On Inverses and Generalized Inverses of Hessenberg Matrices

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ABSTRACT

In this paper, new algorithms for computing the inverses and the generalized inverses of a class of real Hessenberg matrices are given. The inverses and the generalized inverses that satisfy one, several or even all of the Penrose equations are shown to have a simple structure, and can be constructed from the inverses of triangular matrices of lower order and some recursive relations.

1. INTRODUCTION

A matrix $H = (h_{ij})_{n \times n}$ is called a lower (upper) Hessenberg matrix if $h_{ij} = 0$ for all pairs $(i, j)$ such that $i + 1 < j$ ($j + 1 < i$). Y. Ikebe in [2] shows that the upper half of the inverse of a lower Hessenberg matrix has a simple structure, and applies the result to find an algorithm for inverting a tridiagonal matrix. This paper presents new kinds of algorithm for computing inverses and generalized inverses of a class of real Hessenberg matrices. The inverses and the generalized inverses that satisfy one, several or even all of the Penrose equations are shown to have a simple structure, and can be constructed from the inverses of triangular matrices of lower order and some recursive relations. Since, by [1], every square matrix can be reduced to Hessenberg form by the method of Householder and Bauer, it is important to compute the inverses or the generalized inverses of Hessenberg matrices.

2. THE INVERSES OF A CLASS OF HESSENBERG MATRICES

Consider a lower Hessenberg matrix of order \( n \)

\[
H = \begin{pmatrix}
 h_{11} & \alpha_1 & 0 & 0 & \cdots & 0 \\
 h_{21} & h_{22} & \alpha_2 & 0 & \cdots & 0 \\
 & & & & \ddots & \ddots \\
 h_{n1} & h_{n2} & h_{n3} & h_{n4} & \cdots & h_{nn}
\end{pmatrix}
\]

Let the matrix \( H \) be partitioned into the form

\[
H = \begin{pmatrix}
 C_{n-1} & P_{n-1} \\
 h_{n1} & R_{n-1}^T
\end{pmatrix},
\]

where

\[
C_{n-1} = (h_{11} \ h_{21} \ \cdots \ h_{n-1,1})^T,
\]

\[
R_{n-1}^T = (h_{n2} \ h_{n3} \ \cdots \ h_{nn}),
\]

\[
P_{n-1} = \begin{pmatrix}
 \alpha_1 & 0 & 0 & \cdots & 0 \\
 h_{22} & \alpha_2 & 0 & \cdots & 0 \\
 & & & & \ddots \\
 h_{n-1,2} & h_{n-1,3} & h_{n-1,4} & \cdots & \alpha_{n-1}
\end{pmatrix}
\]

**Lemma 1.** If \( \alpha_i \neq 0 \) (\( i = 1, 2, \ldots, n - 1 \)) and

\[
x_1 = 1
\]

\[
x_i = -\frac{\sum_{k=1}^{i-1} h_{i-1,k} x_k}{\alpha_{i-1}}. \quad i = 2, 3, \ldots, n.
\] (2.1)

then \( H \) is nonsingular if and only if

\[
(h_{n1} \ h_{n2} \ \cdots \ h_{nn})(x_1 \ x_2 \ \cdots \ x_n)^T = \sum_{k=1}^{n} h_{n,k} x_k \neq 0.
\]
Proof. Let $\Delta_i$ ($i = 1, 2, \ldots, n$) be the ordinal principal minors of the matrix $H$, and let $x_1 = 1$. By induction, expanding $\Delta_{i-1}$ ($i = 2, 3, \ldots, n$) using the last row, we can find

$$\begin{align*}
x_i &= \frac{(-1)^{i-1} \Delta_{i-1}}{\prod_{j=1}^{i-1} \alpha_j}, \quad i = 2, 3, \ldots, n.
\end{align*}$$

Similarly, expanding $\det(H)$ by the last row, we have

$$\begin{align*}
\sum_{k=1}^{n} h_{n,k} x_k &= \frac{(-1)^{n-1} \det(H)}{\prod_{j=1}^{n-1} \alpha_j},
\end{align*}$$

which completes the proof.

Remark 1. Lemma 1 can be used to calculate the determinant of the matrix $H$.

Theorem 1. If $\alpha_i \neq 0$ ($i = 1, 2, \ldots, n - 1$) and the matrix $H$ is nonsingular, then

$$H^{-1} = \begin{pmatrix} 0 & 0 \\ P_{n-1} & 0 \\ \vdots & \vdots \\ h_{n-1} & 0 \end{pmatrix} + (x_1 \ x_2 \ \cdots \ x_n)^T (w_1 \ w_2 \ \cdots \ w_n), \quad (2.2)$$

where $x_i$ ($i = 1, 2, \ldots, n$) are defined by (2.1), and $w_i$ ($i = 1, 2, \ldots, n$) can be defined recursively as

$$\begin{align*}
w_n &= \frac{1}{\sum_{k=1}^{n} h_{n,k} x_k}, \\
w_i &= -\frac{\sum_{k=i+1}^{n} h_{k,i+1} w_k}{\alpha_i}, \quad i = n-1, \ldots, 1.
\end{align*}$$

Proof. Let $H^{-1} = (f_{ij})_{n \times n}$, and let the first row of $H^{-1}$ be denoted by $(w_1 \ w_2 \ \cdots \ w_n)$. Writing $HH^{-1} = I_n$ as

$$\begin{pmatrix} C_{n-1} & P_{n-1} \\ h_{n1} & R_{n-1}^T \end{pmatrix} \begin{pmatrix} w_1 & w_2 & \cdots & w_n \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $C_{n-1}$, $P_{n-1}$, $h_{n1}$, $R_{n-1}^T$, $f_{ij}$, $w_i$, and $w_n$ are defined as in the proof.
we get

\[
C^{-1}_{n-1}(w_1, w_2, \cdots, w_n) + P^{-1}_{n-1} \begin{pmatrix}
  f_{21} & f_{22} & \cdots & f_{2n} \\
  \cdots & \cdots & \cdots & \cdots \\
  f_{n1} & f_{n2} & \cdots & f_{nn}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  1 & 0 & \cdots & 0 & 0 \\
  0 & 1 & \cdots & 0 & 0 \\
  0 & 0 & \cdots & 1 & 0
\end{pmatrix}.
\]

Since \( P_{n-1} \) is nonsingular, we have

\[
\begin{pmatrix}
  f_{21} & f_{22} & \cdots & f_{2n} \\
  \cdots & \cdots & \cdots & \cdots \\
  f_{n1} & f_{n2} & \cdots & f_{nn}
\end{pmatrix} = (P_{n-1}^{-1} 0) - P_{n-1}^{-1} C^{-1}_{n-1}(w_1, w_2, \cdots, w_n).
\]

Let \( x_1 = 1 \), and let

\[
(x_2, x_3, \cdots, x_n)^T = -P_{n-1}^{-1} C_{n-1}.
\]

It is obvious that \( x_i \) (\( i = 1, 2, \ldots, n \)) can be determined by (2.1). The \( w_i \) (\( i = 1, 2, \ldots, n \)) can be determined as follows: Examining the last column in \( HH^{-1} = I_n \), we get

\[
H(x_1, x_2, \cdots, x_n)^T = (0 \cdots 0 1/w_n)^T.
\]

Thus

\[
w_n = \frac{1}{\sum_{k=1}^{n} h_{n,k} x_k}.
\]

Examining the first row in \( H^{-1}H = I_n \), we find

\[
(w_1, w_2, \cdots, w_n)H = (1 \ 0 \ \cdots \ 0).
\]

Thus

\[
w_i = -\frac{\sum_{k=i+1}^{n} h_{k,i+1} w_k}{\alpha_i}, \quad i = 1, 2, \ldots, n-1.
\]

The proof is done.
REMARK 2. The upper half of $H^{-1}$ in the Theorem 1 is the same as the result in [2].

With minor modifications, the techniques can be applied to block Hessenberg matrices. Consider a block lower Hessenberg matrix $H = (H_{ij})_{n \times n}$ ($H_{ij} = 0$ for $i + 1 < j$), where $H_{ij}$ ($i, j = 1, 2, \ldots, n$) are square matrices of a fixed order, say $m$. Suppose the superdiagonal $H_{i,i+1}$ ($i = 1, 2, \ldots, n - 1$) are nonsingular.

**Lemma 2.** Let

$$X_1 = I_m$$

$$X_i = -H_{i-1,i}^{-1} \sum_{k=1}^{i-1} H_{i-1,k}X_k, \quad i = 2, 3, \ldots, n. \quad (2.3)$$

Then the matrix $H$ is nonsingular if and only if

$$\det \left( \sum_{k=1}^{n} H_{n,k}X_k \right) \neq 0.$$

**Proof.** Let $D_i$ ($i = 1, 2, \ldots, n$) be the block ordinal principal minors of the matrix $H$. By induction, we can prove

$$D_i = (-1)^{(i-1)m + 1} \det \left( \sum_{k=1}^{i} H_{ik}X_k \right) \prod_{k=1}^{i-1} \det(H_{k,k+1}).$$

In particular,

$$\det(H) = (-1)^{(n-1)m + 1} \det \left( \sum_{k=1}^{n} H_{nk}X_k \right) \prod_{k=1}^{n-1} \det(H_{k,k+1}),$$

which completes the proof. \qed

**Theorem 2.** If $H$ is nonsingular, then

$$H^{-1} = \begin{pmatrix} 0 & 0 \\ \tilde{p}_{n-1}^{-1} & 0 \end{pmatrix} + \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}(Y_1 \cdots Y_n), \quad (2.4)$$
where

\[ \tilde{P}_{n-1} = \begin{pmatrix}
    H_{12} & 0 & \cdots & 0 \\
    H_{22} & H_{23} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    H_{n-1,2} & H_{n-1,3} & \cdots & H_{n-1,n}
\end{pmatrix}, \]

\[ X_i \ (i = 1, 2, \ldots, n) \] can be determined by (2.3), and \( Y_i \ (i = 1, 2, \ldots, n) \) are defined recursively as

\[ Y_n = \left( \sum_{k=1}^{n} H_{n,k}X_k \right)^{-1} \]

\[ Y_i = - \left( \sum_{k=i+1}^{n} Y_k H_{k,i+1} \right) H_i^{-1}, \quad i = n - 1, \ldots, 1. \quad (2.5) \]

**Proof.** Similar to the proof of Theorem 1.

3. THE GENERALIZED INVERSES OF A CLASS OF SINGULAR HESSENBERG MATRICES

We shall constantly use the following notation as in [3]. For a matrix \( H \), let \( H^{(i,j,\ldots,l)} \) denote the generalized inverse which satisfies equations \((i), (j), \ldots, (l)\) from among the Penrose equations: (1) \( HXH = H \); (2) \( XHX = X \), (3) \( (HX)^T = HX \), (4) \( (XH)^T = XH \). In particular, we denote \( H^{(1,2,3,4)} \) by \( H^+ \).

Consider the \( n \times n \) lower Hessenberg matrix

\[ H = \begin{pmatrix}
    C_{n-1} & P_{n-1} \\
    h_n & R_{n-1}^T
\end{pmatrix}, \]

where the symbols \( C_{n-1}, \ P_{n-1}, \ R_{n-1}^T \) mean the same as in Section 2. Throughout this section, we suppose that the matrix \( H \) is singular and \( \alpha_i \neq 0 \) \((i = 1, 2, \ldots, n - 1)\).

**Lemma 3.** Let

\[ y_n = 1 \]

\[ y_i = - \frac{\sum_{k=i+1}^{n} y_k h_{k,i+1}}{\alpha_i}, \quad i = n - 1, \ldots, 1. \quad (3.1) \]
Then
\[(y_1 \cdots y_n)(h_{11} \ h_{21} \cdots h_{n1})^T = \sum_{k=1}^{n} y_k h_{k1} = 0.\]

**Proof.** Similar to the proof of Lemma 1.

**Theorem 3.**

\[
H^{(1)} = \begin{pmatrix}
0 & 0 \\
p \cdot \cdots \\
0 & 0
\end{pmatrix} + (x_1 \cdots x_n)^T (f_{11} \ f_{12} \cdots f_{1n})
+ (f_{1n} \ f_{2n} \cdots f_{nn})^T (y_1 \cdots y_n)
- f_{1n} (x_1 \cdots x_n)^T (y_1 \cdots y_n),
\]

(3.2)

where \(x_i, y_i (i = 1, 2, \ldots, n)\) are defined by (2.1) and (3.1), respectively, and \(f_{11}, f_{in} (i = 1, 2, \ldots, n)\) are arbitrary.

**Proof.** Let \(H^{(1)} = (f_{ij})_{n \times n}\). Writing \(HH^{(1)}H = H\) as

\[
\begin{pmatrix}
C_{n-1} & P_{n-1} \\
h_n & R_{n-1}^T
\end{pmatrix}
\begin{pmatrix}
f_{11} & \cdots & f_{1,n-1} & f_{1n} \\
f_{21} & \cdots & f_{2,n-1} & f_{2n} \\
\vdots & \ddots & \vdots & \vdots \\
f_{n1} & \cdots & f_{n,n-1} & f_{nn}
\end{pmatrix}
\begin{pmatrix}
C_{n-1} & P_{n-1} \\
h_n & R_{n-1}^T
\end{pmatrix}
\]

we get

\[
C_{n-1}(f_{11} \cdots f_{1,n-1})P_{n-1}P_{n-1} + P_{n-1}(f_{21} \cdots f_{2,n-1})P_{n-1}
+ f_{1n}R_{n-1}^T = P_{n-1}.
\]
Since $P_{n-1}$ is nonsingular, we have

$$
\begin{pmatrix}
  f_{21} & \cdots & f_{2,n-1} \\
  \vdots & \ddots & \vdots \\
  f_{n1} & \cdots & f_{n,n-1}
\end{pmatrix} = P_{n-1}^{-1} - P_{n-1}^{-1}C_{n-1}(f_{11} \cdots f_{1,n-1})
$$

$$
- (f_{2n} \cdots f_{nn})^T R_{n-1}^T P_{n-1}^{-1}
$$

$$
- f_{1n} P_{n-1}^{-1} C_{n-1} R_{n-1}^T P_{n-1}^{-1}.
$$

Let $x_1 = 1$, $y_n = 1$, and let

$$
(x_2 \cdots x_n)^T = - P_{n-1}^{-1} C_{n-1},
$$

$$
(y_1 \cdots y_{n-1}) = - R_{n-1}^T P_{n-1}^{-1}.
$$

Then $x_i, y_i$ ($i = 1, 2, \ldots, n$) can be determined by (2.1) and (3.1), respectively. Therefore,

$$
\begin{pmatrix}
  f_{11} & \cdots & f_{1n} \\
  \vdots & \ddots & \vdots \\
  f_{n1} & \cdots & f_{nn}
\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ P_{n-1}^{-1} & 0 \end{pmatrix} + \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (f_{11} \cdots f_{1n})
$$

$$
+ \begin{pmatrix} f_{1n} \\ \vdots \\ f_{nn} \end{pmatrix} (y_1 \cdots y_n) - f_{1n} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (y_1 \cdots y_n).
$$

It is readily verified by Lemma 1 and 3 that the expression satisfies the matrix equation $HXH = H$ no matter how the $f_{1i}, f_{in}$ ($i = 1, 2, \ldots, n$) are chosen. The proof is done.

For convenience sake, let

$$
X = (x_2 \cdots x_n)^T,
$$

$$
Y = (y_1 \cdots y_{n-1})^T.
$$
It follows from the proof of Theorem 3 that

\[ C_{n-1} = -P_{n-1}X, \quad R_{n-1}^T = -Y^TP_{n-1}, \]
\[ h_{n1} = -R_{n-1}^TX = -Y^TC_{n-1} = Y^TP_{n-1}X. \]  
(3.3)

**Theorem 4.** If \( f_{i1}, f_{in} (i = 1, 2, \ldots, n) \) satisfy

\[
(f_{11} \ldots f_{in})H(f_{1n} \ldots f_{nn})^T = f_{1n},
\]  
(3.4)

then \( H^{(1)} \) defined by (3.2) satisfies the equation \( H^{(1)}H^{(1)} = H^{(1)} \), that is,

\[ H^{(1)} = H^{(1,2)}. \]

**Proof.** It follows from (2.1), (3.1), (3.2), (3.4) that

\[
H^{(1)}H^{(1)} = H^{(1)} + \left( \frac{1}{X} \right) \left[ (f_{11} \ldots f_{in})H(f_{1n} \ldots f_{nn})^T - f_{1n} \right] (Y^T - 1),
\]

which completes the proof.

**Theorem 5.** If \( f_{i1} (i = 1, 2, \ldots, n-1) \) satisfy

\[
(f_{11} \ldots f_{i-1, n-1}) = f_{in}X^{-} - \frac{X^TP_{n-1}^{-1}}{1 + X^TX},
\]  
(3.5)

and \( f_{in} (i = 1, 2, \ldots, n) \) are arbitrary, then the \( H^{(1)} \) defined by (3.2) satisfies the equation \( (H^{(1)}H)^T = H^{(1)}H \), that is,

\[ H^{(1)} = H^{(1,4)}. \]

**Proof.** By (2.1), (3.1), (3.2), (3.4), we have

\[
H^{(1)}H = \begin{pmatrix} 0 & 0 \\ -X & I_{n-1} \end{pmatrix} + \left( \frac{1}{X} \right) (f_{11} \ldots f_{in})H.
\]
Thus, it follows from the equation \((H^{(1)}H)^T = H^{(1)}H\) that

\[
\begin{pmatrix}
1 \\
X
\end{pmatrix}
(f_{11} \cdots f_{1n})
\begin{pmatrix}
C_{n-1} & P_{n-1} \\
R^T_{n-1}
\end{pmatrix}
= \begin{pmatrix}
C^T_{n-1} & h_{n1} \\
P^T_{n-1} & R_{n-1}
\end{pmatrix}
\begin{pmatrix}
f_{11} \\
\vdots \\
f_{1n}
\end{pmatrix}
\begin{pmatrix}
1 \\
X^T
\end{pmatrix}
= \begin{pmatrix}
0 \\
X
\end{pmatrix}
- X^T.
\]

(3.6)

It is easily verified that if we have

\[
(f_{11} \cdots f_{1n})
\begin{pmatrix}
P_{n-1} \\
R^T_{n-1}
\end{pmatrix}
- \begin{pmatrix}
C^T_{n-1} & h_{n1} \\
P^T_{n-1} & R_{n-1}
\end{pmatrix}
X^T = - X^T,
\]

then (3.6) holds. From (3.3) and (3.7), we get

\[
(f_{11} \cdots f_{1,n-1})P_{n-1}(I_{n-1} + XX^T) = f_{1n}Y^TP_{n-1}(I_{n-1} + XX^T) - X^T.
\]

(3.8)

The matrix \(I_{n-1} + XX^T\) is positive definite, and its inverse is, by the Sherman-Morrison formula,

\[
(I_{n-1} + XX^T)^{-1} = I_{n-1} - \frac{XX^T}{1 + X^TX}.
\]

Substituting it in (3.8) gives (3.5), completing the proof.

\[\blacksquare\]

**Theorem 6.** If \(f_{in} \ (i = 2, \ldots, n)\) satisfy

\[
(f_{2n} \cdots f_{nn})^T = f_{1n}X - \frac{P_{n-1}Y}{1 + Y^TY},
\]

(3.9)

and \(f_{1i} \ (i = 1, 2, \ldots, n)\) are arbitrary, then \(H^{(1)}\) defined by (3.2) satisfies the equation \((HH^{(1)})^T = HH^{(1)}\), that is,

\[H^{(1)} = H^{(1,3)}.
\]
Proof. Similar to the proof of Theorem 5.

**Theorem 7.** If $f_{i1}$, $f_{in}$ ($i = 1, 2, \ldots, n$) satisfy

$$
(f_{i1} \cdots f_{in-1}) = - \frac{X^TP_{n-1}^{-1}}{1 + X^TX} \left( I_{n-1} - \frac{YY^T}{1 + Y^TY} \right),
$$

$$
(f_{2n} \cdots f_{nn})^T = - \left( I_{n-1} - \frac{XX^T}{1 + X^TX} \right) \frac{P_{n-1}^{-1}Y}{1 + Y^TY},
$$

$$
f_{1n} = - (f_{i1} \cdots f_{in-1}) Y, \quad (3.10)
$$

then the $H^{(1)}$ defined by (3.2) is the Moore-Penrose inverse, that is,

$$
II^{(1)} = II^+.
$$

Proof. Follows from (3.3), (3.4), (3.5), (3.9).

As a simple example of the above results, consider a Hessenberg matrix

$$
\begin{pmatrix}
  h & 1 & & \\
  h^2 & h & 1 & \\
  h^{n-1} & h^{n-2} & h^{n-3} & \cdots & \cdots & 1 \\
  h^n & h^{n-1} & h^{n-2} & \cdots & h & \\
\end{pmatrix} \quad (n \geq 3),
$$

where $h$ is an arbitrary real number, which was presented by I. Singh, G. Poole, and T. Boullion [4] for adoption as a test matrix. Applying Lemma 1 above,

$$
x_1 = 1, \quad x_2 = -h, \quad x_3 = \cdots = x_n = 0, \quad \sum_{k=1}^{n} h_{n,k} x_k = 0;
$$

thus $H_n$ is a singular matrix. From (3.1), we have

$$
y_n = 1, \quad y_{n-1} = -h, \quad y_{n-2} = \cdots = y_1 = 0.
$$

It follows from

$$
P_{n-1} = \begin{pmatrix} 
1 & & \\
& h & 1 \\
& h^2 & h & 1 \\
& h^{n-2} & h^{n-3} & h^{n-4} & \cdots & h & 1 \\
\end{pmatrix}
$$
that

\[ p_{n-1}^{-1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -h & -h & \cdots & -h \\ & \ddots & \ddots & \ddots \\ & & -h & 1 \\ \end{pmatrix}. \]

Then, by (3.2), (3.5), (3.9), (3.10), we have

\[ H^{(1)}_n = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ -h \\ \end{pmatrix} + \begin{pmatrix} 1 \\ -h \\ \vdots \\ 0 \\ \end{pmatrix} (f_{11}, \ldots, f_{1n}) + \begin{pmatrix} f_{1n} \\ f_{2n} \\ \vdots \\ f_{nn} \end{pmatrix} (0, \ldots, 0, -h, 1) - f_{1n} (0, \ldots, 0, -h, 1), \]

where \( f_{ii}, f_{in} (i = 1, 2, \ldots, n) \) are arbitrary;

\[ H^{(1.4)}_n = \begin{pmatrix} h \\ 1 \\ \vdots \\ -h \\ \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \end{pmatrix} + \begin{pmatrix} f_{1n} \\ f_{2n} \\ \vdots \\ f_{nn} \end{pmatrix} (0, \ldots, 0, -h, 1), \]
where \( f_{in} \) \((i = 1, 2, \ldots, n)\) are arbitrary;

\[
H_{n}^{(1,3)} = \begin{pmatrix}
0 & 0 & & & & \\
1 & 0 & & & & \\
-\frac{1}{h} & 1 & 0 & & & \\
& \ddots & \ddots & \ddots & & \\
& & -\frac{1}{h} & 1 & 0 & \\
& & & -\frac{1}{h} & \frac{1}{1+h^2} & \frac{h}{1+h^2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
-\frac{1}{h} \\
0 \\
\vdots \\
0
\end{pmatrix}
(f_{11} \quad \cdots \quad f_{in}),
\]

where \( f_{ii} \) \((i = 1, 2, \ldots, n)\) are arbitrary; and

\[
H_{n}^{+} = \begin{pmatrix}
\frac{h}{1+h^2} & \frac{1}{1+h^2} & 0 & & & \\
\frac{1}{1+h^2} & 0 & & & & \\
-\frac{1}{h} & 1 & 0 & & & \\
& \ddots & \ddots & \ddots & & \\
& & -\frac{1}{h} & 1 & 0 & \\
& & & -\frac{1}{h} & \frac{1}{1+h^2} & \frac{h}{1+h^2}
\end{pmatrix}
\]

Similar results of Section 2 and 3 can be derived for an upper Hessenberg matrix.

The requirement that the superdiagonal elements be nonzero is not overly restrictive, since every lower Hessenberg matrix is a block triangular matrix where the diagonal blocks have nonzero superdiagonal elements. Thus, we can use the method of inverting a block triangular matrix and the method of this paper to compute the inverse or the generalized inverse of the Hessenberg matrix.

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