(s) = V(s;t,x,v)$. Notice that bi-characteristics equation (2.3) can be rewritten as the following integral form:

$$\begin{cases} V(s) = v - \displaystyle\int_s^t E\big(\tau, X(\tau), V(\tau)\big)\, d\tau, \\ X(s) = x - v(t-s) + \displaystyle\int_s^t \int_\theta^t E\big(\tau, X(\tau), V(\tau)\big)\, d\tau\, d\theta. \end{cases} \tag{2.5}$$

Thanks to this integral form, it is easy to prove the following estimates for characteristics (for details, see, e.g.: [15]).

Lemma 2.1. *Suppose that $E(t,x,v) \in C([0,\infty), C_b^1(\mathbb{R}_x^3 \times \mathbb{R}_v^3))$ and satisfies (1.4), then we have that for any (t,x,v) in $[0,T) \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$,*

$$\big|V(s) - v\big| \leqslant E_T, \qquad \big|X(s) - \big[x - v(t-s)\big]\big| \leqslant |t-s| E_T.$$

Moreover, we have

$$\big|V(s) - V(0)\big| \leqslant E_T, \qquad \big|X(s) - X(0) - vs\big| \leqslant |s| E_T.$$

Let $f(v)$ be a spatially homogeneous kinetic density and let ρ, u and θ be the mass density, bulk velocity and temperature of f. The following two lemmas on L^p estimates for local Maxwellians were obtained in Refs. [19,21,28] by interpolation method.

Lemma 2.2. *Suppose that $f(v) \in L^p(\mathbb{R}^3) \cap L_2^1(\mathbb{R}^3)_+$, $1 < p \leqslant \infty$ and $1/p + 1/p' = 1$, then for $q = 0$ or $1 \leqslant q \leqslant 3/p'$ or $q \geqslant 3/p' + 2$,*

$$\big\| |v|^q M[f] \big\|_p \leqslant C(p,q) \big\| |v|^q f \big\|_p. \tag{2.6}$$

Lemma 2.3. *Suppose that $f(v) \in L^p(\mathbb{R}^3) \cap L_2^1(\mathbb{R}^3)_+$, $1 < p \leqslant \infty$ and $1/p + 1/p' = 1$. Then, for $q \geqslant 3/p' + 2$ and $\alpha \in [0, q - 3/p']$,*

$$\rho\theta^{\alpha/2} \leqslant C(p,q,\alpha) \big\| \big(1 + |v|^q\big) f \big\|_p. \tag{2.7}$$

For $q \geqslant 3/p' + 2$, $\beta \in [0,1]$ and $\alpha \in [0, q - 3/p'(1-\beta)]$,

$$\frac{\rho |u|^\alpha}{\theta^{3\beta/2p'}} \leqslant C(p,q,\alpha,\beta) \big\| \big(1 + |v|^q\big) f \big\|_p. \tag{2.8}$$

In order to prove our main results, we also need two technical results—the velocity moments lemma and the velocity averaging lemma.

Lemma 2.4. *Let $E(t,x,v) \in C([0,\infty), C_b^1(\mathbb{R}_x^3 \times \mathbb{R}_v^3))$ and $\nabla_v \cdot E = 0$. Suppose that $f \in C([0,T), L^1(\mathbb{R}_x^3 \times \mathbb{R}_v^3))$ is the unique distributional solution of the following Cauchy problem*

$$\begin{aligned} &\frac{\partial f}{\partial t} + v \cdot \nabla_x f + E(t,x,v) \cdot \nabla_v f = g - f, \\ &f(0,x,v) = f_0(x,v) \end{aligned} \tag{2.9}$$

with $g \geqslant 0$, $f_0 \geqslant 0$. Assume that there exists a positive constant $C(T)$ such that

$$\int_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \left(1+|v|^2\right)\left(g(t,x,v)+f_0(x,v)\right)dx\,dv \leqslant C(T), \quad t\in[0,T].$$

Then, for any bounded $K_x \Subset \mathbb{R}^3$ we have

$$\int_0^T dt \int_{K_x\times\mathbb{R}^3} |v|^3 f(t,x,v)\,dx\,dv \leqslant C\left(T,\operatorname{diam}(K_x)\right),$$

where $\operatorname{diam}(K_x)$ *is the diameter of the set* K_x.

Proof. To shorten the presentation of the paper, we omit the proof (for the details, we refer the readers to [5]). We only mention that the constant $C(T,\operatorname{diam}(K_x))$ can be exactly computed, namely

$$C\left(T,\operatorname{diam}(K_x)\right) = \left(1+\operatorname{diam}(K_x)^2\right)^{3/2} C(T)\left(2+\int_0^T \left\|E(t)\right\|_\infty dt\right),$$

which will be used in Sections 4 and 5. □

Similar to the proof of Refs. [5,7,14], the velocity averaging lemma can be written as follows:

Lemma 2.5. *Let $E_n(t,x,v)\in C([0,\infty), C_b^1(\mathbb{R}_x^3\times\mathbb{R}_v^3))$ and $\nabla_v\cdot E_n=0$. Suppose that the sequence $f_n(t,x,v)\in L^1((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$ is weakly compact, and the sequence $g_n(t,x,v)\in L^1((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$ is also locally weakly compact such that*

$$f_n+\partial_t f_n + v\cdot\nabla_x f_n + E_n(t,x,v)\cdot\nabla_v f_n = g_n \tag{2.10}$$

in the distributional sense. Then for any bounded sequence $\psi_n(t,x,v)\in L^\infty([0,T]\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$ that converges almost everywhere, the sequence $\int_{\mathbb{R}^3} f_n(t,x,v)\psi_n(t,x,v)\,dv$ is compact in $L^1((0,T)\times\mathbb{R}^3)$.

Corollary 2.1. *Let $E_n(t,x,v)\in C([0,\infty), C_b^1(\mathbb{R}_x^3\times\mathbb{R}_v^3))$ and $\nabla_v\cdot E_n=0$. Suppose that the sequence $f_n(t,x,v)\in L^1((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$ is weakly compact in $L^1((0,T)\times K_x\times\mathbb{R}^3)$ for any compact set $K_x\Subset\mathbb{R}^3$, and the sequence $g_n(t,x,v)\in L^1_{loc}((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$ is also locally weakly compact such that*

$$f_n+\partial_t f_n + v\cdot\nabla_x f_n + E_n(t,x,v)\cdot\nabla_v f_n = g_n$$

in the distributional sense. Then for any bounded sequence $\psi_n(t,x,v)\in L^\infty([0,T]\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$ that converges almost everywhere and for any compact set $K_x\Subset\mathbb{R}_x^3$, the sequence $\int_{\mathbb{R}_v^3} f_n(t,x,v)\psi_n(t,x,v)\,dv$ is compact in $L^1((0,T)\times K_x)$.

3. Weighted L^∞ bounds

In this section, we will prove the existence and uniqueness of the solutions by the contraction mapping principle in the weighted L^∞ space, and obtain some weighted L^∞ estimates. Throughout this section, we assume $E(t,x,v)\in C([0,\infty), C_b^1(\mathbb{R}_x^3\times\mathbb{R}_v^3))$ satisfies (1.4) and $\nabla_v\cdot E=0$.

Firstly, we introduce some norms. Let $m,k>0$. We define the weighted L^∞ norm as follows:

$$\begin{aligned}
N_m(f)(t) &= \sup_{(x,v)\in\mathbb{R}_x^3\times\mathbb{R}_v^3}\left(1+|v|^m\right)\left|f(t,x,v)\right|,\\
N_{m,k}(f)(t) &= \sup_{(x,v)\in\mathbb{R}_x^3\times\mathbb{R}_v^3}\left(1+|v|^m\right)\left(1+|x|^k\right)\left|f(t,x,v)\right|,\\
N'_m(f)(t) &= \sup_{(x,v)\in\mathbb{R}_x^3\times\mathbb{R}_v^3}\left(1+\left|V(0)\right|^m\right)\left|f(t,x,v)\right|,\\
N'_{m,k}(f)(t) &= \sup_{(x,v)\in\mathbb{R}_x^3\times\mathbb{R}_v^3}\left(1+\left|V(0)\right|^m\right)\left(1+\left|X(0)\right|^k\right)\left|f(t,x,v)\right|.
\end{aligned}$$

Obviously,

$$N_m(f_0)=N'_m(f_0), \qquad N_{m,k}(f_0)=N_{m,k}(f_0).$$

Then, we have

Lemma 3.1. *For any $T<\infty$, we have*

$$N_{m+k}(f)(t)\leqslant C,\qquad N_{m,k}(f)(t)\leqslant C,\quad t\in[0,T]$$

if and only if

$$N'_{m+k}(f)(t)\leqslant C,\qquad N'_{m,k}(f)(t)\leqslant C,\quad t\in[0,T],$$

where constant C is only depending on T, m, E_T and k.

Proof. By the definitions, we have

$$\begin{aligned}N'_{m+k}(f)&=\sup_{(x,v)\in\mathbb{R}^3_x\times\mathbb{R}^3_v}\bigl(1+\bigl|V(0)\bigr|^{m+k}\bigr)\bigl|f(t,x,v)\bigr|\\&\leqslant\sup_{(x,v)\in\mathbb{R}^3_x\times\mathbb{R}^3_v}\bigl(1+\bigl(|v|+E_T\bigr)^{m+k}\bigr)\bigl|f(t,x,v)\bigr|\\&\leqslant\sup_{(x,v)\in\mathbb{R}^3_x\times\mathbb{R}^3_v}C(m,k,E_T)\bigl(1+|v|^{m+k}\bigr)\bigl|f(t,x,v)\bigr|=C(m,k,E_T)N_{m+k}(f).\end{aligned}$$

Similarly,

$$\begin{aligned}N'_{m,k}(f)&=\sup_{(x,v)\in\mathbb{R}^3_x\times\mathbb{R}^3_v}\bigl(1+\bigl|V(0)\bigr|^{m}\bigr)\bigl(1+\bigl|X(0)\bigr|^{k}\bigr)\bigl|f(t,x,v)\bigr|\\&\leqslant\sup_{(x,v)\in\mathbb{R}^3_x\times\mathbb{R}^3_v}\bigl(1+\bigl(|v|+E_T\bigr)^{m}\bigr)\bigl(1+\bigl(|x-vt|+TC(T)\bigr)^{k}\bigr)\bigl|f(t,x,v)\bigr|\\&\leqslant\sup_{(x,v)\in\mathbb{R}^3_x\times\mathbb{R}^3_v}C(m,k,E_T)\bigl(1+|v|^{m}\bigr)\bigl(1+|x|^{k}\bigr)\bigl|f(t,x,v)\bigr|+C(m,k,T,E_T)\bigl(1+|v|^{m+k}\bigr)\bigl|f(t,x,v)\bigr|\\&=C(m,k,T,E_T)\bigl(N_{m,k}(f)+N_{m+k}(f)\bigr).\end{aligned}$$

We have the same kind of estimates by inversion of characteristics. Then, the proof of Lemma 3.1 is completed. □

In this section, we assume that the initial datum $f_0(x,v)$ satisfies the following two conditions:

(A1) There exist $m>5$, $k>3$ such that

$$N_{m+k}(f_0)<\infty,\qquad N_{m,k}(f_0)<\infty.\tag{3.1}$$

(A2) There exist a function $\varphi(v)\in L^1(\mathbb{R}^3)$ and a constant $\delta>0$ such that $|\varphi(v)|\geqslant\delta$ for $|v|<1$, and $(1+|x|^k)f_0(x,v)\geqslant\varphi(v)$ for any $(x,v)\in\mathbb{R}^3_x\times\mathbb{R}^3_v$.

Obviously, the condition (A1) ensures that $(1+|v|^2)f_0(x,v)\in L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)$. Therefore, we can define the initial mass density $\rho(0,x)$, the mean velocity $u(0,x)$, the temperature $\theta(0,x)$ and the Maxwell distribution $M[f_0]$. With the above notations and assumptions, the main results of this section can be described by follows.

Theorem 3.1. *Suppose that the external force $E(t,x,v)\in C([0,\infty),C^1_b(\mathbb{R}^3_x\times\mathbb{R}^3_v))$, $\nabla_v\cdot E=0$ and satisfies* (1.4)*. Let the initial datum $f_0(x,v)$ be a nonnegative function and satisfies conditions* (A1) *and* (A2)*. Then there exists one and only one mild solution $f(t,x,v)\in C([0,\infty),L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v,(1+|v|^2)\,dx\,dv))$ to the initial value problem* (1.1), (1.2) *and* (1.3)*. Moreover, there exist two nonnegative functions $C_1(t)$, $C_2(t)$ such that*

$$N'_{m+k}\bigl(f(t)\bigr),N'_{m,k}\bigl(f(t)\bigr)\leqslant C_1(t)<\infty,\tag{3.2}$$

$$\rho(t,x),\bigl|u(t,x)\bigr|,\theta(t,x)\leqslant C_1(t)<\infty,\tag{3.3}$$

$$\bigl(1+|x|^k\bigr)\rho(t,x),\theta(t,x)\geqslant C_2(t)>0.\tag{3.4}$$

Proof. A nonnegative function $f(t,x,v)\in C([0,\infty),L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v,(1+|v|^2)\,dx\,dv))$ is a mild solution if and only if f satisfies the integral equation

$$f(t,x,v)=f_0\bigl(X(0),V(0)\bigr)e^{-t}\int_0^t e^{(s-t)}M[f]\bigl(s,X(s),V(s)\bigr)\,ds.$$

We define a nonlinear operator F by

$$Ff = f_0\big(X(0), V(0)\big)e^{-t} + \int_0^t e^{(s-t)} M[f]\big(s, X(s), V(s)\big)\, ds. \tag{3.5}$$

Then, we only need to show that the operator F has a unique fixed point in the function space $C([0,T], L^1(\mathbb{R}_x^3 \times \mathbb{R}_v^3, (1+|v|^2)\,dx\,dv))$ and satisfies (3.2), (3.3) and (3.4). Now, we consider the function space X:

$$X = \left\{ \begin{array}{c} f(t,x,v) \in C([0,T], L^1(\mathbb{R}_x^3 \times \mathbb{R}_v^3, (1+|v|^2)\,dx\,dv))_+, \\ N'_{m+k}(f(t)) \leqslant e^{Ct} N_{m+k}(f_0), \\ N'_{m,k}(f(t)) \leqslant e^{Ct}[N_{m+k}(f_0) + N_{m,k}(f_0)], \\ (1+|x|^k)\rho(t,x) \geqslant C(T,k,E_T) > 0, \theta(t,x) > 0 \end{array} \right\} \tag{3.6}$$

where C is sufficiently large. Define metric in X as follows

$$d(f_1, f_2) = \sup_{t\in[0,T]} \exp(-Gt) \int_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \big|f_1(t,x,v) - f_2(t,x,v)\big|\big(1+|v|^2\big)\,dx\,dv,$$

where G is sufficiently large. Then the function space X with the metric d is a complete metric space.

Firstly, we will show that Ff satisfies the following for $f \in X$,

$$N'_{m+k}(Ff)(t) \leqslant e^{Ct} N_{m+k}(f_0), \tag{3.7}$$

$$N'_{m,k}(Ff)(t) \leqslant e^{Ct}\big[N_{m+k}(f_0) + N_{m,k}(f_0)\big], \tag{3.8}$$

$$\big(1+|x|^k\big)\rho(Ff)(t,x) \geqslant C(T,k,E_T) > 0, \tag{3.9}$$

$$\theta(Ff)(t,x) \geqslant C(T,k,E_T) > 0. \tag{3.10}$$

Multiplying both sides of (3.5) by $1+|V(0)|^{m+k}$, we obtain

$$\begin{aligned} 0 \leqslant \big(1+\big|V(0)\big|^{m+k}\big)(Ff)(t,x,v) &= e^{-t}\big(1+\big|V(0)\big|^{m+k}\big) f_0\big(X(0), V(0)\big) \\ &\quad + \int_0^t e^{s-t}\big(1+\big|V(0)\big|^{m+k}\big) M[f]\big(s, X(s), V(s)\big)\, ds. \end{aligned}$$

By Lemmas 2.1, 2.2 and 3.1, we have

$$\begin{aligned} N'_{m+k}(Ff)(t) &\leqslant e^{-t} N_{m+k}(f_0) + \int_0^t e^{s-t}\big(1+\big(\big|V(s)\big| + C(T)\big)^{m+k}\big) M[f]\big(s, X_n(s), V_n(s)\big)\, ds \\ &\leqslant e^{-t} N_{m+k}(f_0) + C(m,k,E_T) \int_0^t e^{s-t} N_{m+k}\big(M[f]\big)\, ds \\ &\leqslant e^{-t} N_{m+k}(f_0) + C(m,k,E_T) \int_0^t e^{s-t} N'_{m+k}\big(f(s)\big)\, ds \\ &\leqslant e^{-t} N_{m+k}(f_0) + C(m,k,E_T) N_{m+k}(f_0) \int_0^t e^{s-t} e^{Cs}\, ds \\ &= e^{-t} N_{m+k}(f_0) + \frac{C(m,k,E_T)}{C+1} N_{m+k}(f_0)\big(e^{Ct} - e^{-t}\big). \end{aligned}$$

Taking $C \geqslant C(m,k,E_T)$, the above inequalities imply that

$$N'_{m+k}(Ff)(t) \leqslant e^{Ct} N_{m+k}(f_0).$$

Multiplying both sides of (3.5) by $(1+|V(0)|^m)(1+|X(0)|^k)$ and computing in the same way, we can get

$$N'_{m,k}(Ff)(t) \leqslant e^{Ct}\big(N_{m+k}(f_0) + N_{m,k}(f_0)\big).$$

Next, we show (3.9). For any $t \in [0, T]$ and $x \in \mathbb{R}^3$,

$$\begin{aligned}\rho(Ff)(t,x) &\geqslant e^{-t}\int_{\mathbb{R}^3_v} f_0\big(X_n(0), V_n(0)\big)\,dv \geqslant e^{-t}\int_{\mathbb{R}^3_v} \frac{\varphi(V_n(0))}{1+|X_n(0)|^k}\,dv\\ &\geqslant e^{-t}\int_{|V_n(0)|\leqslant 1} \frac{\delta}{1+|X_n(0)|^k}\,dv\\ &\geqslant e^{-t}\int_{|v|\leqslant|1-E_T|} \frac{\delta}{[1+(|x|+|tv|+CE_T)^k]}\,dv\\ &\geqslant C(k,E_T)e^{-T}\int_{|v|\leqslant|1-E_T|} \frac{\delta}{2^k(1+t^k)(1+|x|^k)}\,dv\\ &\geqslant \frac{C(k,T,E_T)}{(1+|x|^k)}.\end{aligned}$$

Then we obtain (3.9).

By Lemma 2.3 and (3.9), $\theta(Ff)$ can be estimated as follows:

$$\frac{1}{\theta(Ff)} = \big(1+|x|^k\big)^{2/3}\frac{\rho^{2/3}(Ff)}{\theta(Ff)}\cdot\frac{1}{((1+|x|^k)\rho(Ff))^{2/3}} \leqslant C(k,T,E_T).$$

Then, (3.10) holds. Consequently, the operator F maps X into itself.

Secondly, we will show that the operator F is a contraction from X to itself. By Lemma 2.3, it has been proved in [21] that there exists a positive constant L such that for any $f_1, f_2 \in X$

$$\int_{\mathbb{R}^3_x\times\mathbb{R}^3_v} \big|M[f_1]-M[f_2]\big|\big(1+|v|^2\big)\,dx\,dv \leqslant L\int_{\mathbb{R}^3_x\times\mathbb{R}^3_v} |f_1-f_2|\big(1+|v|^2\big)\,dx\,dv.$$

Then for any $f_1, f_2 \in X$, we obtain

$$d(Ff_1, Ff_2) \leqslant \frac{L}{G+1}d(f_1, f_2).$$

Taking $G \geqslant L$, we obtain that the operator F is a contraction. According to the contraction mapping theorem, the operator F has a unique fixed point in X. Obviously, the fixed point is the unique solution to the BGK equation verifying (3.7), (3.8), (3.9) and (3.10). □

Remark 3.1. By Lemma 3.1, Theorem 3.2 implies that

$$N_{m+k}\big(f(t)\big), N_{m,k}\big(f(t)\big) \leqslant C_1(t) < \infty, \quad \forall t \in [0, T].$$

4. Proof of Theorem 1.1

In this section, we will prove the existence theorem of the L^1 solutions by using compactness method. So we are in a position to construct approximate solutions to (1.1)–(1.3). We always assume that the initial datum f_0 satisfies (1.5). Following the method in [7], we have

Lemma 4.1. *Suppose that the initial datum f_0 satisfies (1.5), $k > 5$, then there exist sequences $f_0^n \in C^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_v)$, $\bar{f}_0^n \in C_c^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_v)$ and positive constant C_0 such that*

$$f_0^n = \bar{f}_0^n + \frac{\exp(-|v|^2)}{n(1+|x|^k)} \geqslant \frac{\exp(-|v|^2)}{n(1+|x|^k)}, \tag{4.1}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3_x\times\mathbb{R}^3_v} \big(1+|x|^2+|v|^2\big)\big|f_0^n-f_0\big|\,dx\,dv = 0, \tag{4.2}$$

$$\int_{\mathbb{R}^3_x\times\mathbb{R}^3_v} \big|\log f_0^n\big|f_0^n\,dx\,dv \leqslant C_0, \tag{4.3}$$

$$\lim_{n\to\infty} H\big(f_0^n\big) = H(f_0). \tag{4.4}$$

We also assume that $E_n(t,x,v) = E(t,x,v) * J_n$, where $J_n(t,x,v)$ are mollifying functions. Then we have $E_n(t,x,v) \in C^\infty([0,\infty) \times \mathbb{R}_x^3 \times \mathbb{R}_v^3)$, $\int_0^T \|E_n(t)\|_\infty \, dt \leqslant E_T$ and $\nabla_v \cdot E_n(t,x,v) = 0$. So E_n ensures the existence of characteristics equations (2.1), (2.3) and satisfies all assumptions of Lemmas 2.1–2.5. And it is obvious that for $n = 1, 2, \ldots$, f_0^n and E_n satisfy all assumptions of Theorem 3.1. Hence, the approximate BGK equation

$$\begin{cases} \dfrac{\partial f_n}{\partial t} + v \cdot \nabla_x f_n + E_n(t,x,v) \cdot \nabla_v f_n = M[f_n] - f_n, \\ f_n(0,x,v) = f_0^n(x,v), \end{cases} \tag{4.5}$$

has a unique global solution $f_n(t,x,v)$, where

$$\begin{pmatrix} \rho_n \\ \rho_n u_n \\ \rho|u_n|^2 + 3\rho_n\theta_n \end{pmatrix}(t,x) = \int\limits_{\mathbb{R}_v^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_n(t,x,v) \, dv, \tag{4.6}$$

and

$$M[f_n](t,x,v) = \frac{\rho_n(t,x)}{(2\pi\theta_n(t,x))^{3/2}} \exp\left(-\frac{|v - u_n(t,x)|^2}{2\theta_n(t,x)}\right). \tag{4.7}$$

By Theorem 3.1 and Remark 3.1, the solution satisfies

$$N_{m,k}(f_n)(t) \leqslant C(T)\big(N_{m+k}\big(f_0^n\big) + N_{m,k}\big(f_0^n\big)\big). \tag{4.8}$$

Obviously, for $m, k > 5$, we have

$$\int\limits_{\mathbb{R}_x^3 \times \mathbb{R}_v^3} \big(1 + |x|^2 + |v|^2\big) f_n(t,x,v) \, dx \, dv \leqslant C(n, T, f_0, E_T), \quad \forall t \leqslant T,$$

and

$$\big\| f_n(t) \big\|_\infty \leqslant C(n, T, f_0, E_T), \quad \forall t \leqslant T.$$

Then for any $t \leqslant T$, we have

$$\int\limits_{\mathbb{R}_x^3 \times \mathbb{R}_v^3} \big(1 + |x|^2 + |v|^2 + \big|\log f_n(t,x,v)\big|\big) f_n(t,x,v) \, dx \, dv \leqslant C(n, T, f_0, E_T). \tag{4.9}$$

Notice that since $N_{m+k}(f_0^n)$ and $N_{m,k}(f_0^n)$ are dependent on n, (4.9) is not a uniform estimate. In fact, we will show that the constant in (4.9) is not dependent on n, for that we need the following lemma.

Lemma 4.2. *(See [5].) Suppose that $f_n \in C([0,\infty), L^1(\mathbb{R}_x^3 \times \mathbb{R}_v^3))$ is a solution of the BGK equation* (4.5), (4.6) *and* (4.7) *in distributional sense and satisfies condition* (4.9)*, then we have*

$$\partial_t(f_n \log f_n) + v \cdot \nabla_x(f_n \log f_n) + E_n \cdot \nabla_v(f_n \log f_n) = \big(M[f_n] - f_n\big)(1 + \log f_n)$$

in the distributional sense. Moreover, for any $0 \leqslant t_1 \leqslant t_2$, we have

$$\begin{aligned} H(f_n)(t_1) - H(f_n)(t_2) &= \int\limits_0^t \int\limits_{\mathbb{R}_x^3 \times \mathbb{R}_v^3} \big(f_n - M[f_n]\big) \log f \, dx \, dv \, dt \\ &= \int\limits_0^t \int\limits_{\mathbb{R}_x^3 \times \mathbb{R}_v^3} \big(f_n - M[f_n]\big)\big(\log f_n - \log M[f_n]\big) \, dx \, dv \, dt \geqslant 0. \end{aligned}$$

Now, we give the uniform estimate of the approximate solutions.

Lemma 4.3. *Suppose that $f_n \in C([0,\infty), L^1(\mathbb{R}_x^3 \times \mathbb{R}_v^3))$ is a distribution solution of the BGK equation* (4.5), (4.6), (4.7) *with condition* (1.5) *and satisfies condition* (4.9)*, then we have*

$$\int\limits_{\mathbb{R}_x^3 \times \mathbb{R}_v^3} \big(1 + |x|^2 + |v|^2 + \big|\log f_n(t,x,v)\big|\big) f_n(t,x,v) \, dx \, dv \leqslant C(T, f_0, E_T), \quad \forall t \leqslant T.$$

Proof. Using the characteristics flow (2.1), the BGK equation can be rewritten as

$$\frac{\partial f_n^\sharp}{\partial t} = M[f_n]^\sharp - f_n^\sharp, \tag{4.10}$$

where $f_n^\sharp(t,x,v) = f(t, X_n^t(x,v), V_n^t(x,v))$, $M[f_n]^\sharp(t,x,v) = M[f_n](t, X_n^t(x,v), V_n^t(x,v))$. Integrating both hand sides in $\mathbb{R}_x^3 \times \mathbb{R}_v^3$, we have

$$\int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} f_n(t,x,v)\,dx\,dv = \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} f_0^n(x,v)\,dx\,dv \leqslant C(f_0). \tag{4.11}$$

We multiply both sides by $|V_n^t(x,v)|^2$, and integrate in $\mathbb{R}_x^3 \times \mathbb{R}_v^3$,

$$\int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \frac{\partial f_n^\sharp}{\partial t} \left|V_n^t(x,v)\right|^2 dx\,dv = \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \left|V_n^t(x,v)\right|^2 \left(M[f_n]^\sharp - f_n^\sharp\right) dx\,dv.$$

Obviously, the right-hand side can be easily computed by

$$\int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \left|V_n^t(x,v)\right|^2 \left(M[f_n]^\sharp - f_n^\sharp\right) dx\,dv = \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} |v|^2 \left(M[f_n] - f_n\right)(t,x,v)\,dx\,dv = 0,$$

the left-hand side can be computed by

$$\int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \frac{\partial f_n^\sharp}{\partial t} \left|V_n^t(x,v)\right|^2 dx\,dv = \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \frac{\partial}{\partial t}\left(f_n^\sharp \left|V_n^t(x,v)\right|^2\right) dx\,dv - 2 \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \left(E_n\left(t, X_n^t, V_n^t\right) \cdot V_n^t(x,v)\right) f_n^\sharp\,dx\,dv.$$

Therefore, we get

$$\begin{aligned}
\frac{\partial}{\partial t} \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} |v|^2 f_n(t,x,v)\,dx\,dv &= \frac{\partial}{\partial t} \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \left|V_n^t(x,v)\right|^2 f_n^\sharp\,dx\,dv \\
&\leqslant 2 \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \left|E_n\left(t, X_n^t, V_n^t\right) \cdot V_n^t(x,v)\right| f_n^\sharp\,dx\,dv \\
&\leqslant \left\|E_n(t)\right\|_\infty \left(\int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} f_n^\sharp\,dx\,dv + \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \left|V_n^t(x,v)\right|^2 f_n^\sharp\,dx\,dv \right) \\
&\leqslant \left\|E_n(t)\right\|_\infty \left(\int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} f_n(t,x,v)\,dx\,dv + \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} |v|^2 f_n(t,x,v)\,dx\,dv \right) \\
&\leqslant \left\|E_n(t)\right\|_\infty \left(C(f_0) + \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} |v|^2 f_n(t,x,v)\,dx\,dv \right).
\end{aligned}$$

Applying Gronwall type estimates, we have

$$\int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} |v|^2 f_n(t,x,v)\,dx\,dv \leqslant e^{\int_0^T \|E_n(t)\|_n\,dt} \left(C(f_0) + \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} |v|^2 f_0^n(x,v)\,dx\,dv \right) \leqslant C(T, f_0, E_T). \tag{4.12}$$

Analogously,

$$\begin{aligned}
\frac{\partial}{\partial t} \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} |x|^2 f_n(t,x,v)\,dx\,dv &= \frac{\partial}{\partial t} \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \left|X_n^t(x,v)\right|^2 f_n^\sharp\,dx\,dv \\
&\leqslant 2 \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \left|V_n^t(x,v) \cdot X_n^t(x,v)\right| f_n^\sharp\,dx\,dv
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int\limits_{\mathbb{R}^3_x\times\mathbb{R}^3_v} \left|X^t_n(x,v)\right|^2 f^\sharp_n\,dx\,dv + \int\limits_{\mathbb{R}^3_x\times\mathbb{R}^3_v} \left|V^t_n(x,v)\right|^2 f^\sharp_n\,dx\,dv\\
&\leqslant \int\limits_{\mathbb{R}^3_x\times\mathbb{R}^3_v} |x|^2 f_n\,dx\,dv + \int\limits_{\mathbb{R}^3_x\times\mathbb{R}^3_v} |v|^2 f_n\,dx\,dv\\
&\leqslant C(T,f_0,E_T) + \int\limits_{\mathbb{R}^3_x\times\mathbb{R}^3_v} |x|^2 f_n(t,x,v)\,dx\,dv.
\end{aligned}
$$

Then

$$
\int\limits_{\mathbb{R}^3_x\times\mathbb{R}^3_v} |x|^2 f_n(t,x,v)\,dx\,dv \leqslant C(T,f_0,E_T). \tag{4.13}
$$

Thanks to Lemma 4.1 and Lemma 4.2, we obtain

$$
\int\limits_{\mathbb{R}^3_x\times\mathbb{R}^3_v} f_n \log f_n(t,x,v)\,dx\,dv \leqslant \int\limits_{\mathbb{R}^3_x\times\mathbb{R}^3_v} f^n_0 \log f^n_0(x,v)\,dx\,dv \leqslant C_0.
$$

On the other hand, due to the above inequalities (4.12) and (4.13), we have

$$
\begin{aligned}
&-\int\limits_{f_n\leqslant 1} f_n \log f_n(t,x,v)\,dx\,dv\\
&\quad= -\int\limits_{\exp(-(|x|^2+|v|^2))f_n\leqslant 1} f_n \log f_n(t,x,v)\,dx\,dv - \int\limits_{f_n\leqslant \exp(-(|x|^2+|v|^2))} f_n \log f_n(t,x,v)\,dx\,dv\\
&\quad\leqslant \int\limits_{\mathbb{R}^3_x\times\mathbb{R}^3_v} \big(|x|^2+|v|^2\big) f_n(t,x,v)\,dx\,dv + \int\limits_{f_n\leqslant \exp(-(|x|^2+|v|^2))} f_n^{1/2}(t,x,v)\,dx\,dv\\
&\quad\leqslant C(T,f_0,E_T).
\end{aligned}
$$

Here, we use the inequality $f^{1/2}\log f^{-1} < 1$ for $f < 1$. Therefore,

$$
\int\limits_{\mathbb{R}^3_x\times\mathbb{R}^3_v} f_n|\log f_n|(t,x,v)\,dx\,dv = \int\limits_{\mathbb{R}^3_x\times\mathbb{R}^3_v} f_n \log f_n(t,x,v)\,dx\,dv - \int\limits_{f_n\leqslant 1} f_n \log f_n(t,x,v)\,dx\,dv \leqslant C(T,f_0,E_T).
$$

Then the proof of Lemma 4.3 is completed. □

Proof of Theorem 1.1. It follows from Lemma 4.3 and the Dunford–Pettis theorem [11] that $\{f_n(t,x,v)\colon\ n=1,2,\dots\}$ is weakly compact in $L^1((0,T)\times\mathbb{R}^3_x\times\mathbb{R}^3_v)$ for any $T>0$. Hence, there exists a nonnegative function $f(t,x,v)\in L^1((0,T)\times\mathbb{R}^3_x\times\mathbb{R}^3_v)$ such that

$$
f_n(t,x,v) \rightharpoonup f(t,x,v) \quad (n\to\infty)
$$

in $L^1((0,T)\times\mathbb{R}^3_x\times\mathbb{R}^3_v)$ (choosing a subsequence if necessary), we will show that the weak limit f is the distribution solution of the BGK equation. Here and below in this paper, we use the symbols $\rightharpoonup$, $\stackrel{*}{\rightharpoonup}$ and $\to$ to represent the weak convergence, weak* convergence, and strong convergence.

For any $t\geqslant 0$, denote

$$
\big(\rho_n,\rho_n u_n,\rho_n|u_n|^2+3\rho_n\theta_n\big)(t,x) = \int\limits_{\mathbb{R}^3_v} \big(1,v,|v|^2\big) f_n(t,x,v)\,dv, \tag{4.14}
$$

$$
\big(\rho,\rho u,\rho|u|^2+3\rho\theta\big)(t,x) = \int\limits_{\mathbb{R}^3_v} \big(1,v,|v|^2\big) f(t,x,v)\,dv. \tag{4.15}
$$

By Lemmas 2.4, 2.5 and the standard procedure developed in [20], we can show that for any $T>0$ and $K_x\Subset\mathbb{R}^3_x$,

$$\begin{cases} \rho_n \to \rho, & \text{in } L^1\big((0,T)\times\mathbb{R}_x^3\big), \\ \rho_n u_n \to \rho u, & \text{in } L^1\big((0,T)\times\mathbb{R}_x^3\big), \\ \rho_n|u_n|^2 + 3\rho_n\theta_n \to \rho|u|^2 + 3\rho\theta, & \text{in } L^1\big((0,T)\times K_x\big). \end{cases} \tag{4.16}$$

Hence, we have for almost all $(t,x)\in(0,T)\times\mathbb{R}_x^3$ (choosing a subsequence if necessary)

$$\lim_{n\to\infty}\big(\rho_n, \rho_n u_n, \rho_n|u_n|^2 + 3\rho_n\theta_n\big) = \big(\rho, \rho u, \rho|u|^2 + 3\rho\theta\big)(t,x).$$

Let $G = \{(t,x)\in(0,T)\times\mathbb{R}_x^3\colon\ \rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv \neq 0\}$, then we have

$$\lim_{n\to\infty}(\rho_n, u_n, \theta_n)(t,x) = (\rho, u, \theta)(t,x), \quad \theta(t,x) > 0$$

for almost all $(t,x)\in G$. Consequently,

$$M[f_n](t,x,v) \to M[f](t,x,v) \quad \text{in } G\times\mathbb{R}_v^3, \text{ a.e.} \tag{4.17}$$

On the other hand, it is easy to prove that

$$\int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \big(1 + |x|^2 + |v|^2 + \big|\log M[f_n]\big|\big) M[f_n](t,x,v)\,dx\,dv \leqslant C(T, f_0, E_T), \quad \forall t \leqslant T. \tag{4.18}$$

This implies that $\{M[f_n]\colon n = 1,2,\ldots\}$ is weakly compact in $L^1((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$. By the Dunford–Pettis theorem, we have

$$M[f_n](t,x,v) \rightharpoonup m(t,x,v) \tag{4.19}$$

in $L^1((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$ (choosing subsequence if necessary), where $m(t,x,v)\in L^1((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$. Then, $\rho_n = \int_{\mathbb{R}^3} M[f_n]\,dv$ converges weakly to $\int_{\mathbb{R}^3} m\,dv$ in $L^1((0,T)\times\mathbb{R}_x^3)$. Noting that ρ_n converges strongly to ρ in $L^1((0,T)\times\mathbb{R}_x^3)$, we obtain

$$\int\limits_{\mathbb{R}^3} M[f_n]\,dv = \rho_n \to \int\limits_{\mathbb{R}^3} m\,dv = \rho \quad \text{in } L^1\big((0,T)\times\mathbb{R}_x^3\big).$$

Since $\rho(t,x) = 0$ for almost all $(t,x)\in G^c$, we get $\int_{\mathbb{R}^3} M[f_n]\,dv \to 0$ in $L^1(G^c)$. So, $M[f_n]\to 0$ in $L^1(G^c\times\mathbb{R}_v^3)$. This implies that (choosing a subsequence if necessary)

$$M[f_n] \to 0 \quad \text{in } G^c\times\mathbb{R}_v^3, \text{ a.e.} \tag{4.20}$$

By (4.17) and (4.20), we obtain

$$M[f_n] \to M[f] \quad \text{in } (0,T)\times\mathbb{R}_v^3\times\mathbb{R}_x^3, \text{ a.e.} \tag{4.21}$$

It follows from (4.19) and (4.21) that $m(t,x,v) = M[f](t,x,v)$ and $M[f_n](t,x,v) \to M[f](t,x,v)$ in $L^1((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$.

Now, by passing to the limit in the BGK equation (4.5)–(4.7), we know that $f(t,x,v)$ is a solution to (1.1)–(1.3) in the distributional sense. □

5. Proof of Theorem 1.2

In this section, we will prove the existence theorem of the L^p solutions and establish the propagation properties of some L^p moments. Let f_0 satisfies the assumption of Theorem 1.2. Following the method in [28], we cutoff the initial datum f_0 and obtain a sequence f_0^n of the cutoff initial data as follows:

$$f_0^n = \mu_n(x,v)\max\big\{f_0(x,v), n\big\} + \frac{\exp(-|v|^2)}{n(1+|x|^k)} \geqslant \frac{\exp(-|v|^2)}{n(1+|x|^k)}, \tag{5.1}$$

where μ_n is any cutoff function such that $0 \leqslant \mu_n \leqslant 1$ and $\mu_n = 0$ for $|x|^2 + |v|^2 \geqslant n^2$, furthermore, $\lim_{n\to\infty}\mu_n(x,v) = 1$ and

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \big(1 + |v|^2\big)\big|f_0^n - f_0\big|(x,v)\,dx\,dv = 0, \qquad \lim_{n\to\infty}\big\|f_0^n - f_0\big\|_{L^p} = 0. \tag{5.2}$$

We also assume that $E_n(t,x,v) = E(t,x,v) * J_n$, where $J_n(t,x,v)$ are mollifying functions. Then for $n = 1,2,\ldots,$ f_0^n and E_n satisfy all assumptions of Theorem 3.1. Hence, the approximate BGK equation

$$\begin{cases} \dfrac{\partial f_n}{\partial t} + v\cdot\nabla_x f_n + E_n(t,x,v)\cdot\nabla_v f_n = M[f_n] - f_n, \\ f_n(0,x,v) = f_0^n(x,v) \end{cases} \tag{5.3}$$

has a unique global positive solution $f_n(t,x,v)$. From Lemma 4.3,

$$\int\limits_{\mathbb{R}_x^3\times\mathbb{R}_v^3} \big(1+|x|^2+|v|^2\big) f_n(t,x,v)\,dx\,dv \leqslant C(T,f_0,E_T), \quad \forall t\leqslant T. \tag{5.4}$$

Proof of Theorem 1.2. Firstly, we will prove that

$$\big\|f_n(t)\big\|_p \leqslant C(T,f_0,p), \quad \forall t\leqslant T. \tag{5.5}$$

In fact, we notice that the mild solution form of the BGK equation can be written as

$$f_n(t,x,v) = f_0^n\big(X_n(0),V_n(0)\big)e^{-t} + \int\limits_0^t e^{(s-t)} M[f_n]\big(s,X_n(s),V_n(s)\big)\,ds. \tag{5.6}$$

Taking L^p norms on both sides of (5.6), we have

$$\big\|f_n(t)\big\|_p \leqslant \exp(-t)\big\|f_0^n\big\|_p + \int\limits_0^t \exp(s-t)\big\|M[f_n](s)\big\|_p\,ds,$$

where we use that the mapping $(x,v)\to(X_n(s),V_n(s))$ preserves the measure for any $s,t\geqslant 0$. Using the L^p estimate $\|M[f_n](s)\|_p\leqslant C(p)\|f_n(s)\|_p$ by Lemma 2.2, we further obtain

$$\big\|f_n(t)\big\|_p \leqslant \exp(-t)\big\|f_0^n\big\|_p + C(p)\int\limits_0^t \exp(s-t)\big\|f_n(s)\big\|_p\,ds.$$

Then, the Gronwall Lemma implies that

$$\big\|f_n(t)\big\|_p \leqslant \exp\big(\big(C(p)-1\big)t\big)\big\|f_0^n\big\|_p. \tag{5.7}$$

On the other hand, from the definition of f_0^n, we have

$$\big\|f_0^n\big\|_p \leqslant C(p)\big(\|f_0\|_p+1\big). \tag{5.8}$$

Thus, combining (5.7) and (5.8), we get the estimate of (5.5).

Following from the estimates of (5.4) and (5.5), we can obtain some important conclusions. Firstly, for any $T>0$, if $1<p<\infty$, $L^p((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$ is reflexive, (5.5) obviously implies that $\{f_n(t,x,v)\colon n=1,2,\dots\}$ is weakly compact in $L^p((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$; if $p=\infty$, it follows from the Banach–Alauglu theorem [11] and (5.5) that $\{f_n(t,x,v)\colon n=1,2,\dots\}$ is relatively compact in $L^\infty((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$ in the weak* topology. Hence, there exists a nonnegative function $f(t,x,v)\in L^p((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$ such that for $1<p<\infty$,

$$f_n(t,x,v)\rightharpoonup f(t,x,v), \quad \text{in } L^p\big((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3\big),$$

and for $p=\infty$,

$$f_n(t,x,v)\rightharpoonup f(t,x,v), \quad \text{in } L^\infty\big((0,T)\times\mathbb{R}_x^3\times\mathbb{R}_v^3\big).$$

Secondly, it follows from (5.4) and (5.5) that for any given $K_x\Subset\mathbb{R}_x^3$, $\{f_n(t,x,v)\colon n=1,2,\dots\}$ is weakly compact in $L^1((0,T)\times K_x\times\mathbb{R}_v^3)$. Then we can obtain that for any $K_x\Subset\mathbb{R}_x^3$

$$\rho_n(t,x)\big|u_n(t,x)\big|^2+3\rho_n(t,x)\theta_n(t,x)\rightharpoonup\rho(t,x)\big|u(t,x)\big|^2+3\rho(t,x)\theta(t,x)$$

in $L^1((0,T)\times K_x)$.

On the other hand, by Lemma 2.2 and (5.5), we get

$$\big\|M[f_n](t)\big\|_p \leqslant C(p)\big\|f_n(t)\big\|_p \leqslant C(p,T,f_0), \quad t\in[0,T]. \tag{5.9}$$

Then, Lemma 4.3 and the inequalities (5.9) imply that for any $K_x\Subset\mathbb{R}_x^3$, $\{M[f_n](t,x,v)\colon n=1,2,\dots\}$ is weakly compact in $L^1((0,T)\times K_x\times\mathbb{R}_v^3)$. By Corollary 2.1, we obtain that for any $\psi(t,x,v)\in L^\infty([0,T]\times\mathbb{R}_x^3\times\mathbb{R}_v^3)$, the sequence $\int_{\mathbb{R}_v^3} f_n(t,x,v)\psi(t,x,v)\,dv$ is compact in $L^1((0,T)\times K_x)$. Furthermore, we have that for any $K_x\Subset\mathbb{R}_x^3$

$$\rho_n(t,x)\big|u_n(t,x)\big|^2+3\rho_n(t,x)\theta_n(t,x)\to\rho(t,x)\big|u(t,x)\big|^2+3\rho(t,x)\theta(t,x)$$

in $L^1((0,T)\times K_x)$. Similar to Section 4, we can get

$$M[f_n]\to M[f]\quad \text{in } L^1\big((0,T)\times K_x\times\mathbb{R}^3\big).$$

Passing to the limit in the BGK equation (5.3) for $n\to\infty$, we know that $f(t,x,v)$ is a distributional solution to (1.1)–(1.3).

The inequalities (1.8) and (1.9) are obvious. In fact, by the definition of mild solution of BGK equation and inequalities (5.4) and (5.5) for any $t_1,t_2\in[0,T]$ and $1\leqslant p\leqslant\infty$,

$$\big\|f_n(t_1,x,v)-f_n(t_2,x,v)\big\|_p\leqslant C(T,f_0,E_T)|t_1-t_2|,\quad n=1,2,\dots \tag{5.10}$$

then we have

$$f_n(t,x,v)\rightharpoonup(\text{or }\rightharpoonup)f(t,x,v)\quad \text{in } L^p\big(\mathbb{R}_x^3\times\mathbb{R}_v^3\big),\ \forall t\geqslant 0. \tag{5.11}$$

By (5.11), (5.4) and (5.5), we get

$$\int_{\mathbb{R}_x^3}\int_{|v|\leqslant R}\big(1+|v|^2\big)f(t,x,v)\,dx\,dv=\lim_{n\to\infty}\int_{\mathbb{R}_x^3}\int_{|v|\leqslant R}\big(1+|v|^2\big)f_n(t,x,v)\,dx\,dv\leqslant C(T,f_0,E_T),$$

and for any $g(x,v)\in L^{p'}(\mathbb{R}_x^3\times\mathbb{R}_v^3)$ $(1/p+1/p'=1)$,

$$\int_{\mathbb{R}_x^3\times\mathbb{R}_v^3}f(t,x,v)g(x,v)\,dx\,dv=\lim_{n\to\infty}\int_{\mathbb{R}_x^3\times\mathbb{R}_v^3}f_n(t,x,v)g(x,v)\,dx\,dv\leqslant\lim_{n\to\infty}\big\|f_n(t)\big\|_p\|g\|_{p'}\leqslant C(T,p,f_0,E_T)\|g\|_{p'}.$$

Thus,

$$\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}_x^3\times\mathbb{R}_v^3}\big(1+|v|^2\big)f(t,x,v)\,dx\,dv\leqslant C(T,f_0,E_T),$$
$$\sup_{0\leqslant t\leqslant T}\big\|f(t)\big\|_p\leqslant C(T,p,f_0,E_T).$$

Now, we will prove the inequalities (1.11). Firstly, we establish the weighted L^p estimates of the approximate solutions f_n. By the mild form (2.2) of the BGK equation, we have

$$f_n(t,x,v)=f_0^n\big(X_n(0),V_n(0)\big)e^{-t}+\int_0^t e^{(s-t)}M[f_n]\big(s,X_n(s),V_n(s)\big)\,ds. \tag{5.12}$$

Multiplying both sides of (5.12) by $|v|^q$ and taking L^p norms in $\mathbb{R}_x^3\times\mathbb{R}_v^3$, we obtain

$$\big\||v|^q f_n(t)\big\|_p\leqslant\big\||v|^q f_0^n\big(X_n(0),V_n(0)\big)\big\|_p+\int_0^t\big\||v|^q M[f_n]\big(s,X_n(s),V_n(s)\big)\big\|_p\,ds.$$

It follows from Lemma 2.1 and Ref. [28] ($\|(|v|^q+|x|^q)f_0^n(x,v)\|_p\leqslant C(p,q,f_0,E_T)$) that

$$\begin{aligned}\big\||v|^q f_0^n\big(X_n(0),V_n(0)\big)\big\|_p&\leqslant\big\|\big(\big|V_n(0)\big|+E_T\big)^q f_0^n\big(X_n(0),V_n(0)\big)\big\|_p\\&\leqslant C(q,E_T)\big\|\big(\big|V_n(0)\big|+1\big)^q f_0^n\big(X_n(0),V_n(0)\big)\big\|_p\\&=C(q,E_T)\big\|\big(|v|+1\big)^q f_0^n(x,v)\big\|_p\leqslant C(p,q,f_0,E_T),\end{aligned}$$

and it follows from Lemma 2.1 and Lemma 2.2 that

$$\begin{aligned}&\int_0^t\big\||v|^q M[f_n]\big(s,X_n(s),V_n(s)\big)\big\|_p\,ds\\&\quad\leqslant\int_0^t\big\|\big(\big|V_n(s)\big|+E_T\big)^q M[f_n]\big(s,X_n(s),V_n(s)\big)\big\|_p\,ds\end{aligned}$$

$$\leqslant C(q, E_T)\int_0^t \left\|\left(\left|V_n(s)\right|+1\right)^q M[f_n]\left(s, X_n(s), V_n(s)\right)\right\|_p ds$$
$$= C(q, E_T)\int_0^t \left\|\left(|v|+1\right)^q M[f_n](s)\right\|_p ds \leqslant C(p,q,T,E_T)\int_0^t \left\|\left(|v|+1\right)^q f_n(s)\right\|_p ds.$$

Combining the above two equalities and using the Gronwall Lemma, we have

$$\left\||v|^q f_n(t)\right\|_p \leqslant C(p,q,T,f_0,E_T). \tag{5.13}$$

Similarly, multiplying both sides of (5.12) by $|x|^q$ and taking L^p norms in $\mathbb{R}^3_x \times \mathbb{R}^3_v$, we obtain

$$\left\||x|^q f_n(t)\right\|_p \leqslant \left\||x|^q f_0^n\left(X_n(0), V_n(0)\right)\right\|_p + \int_0^t \left\||x|^q M[f_n]\left(s, X_n(s), V_n(s)\right)\right\|_p ds.$$

It follows from Lemma 2.1 and Ref. [28] that for $t \in [0,T]$

$$\begin{aligned}\left\||x|^q f_0^n\left(X_n(0), V_n(0)\right)\right\|_p &\leqslant \left\|\left(\left|X(0)\right| + tE_T + t|v|\right)^q f_0^n\left(X_n(0), V_n(0)\right)\right\|_p \\ &\leqslant C(q,T,E_T)\left\|\left(\left(\left|V_n(0)\right|+1\right)^q + \left|X_n(0)\right|^q\right) f_0^n\left(X_n(0), V_n(0)\right)\right\|_p \\ &= C(q,T,E_T)\left\|\left(\left(|v|+1\right)^q + |x|^q\right) f_0^n(x,v)\right\|_p \leqslant C(p,q,T,f_0,E_T),\end{aligned}$$

and

$$\begin{aligned}\int_0^t \left\||x|^q M[f_n]\left(s, X_n(s), V_n(s)\right)\right\|_p ds &\leqslant \int_0^t \left\|\left(\left|X_n(s)\right| + |t-s|E_T + \left|(t-s)v\right|\right)^q M[f_n]\left(s, X_n(s), V_n(s)\right)\right\|_p ds \\ &\leqslant C(q,T,E_T)\int_0^t \left\|\left(\left(\left|X_n(s)\right|+1\right)^q + \left|V_n(s)\right|^q\right) M[f_n]\left(s, X_n(s), V_n(s)\right)\right\|_p ds \\ &= C(q,T,E_T)\int_0^t \left\|\left(\left(|x|+1\right)^q + |v|^q\right) M[f_n](s)\right\|_p ds \\ &\leqslant C(p,q,T,E_T)\int_0^t \left\|\left(\left(|x|+1\right)^q + |v|^q\right) f_n(s)\right\|_p ds.\end{aligned}$$

Combining the above two equalities and (5.13), we have

$$\left\||x|^q f_n(t)\right\|_p \leqslant C(p,q,T,f_0,E_T). \tag{5.14}$$

Hence,

$$\left\|\left(|x|^q + |v|^q\right) f_n(t)\right\|_p \leqslant C(p,q,T,f_0,E_T). \tag{5.15}$$

The proof of $\|(|x|^q + |v|^q) f(t)\|_p \leqslant C(p,q,T,f_0,E_T)$ is the same to the proof of $\|f(t)\|_p \leqslant C(p,T,f_0,E_T)$. Thus, the proof of Theorem 1.2 is completed. □

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References

[1] K. Asano, Local solutions to the initial and initial boundary value problem for the Boltzmann equation with an external force, J. Math. Kyoto Univ. 24 (2) (1984) 225–238.
[2] N. Bellomo, M. Lachowicz, A. Palczewski, G. Toscani, On the initial value problem for the Boltzmann equation with a force term, Transport Theory Statist. Phys. 18 (1) (1989) 87–102.

[3] A. Bellouquid, Existence global et comportement asymptotique du problème de Cauchy pour le modèle de BGK, C. R. Acad. Sci. Sér. I: Math. 321 (1995) 1637–1640.
[4] P.L. Bhatnagar, E.P. Gross, M. Krook, A model for collision processes in gases, Phys. Rev. 94 (1954) 511–514.
[5] R. Bosi, M.J. Cáceres, The BGK model with external confining potential: existence, long-time behaviour and time-periodic Maxwellian equilibria, J. Stat. Phys. 136 (2009) 297–330.
[6] C. Cercignani, The Boltzmann Equation and Its Applications, Springer, New York, 1988.
[7] R.J. Diperna, P.L. Lions, On the Cauchy problem for Boltzmann equations, Global existence and weak stability, Ann. of Math. 130 (1989) 321–366.
[8] R.J. Duan, T. Yang, C.J. Zhu, L1 and BV-type stability of the Boltzmann equation with external forces, J. Differential Equations 227 (1) (2006) 1–28.
[9] R.J. Duan, T. Yang, C.J. Zhu, Global existence to the Boltzmann equation with external forces in infinite vacuum, J. Math. Phys. 46 (2005) 053307.
[10] R.J. Duan, T. Yang, C.J. Zhu, Boltzmann equation with external force and Vlasov–Poisson–Boltzmann system in infinite vacuum, Discrete Contin. Dyn. Syst. 16 (1) (2006) 253–277.
[11] N. Dunford, J.T. Schwartz, Linear Operators, I. General Theory, Interscience, New York, 1958.
[12] A. Glikson, On the existence of general solutions of the initial-value problem for the nonlinear Boltzmann equation with a cut-off, Arch. Ration. Mech. Anal. 45 (1972) 35–46.
[13] A. Glikson, On solution of the nonlinear Boltzmann equation with a cut off in an unbounded domain, Arch. Ration. Mech. Anal. 47 (1972) 389–394.
[14] F. Golse, P.L. Lions, B. Perthame, R. Sentis, Regularity of the moments of the solution of a transport equation, J. Funct. Anal. 76 (1988) 110–125.
[15] Y. Guo, The Vlasov–Poisson–Boltzmann system near vacuum, Comm. Math. Phys. 218 (2) (2001) 293–313.
[16] Y. Guo, The Vlasov–Poisson–Boltzmann system near Maxwellians, Comm. Pure Appl. Math. 55 (9) (2002) 1104–1135.
[17] Y. Guo, The Vlasov–Maxwell–Boltzmann system near Maxwellians, Invent. Math. 153 (2003) 593–630.
[18] W. Greenberg, J. Polewczak, A global existence theorem for the nonlinear BGK equation, J. Stat. Phys. 55 (1989) 1313–1321.
[19] S. Mischler, Uniqueness for the BGK-equation in $\mathbb{R}^3$ and rate of convergence for a semi-discrete scheme, Differential Integral Equations 9 (1996) 1119–1138.
[20] B. Perthame, Global existence to the BGK model of Boltzmann equation, J. Differential Equations 82 (1989) 191–205.
[21] B. Perthame, M. Pulvirenti, Weighted L^∞ bounds and uniqueness for the Boltzmann BGK model, Arch. Ration. Mech. Anal. 125 (1993) 289–295.
[22] J. Polewczak, W. Greenberg, Some remarks about continuity properties of local Maxwellians and an existence theorem for the BGK model of the Boltzmann equation, J. Stat. Phys. 33 (1983) 307–316.
[23] S. Rejeb, On the existence of conditions of a classical solution of BGK–Poisson's equation in finite time, IAENG Int. J. Appl. Math. 39 (2009) 4.
[24] S. Ukai, T. Yang, H.J. Zhao, Global solutions to the Boltzmann equation with external forces, Anal. Appl. 3 (2) (2005) 157–193.
[25] P. Welander, On the temperature jump in a rarefied gas, Ark. Fys. 7 (1954) 507–552.
[26] S. Yun, Cauchy problem for the Boltzmann–BGK model near a global Maxwellian, J. Math. Phys. 51 (2010) 123514.
[27] X.W. Zhang, S.G. Hu, Moment propagation of the BGK equation, Math. Appl. 19 (2006) 857–862.
[28] X.W. Zhang, S.G. Hu, L^p solutions to the Cauchy problem of the BGK equation, J. Math. Phys. 48 (2007) 113304.
[29] X.W. Zhang, On the Cauchy problem of the Vlasov–Poisson–BGK system: global existence of weak solutions, J. Stat. Phys. 141 (2010) 566–588.