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Ordered coloring of grids and related graphs☆

Amotz Bar-Noy [a,b], Panagiotis Cheilaris [c,*], Michael Lampis [b], Valia Mitsou [b], Stathis Zachos [a,b,d]

[a] Brooklyn College, City University of New York, 2900 Bedford Avenue, Brooklyn, NY 11210, USA
[b] Graduate Center, City University of New York, 365 5th Avenue, New York, NY 10016, USA
[c] Center for Advanced Studies in Mathematics, Ben-Gurion University, Be'er Sheva 84105, Israel
[d] National Technical University of Athens, School of Electrical and Computer Engineering, Zografou 15780, Greece

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ABSTRACT

We investigate a coloring problem, called ordered coloring, in grids and some other families of grid-like graphs. Ordered coloring (also known as vertex ranking) has applications, among other areas, in efficient solving of sparse linear systems of equations and scheduling parallel assembly of products. Our main technical results improve upper and lower bounds for the ordered chromatic number of grids and related graphs.

1. Introduction

A (simple) graph is a pair $G = (V, E)$, where V (the vertex set) is a finite set and E (the edge set) is a family of two-element sets of V. A simple path of graph G is a sequence of some of the vertices of G such that consecutive vertices in the sequence are connected with an edge of G and no vertex repeats in the sequence. The notion of *ordered coloring* is defined as follows.

Definition 1. An *ordered coloring* of graph $G = (V, E)$ with k colors is a function $C : V \to \{1, \ldots, k\}$ such that for each simple path p in G the maximum color assigned to vertices of p occurs in *exactly one* vertex of p. The *ordered chromatic number* of a graph G, denoted by $\chi_o(G)$, is the minimum k for which G has an ordered coloring with k colors.

In this paper we focus on the problem of computing ordered colorings (also known as *vertex rankings*) for grids and related graphs, with as few colors as possible.

The problem of computing ordered colorings is a well-known and widely studied problem (see for example [1]) with many applications including VLSI design [2] and parallel Cholesky factorization of matrices [3]. The problem is also interesting for the Operations Research community, because it has applications in planning efficient assembly of products in manufacturing systems [4]. In general, it seems that the ordered coloring problem can model situations where interrelated tasks have to be accomplished fast in parallel (assembly from parts, parallel query optimization in databases, etc.)

In general graphs, finding the exact ordered chromatic number of a graph is NP-complete [5,6] and there is an $O(\log^2 n)$ polynomial time approximation algorithm [7], where n is the number of vertices. Since the problem is generally hard, it makes sense to study specific graph topologies and the focus of this paper is the calculation of the ordered coloring number of several grid-like families of graphs. Our main focus are grid graphs, which can be formally defined as follows.

Definition 2. An $m_1 \times m_2$ *grid*, denoted by G_{m_1,m_2}, is a graph with vertex set $\{0, \ldots, m_1 - 1\} \times \{0, \ldots, m_2 - 1\}$ and edge set $\left\{\{(x_1, y_1), (x_2, y_2)\} \mid |x_1 - x_2| + |y_1 - y_2| \le 1\right\}$.

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* Corresponding author.
E-mail addresses: amotz@sci.brooklyn.cuny.edu (A. Bar-Noy), panagiot@math.bgu.ac.il (P. Cheilaris), mlampis@gc.cuny.edu (M. Lampis), vmitsou@cs.gc.cuny.edu (V. Mitsou), zachos@cs.ntua.gr (S. Zachos).

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When $m_1 = m_2 = m$, we have a *square* grid, denoted by G_m. In a *standard drawing* of the grid graph, vertex (x, y) is drawn at point (x, y) in the plane; vertices with the same x are said to be in the same *column* and ones with the same y to be in the same *row*. A *chain* (or path) graph with n vertices is denoted by P_n and a *ring* (or cycle) graph with n vertices is denoted by C_n. The grid G_{m_1,m_2} can also be defined as the *Cartesian product* of two paths $P_{m_1} \times P_{m_2}$.

It is known from [1] that for general planar graphs the ordered chromatic number is $O(\sqrt{n})$. Grid graphs are planar and therefore the $O(\sqrt{n})$ bound applies. One might expect that, since the graph families we study have a relatively simple and regular structure, it should be easy to calculate their ordered chromatic numbers. This is why it is rather striking that, even though it is not hard to show upper and lower bounds that are only a small constant multiplicative factor apart, the *exact* value of these ordered chromatic numbers is not known. The main contribution of this paper is to further improve on these upper and lower bounds and to the best of our knowledge this is the first such attempt.

Paper organization. In the rest of this section we provide the necessary definitions and some preliminary known results that will prove useful in the remainder. In Section 2 we present our results improving the known upper bounds on the ordered chromatic number of grids, tori and related graphs, while in Section 3 we show lower bounds for square grids and tori. Some discussion and open problems are presented in Section 4.

1.1. Preliminaries

First, let us remark that Definition 1 is not the typical definition of ordered coloring found in the literature. Instead, the following definition is more standard, but it is not hard to show that Definitions 1 and 3 are equivalent (see for example [1]).

Definition 3. An *ordered* k-coloring of a graph $G = (V, E)$ is a function $C: V \to \{1, \ldots, k\}$ such that for every pair of distinct vertices v, v', and every path p from v to v', if $C(v) = C(v')$, there is an internal vertex v'' of p such that $C(v) < C(v'')$.

We note that for the complete graph K_n, we have $\chi_o(K_n) = n$ and for every non-complete graph G on n vertices, we have $\chi_o(G) < n$. Moreover, it is easy to prove that the ordered chromatic number is monotone under taking subgraphs, i.e., if $X \subseteq Y$, then $\chi_o(X) \le \chi_o(Y)$.

A standard concept in graph theory, that will prove useful in the remainder (especially for proving lower bounds), is that of a graph minor.

Definition 4. A graph X is a *minor* of Y, denoted as $X \preccurlyeq Y$, if X can be obtained from Y by a sequence of the following operations: vertex deletion, edge deletion, and edge contraction. Edge contraction is the process of merging both endpoints of an edge into a new vertex, which is connected to all vertices adjacent to the two endpoints.

It is known that the ordered chromatic number is monotone with respect to minors (see for example [8], Lemma 4.3).

Proposition 5. *If* $X \preccurlyeq Y$, *then* $\chi_o(X) \le \chi_o(Y)$.

A similar concept, which is also standard in graph theory, is that of a topological minor.

Definition 6. A graph X is a *topological minor* of Y if X can be obtained from Y by a sequence of the following operations: vertex deletion, edge deletion, and vertex smoothing. Vertex smoothing is the process of deleting a vertex v with exactly two adjacent vertices u and w, deleting its incident edges vu and vw, and adding the edge uw (if it does not exist).

It is known that a topological minor is also a minor (but not necessarily vice versa).

If v is a vertex of graph $G = (V, E)$, we denote by $G - v$ the graph resulting from deleting vertex v from G (and all edges incident to v). We use the notation $G - S$ to denote the deletion of all vertices in set $S \subseteq V$ from G.

Definition 7. A subset $S \subseteq V$ is a *separator* of a connected graph $G = (V, E)$ if $G - S$ is disconnected or empty. A separator S is *inclusion minimal* if no strict subset $S' \subset S$ is a separator.

In the rest of this section we show some simple upper and lower bounds on $\chi_o(G_m)$.

For an ordered coloring of the $m \times m$ grid, G_m, we can use a recursive method of coloring. We use m unique colors for the middle row (these are the highest colors we are going to use) and then at most $m/2$ unique colors for the middle column (we use the same set of at most $m/2$ unique colors above and under the middle row and these are the highest colors we are going to use except for the m colors of the middle row). The grid minus the middle row and the middle column consists of four equal grids of (almost) half side length that we recursively color using exactly the same colors for each of the four grids. This method requires at most $3m$ colors, i.e., $\chi_o(G_m) \le 3m$. However, this coloring remains ordered even if we add two edges in every internal face of the standard drawing of G_m. This indicates that $3m$ is not optimal and in fact, in Section 2, we improve the above upper bound.

There is also a lower bound of $\chi_o(G_m) \ge m$ from [1]. Another proof (in [7]) is immediate from the fact that the *treewidth* and *pathwidth* of a graph G are at most the *minimum elimination tree height* (see [3] for the definition) of G. In the following we provide a new proof, based on a minor argument, that gives a glimpse of our lower bound method used in a following section.

Proposition 8. *For* $m \ge 1$, $\chi_o(G_m) \ge m$.

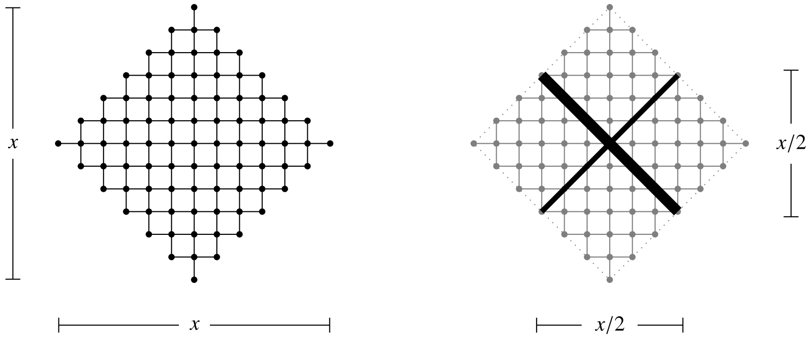

Fig. 1. The rhombus subgraph R_x and its separation.

Table 1
Summary of upper bounds. The last column indicates on which upper bounds each result is based.

Graph	Upper bound	Based on
G_m	$2.519\,m$	R_m, O_m
R_m	$1.500\,m$	–
T_m	$1.118\,m$	R_m
O_m	$1.618\,m$	T_m
$\widehat{G}_m$	$3.500\,m$	R_m

Proof. By induction. Base: For $m = 1$, it is true, as $\chi_o(G_1) = 1$. For the inductive step, with $m > 1$, consider a Hamilton path p of G_m, which always exists. Consider an optimal ordered coloring C of G_m, i.e., one using $\chi_o(G_m)$ colors. In this optimal coloring, there is a vertex v with a uniquely occurring color in p (and thus in the whole G_m). If we restrict coloring C to graph $G_m - v$, we get a coloring using one color less than the coloring for G_m, which is ordered for $G_m - v$. Therefore, $\chi_o(G_m - v) \leq \chi_o(G_m) - 1$. Moreover, $G_m - v$ contains G_{m-1} as a topological minor (and thus also as a minor), because if $v = (x, y)$, one can start with $G_m - v$, and then remove all edges completely in row y, all edges completely in column x, smooth all vertices of row y, and smooth all vertices of column x, so that the resulting graph is a G_{m-1}. Thus, from Proposition 5, $\chi_o(G_m - v) \geq \chi_o(G_{m-1})$ and since $\chi_o(G_m) \geq \chi_o(G_m - v) + 1$, we get $\chi_o(G_m) \geq \chi_o(G_{m-1}) + 1$ and from the inductive hypothesis, $\chi_o(G_m) \geq (m - 1) + 1 = m$. □

In Section 3, we improve the above lower bound. Our improvements are also relying on minor arguments like in the proof above, but involve a careful analysis of separators in grid graphs.

2. Upper bounds

In this section we exhibit ordered colorings for several grid-like families of graphs. We are mainly interested in the $m \times m$ (square) grid, G_m. In order to color the grid efficiently we rely on separators whose removal leaves some subgraphs of the grid to be colored. The subgraphs we will rely on are the rhombus R_x, the wide-side triangle T_x, and the orthogonal triangle O_x. These are depicted in Figs. 1, 2, and 3 and formal definitions similar to Definition 2 are not hard to infer. Another graph topology we will investigate is the *torus*, which is a variation of the grid with wraparound edges added, connecting the last vertex of every row with the first vertex, and the last vertex of every column with the first. The square torus graph $\widehat{G}_m$ can also be defined as the Cartesian product of two cycles $C_m \times C_m$. A summary of our upper bound results can be seen on Table 1. It is interesting that the golden ratio $\phi \approx 1.618$ appears in some of these bounds.

As was evident in the previous section, one strategy for constructing an ordered coloring of a graph is to attempt to find a separator, that is, a set of vertices whose removal disconnects the graph. The vertices of the separator are all assigned distinct colors that will be the maximum colors used in the graph coloring. This way, we can recursively construct a coloring for the connected components that remain after the removal of the separator, such that the same colors are used in each connected component, because paths in the original graph with endvertices in different connected components have a unique maximum color in some vertex of the separator. The problem is, then, to find a separator that has few vertices and the maximum ordered chromatic number over all connected components that remain after removal of the separator is as low as possible.

In the proofs we give below, we partition the graphs with the help of separators. All results are in the order of m, so without further mention we do not include terms logarithmic on m. These terms might be introduced by constant additive terms in a recursive bound. We are also omitting, in most cases, floors and ceilings, because we are interested in asymptotic behavior. In that sense, a result like, for example, $\chi_o(G_m) \leq 2.67m$ should be read as an asymptotic upper bound of $2.67m \pm o(m)$.

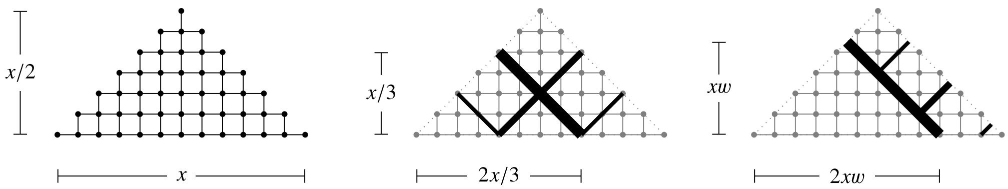

Fig. 2. The wide-side triangle T_x and its separations.

In order to find improved upper bounds we need to find more intricate separators than those of the example of the previous section. The idea is to use separators along diagonals in a standard drawing of the grid. We will also need to find efficient colorings of some subgraphs that are left after we remove diagonal-like separators. This is the reason why we first present efficient colorings for the rhombus and the two triangle subgraphs of the grid.

In the figures of the following sections thicker lines indicate the selection of separator vertices which will receive unique and maximum colors. Thinner lines that lie on different sides of a thicker line may reuse the same color range.

2.1. Rhombi and triangles

The rhombus. The rhombus R_x is the first subgraph of the grid shown in Fig. 1. It has height x. We have the following upper bound.

Proposition 9. $\chi_o(R_x) \le 3x/2 = 1.5x$.

Proof. Use a diagonal separator to cut the rhombus in half ($x/2$ unique colors are used), then cut also the remaining parts in half with a diagonal separator ($x/4$ unique colors, used in both parts). This is shown in Fig. 1. Therefore, we have the recursive formula $\chi_o(R_x) \le x/2 + x/4 + \chi_o(R_{\lfloor x/2 \rfloor})$, which implies $\chi_o(R_x) \le 3x/2$. □

The wide-side triangle. The wide-side triangle T_x is the subgraph of the grid shown in Fig. 2. Its long side has length x. First, we provide a simple upper bound.

Proposition 10. $\chi_o(T_x) \le 7x/6 \approx 1.167x$.

Proof. See the first separation of the wide-side triangle in Fig. 2. Use a separator diagonally, parallel to one of the diagonal sides of the triangle T_x, with $2x/6$ unique colors. In the two remaining parts, separate diagonally by using separators parallel to the other diagonal side of the triangle T_x; each of those separators uses $x/6$ unique colors. With one more use of $x/6$ unique colors, we end up with connected components that are subgraphs of the rhombus $R_{2x/6}$. Therefore, $\chi_o(T_x) \le 2x/6 + x/6 + x/6 + \chi_o(R_{\lfloor 2x/6 \rfloor})$, and since by Proposition 9, $\chi_o(R_x) \le 3x/2$, we have $\chi_o(T_x) \le 7x/6$. □

An improved upper bound can be obtained by the previous one, by making the observation that the graph on the left of the thickest separator in Fig. 2 is also a wide-side triangle. Thus, we may try to color it recursively in the same way. However, this would not improve the bound because the graph that remains on the right side uses $5x/6$ colors anyway. This indicates that the thickest separator would be better positioned if we moved it slightly to the right, since it seems that the remaining graph on the right side requires more colors.

Suppose that we move it slightly to the right, as in the last part of Fig. 2 and that the ratio of its length over the length of the long side of the triangle is $w \in [1/3, 1/2]$ (previously we had $w = 1/3$). We will optimize with respect to this w. Now, the rhombi on the right have length $x(1-2w)$, and the separators between them have length $x(1-2w)/2$. From the previously shown upper bound for the rhombus, and the fact that we need two sets of colors for the separators we conclude that the right part needs at most $\frac{5}{2}x(1-2w)$ colors. Assuming that the two parts are well balanced, the whole triangle needs at most $wx + \frac{5}{2}x(1-2w)$ colors. The triangle formed on the left of the separator has length $2wx$, thus from the above it needs $2w^2x + \frac{5}{2}(2wx)(1-2w)$ and in order for the balancing assumption to hold this must be equal to the number of colors used in the right part. Thus, we get the equation $2w^2 + 5w(1-2w) = \frac{5}{2}(1-2w)$, which has the solution $w = \frac{5-\sqrt{5}}{8} \approx 0.345$ in the interval $[1/3, 1/2]$. It is not hard to verify that using a separator of this length all the above arguments hold. Thus, we reach the following conclusion.

Proposition 11. $\chi_o(T_x) \le \sqrt{5}x/2 \approx 1.118x$.

The orthogonal triangle. The orthogonal triangle O_x is the subgraph of the grid shown in Fig. 3. It has height x. We have the following upper bound.

Proposition 12. $\chi_o(O_x) \le \phi x = \frac{\sqrt{5}+1}{2}x \approx 1.618x$.

Proof. See Fig. 3. Use a separator diagonally to form two wide-side triangles whose long sides are of length x. We have the formula $\chi_o(O_x) \le x/2 + \chi_o(T_x)$ and since by Proposition 11, $\chi_o(T_x) \le \sqrt{5}x/2$, we have $\chi_o(O_x) \le \frac{\sqrt{5}+1}{2}x = \phi x$, where we denote by ϕ the golden ratio. □

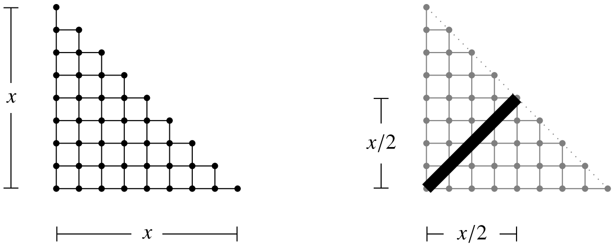

Fig. 3. The orthogonal triangle O_x and its separation.

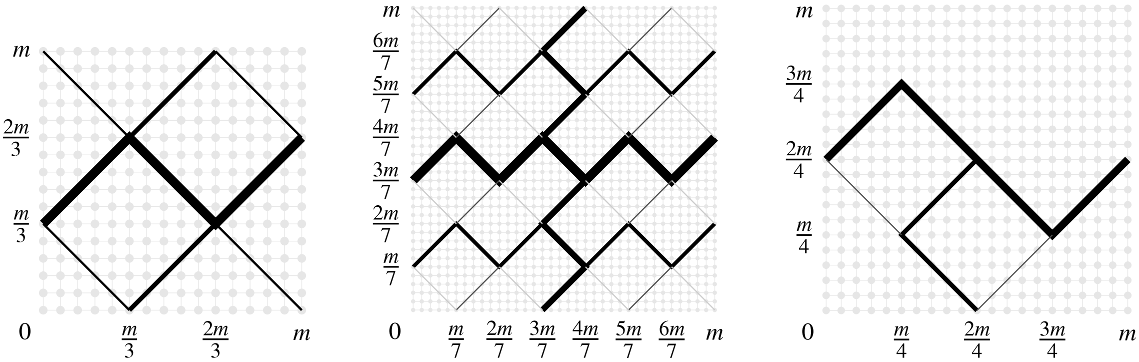

Fig. 4. $8m/3$, $18m/7$ and $(7+2\phi)m/4$ upper bounds.

2.2. Grids and tori

An $8m/3$ *upper bound for square grids.* In the first part of Fig. 4, we show how an $m \times m$ grid has to be partitioned with the help of separators to achieve an $8m/3$ upper bound.

The separators use m, $m/3$, and $m/3$ colors. After the removal of the separators, the remaining components are all subgraphs of a rhombus of height $2m/3$. By Proposition 9, each remaining component can be colored with m colors. In total, $8m/3$ colors are required.

Proposition 13. $\chi_o(G_m) \leq 8m/3 \approx 2.667m$.

An $18m/7$ *upper bound for square grids.* In the second part of Fig. 4, we show how an $m \times m$ grid has to be partitioned with the help of separators to achieve an $18m/7$ upper bound. The separators use m, $3m/7$, $3m/7$, $m/7$, and $m/7$ colors. Then, we have rhombi of height $2m/7$ that remain and, by Proposition 9, each rhombus can be colored with $3m/7$ colors. In total, we have $18m/7$ colors.

Proposition 14. $\chi_o(G_m) \leq 18m/7 \approx 2.571m$.

A $(7+2\phi)m/4$ *upper bound for square grids.* In the third part of Fig. 4 we show how an $m\times m$ grid can be partitioned to achieve a $(7+2\phi)m/4$ upper bound; we show only the partitioning of the subgraph under the first-level separator, since the subgraph over it is done in a symmetric way. We will show in the following section that shrinking this particular partition gives the best currently known result. The separators use $m+m/2+m/4 = 7m/4$ unique colors. The remaining subgraphs of the grid to be colored are rhombi of height $m/2$ and orthogonal triangles of height $m/2$. By Propositions 9 and 12 they can be colored with $3m/4$ and $\phi m/2$ colors, respectively. Therefore the total use of colors is $7m/4 + \max(3m/4, \phi m/2) = (7+2\phi)m/4$.

Proposition 15. $\chi_o(G_m) \leq (7+2\phi)m/4 \approx 2.559m$.

Improving the upper bounds by extending and shrinking colorings. The aforementioned upper bounds may be slightly improved by extending or shrinking the underlying grid. The reason is that, even though for the most part the grid is partitioned into rhombi, different subgraphs are formed along the boundary of the grid (see again Fig. 4).

In the case of the $8m/3$ and $18m/7$ bounds, we can see that wide-side triangles are formed along the boundary of the grid. Each such wide-side triangle has about half the area of a rhombus in the grid. Therefore, it might be beneficial to extend the grid and the coloring by x at every direction (up, down, left, right) to the point where each extended wide-side triangle uses the same number of colors as each rhombus. However, extending the coloring of the grid will also make the separators longer and we must check if the additional colors used in the separators do not cancel any effect from the balancing we did on wide-side triangles and rhombi. The method of extending indeed gives an improvement for the $8m/3$ coloring. The extension by x at every direction is shown in the first part of Fig. 5. The value of extension is chosen so that wide-side triangle $T_{2m/3+2x}$ is using the same number of colors as rhombus $R_{2m/3}$ (we use the colorings for the rhombus and the wide-side triangle from Propositions 9 and 11, respectively) and it is $x = (3\sqrt{5}-5)m/15 \approx 0.114m$. We also check that, since

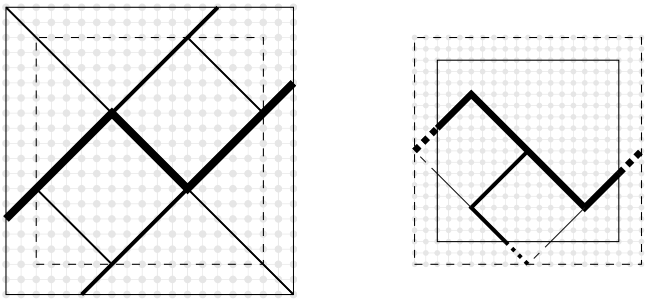

Fig. 5. An $8m/3$ coloring extended and a $(7 + 2\phi)m/4$ coloring shrunk.

$x < m/6$, the two graphs formed at the top-right and bottom-left corner of the first part of Fig. 5 are subgraphs of rhombus $R_{2m/3}$. The extended grid has side length $m' = m + 2x \approx 1.228m$. The first level separator has size $m + 2x$, the second level separator $\frac{1}{3}m + x$ and the third level separator $\frac{1}{3}m + x$. After removal of the separators, only connected components that can be colored with at most m colors are left. In total, $\frac{8}{3}m + 4x = 4(13 + 3\sqrt{5})m'/31$ colors are used and therefore we have the following.

Proposition 16. $\chi_o(G_m) \le 4(13 + 3\sqrt{5})m/31 \approx 2.543m$.

In the case of the $(7 + 2\phi)m/4$ coloring, we follow the opposite approach of shrinking the coloring. Four orthogonal triangles are formed at the corners of the grid, each using more colors than each of the rhombi. Therefore, slightly shrinking the grid so that the orthogonal triangles use the same number of colors as the rhombi improves the result. We shrink the grid by x at every direction (up, down, left, right), as shown in the second part of Fig. 5. The optimal amount of shrinking (so that the rhombus $R_{m/2}$ is using the same number of colors as the orthogonal triangle $O_{(m/2)-2x}$, according to the colorings of Propositions 9 and 12, respectively) is $x = (7 - 3\sqrt{5})m/16 \approx 0.01824m$. The shrunk grid has side $m' = m - 2x \approx 0.9635m$. The first level separator has size $m - 2x$, the second level separator $\frac{1}{2}m - x$ and the third level separator $\frac{1}{4}m - x$. After removal of the separators, only connected components that can be colored with at most $3m/4$ colors are left. In total, $\frac{5}{2}m - 4x = 3(7 + \sqrt{5})m'/11$ colors are used. Thus, we get our best upper bound for the ordered chromatic number of the square grid.

Proposition 17. $\chi_o(G_m) \le 3(7 + \sqrt{5})m/11 \approx 2.519m$.

Torus. An efficient coloring of the torus $\widehat{G}_m$ is as follows: Use the two diagonals as separators (at most $2m$ vertices). The remaining two connected components are subgraphs of the rhombus R_m which can be colored with at most $3m/2$ colors. Therefore, we have the following proposition.

Proposition 18. $\chi_o(\widehat{G}_m) \le 7m/2 = 3.5m$.

Rectangular grids. Intuitively, a rectangular grid begins to resemble a chain when one of its dimensions, say m_1, is much smaller than the other, i.e., $m_1 \ll m_2$. We may attempt to exploit this observation in the following manner: given a grid with m_1 rows and m_2 columns, pick the m_1-th column, the $2m_1$-th column, ..., the $(\lfloor m_2/m_1 \rfloor \cdot m_1)$-th column. These $\lfloor m_2/m_1 \rfloor$ columns will be used as separators, thus partitioning the graph into subgraphs of $m_1 \times m_1$ grids; each subgraph will use the same colors. However, the column separators do not all need distinct colors, because we can color them in a way similar to the coloring of a chain: the middle column receives the highest colors, then we color recursively the columns to the left and those to the right. This results to an upper bound of $\chi_o(G_{m_1,m_2}) \le m_1 \lceil 1 + \log(\lfloor m_2/m_1 \rfloor) \rceil + \chi_o(G_{m_1,m_1})$.

Moreover, the above upper bound can be further improved slightly. Instead of using columns as separators we may use a zig-zag line starting from the top left corner and proceeding diagonally to the right until it hits the bottom, then to the right and up again, and so on. This requires the same number of colors for the separators, since we can still color them in a chain-like fashion, but now wide-side triangles are formed (instead of grids), each of length $2m_1$, for which $\chi_o(T_{2m_1}) \le \sqrt{5}m_1$, from Proposition 11. Thus, we reach the following conclusion.

Proposition 19. $\chi_o(G_{m_1,m_2}) \le m_1 \lceil 1 + \log(\lfloor m_2/m_1 \rfloor) \rceil + \sqrt{5}m_1$.

3. Separators, vertices with unique colors, and lower bounds

In this section we prove lower bounds on the ordered chromatic number of square grids and tori.

An important observation is the following: Consider an optimal ordered coloring of a non-complete connected graph $G = (V, E)$, say with k colors. If the maximum color assigned to more than one vertices is i, then vertices with colors $i + 1, \ldots, k$ must form a separator in G, otherwise between any two vertices colored with i, there exists a path in G with maximum color i occurring two times. In particular, the above set of uniquely colored vertices contains an inclusion minimal separator in G.

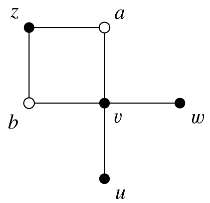

Fig. 6. Vertex v with two grid direction neighbors in the separator.

The following lemma proves useful.

Lemma 20. *Given a connected graph $G = (V, E)$, an optimal ordered coloring C of G, and any set $W \subseteq V$ of vertices with uniquely occurring colors in C,*

$$\chi_o(G) \geq |W| + \chi_o(G - W).$$

Proof. Consider the optimal ordered coloring C of G, using $\chi_o(G)$ colors. The coloring restricted to $V - W$ is an ordered coloring of $G - W$ using $\chi_o(G) - |W|$ colors. Therefore, $\chi_o(G) \geq |W| + \chi_o(G - W)$. □

Thus, we can reason about a lower bound on $\chi_o(G)$ by reasoning about inclusion minimal separators in G: We examine cases on the size and shape of an inclusion minimal separator induced by some of the uniquely occurring colors of an optimal coloring C and then, for each case, argue that the cardinality of a set of vertices W with uniquely occurring colors plus the ordered chromatic number of $G - W$ is at least the desired lower bound (the choice of the set W depends on the inclusion minimal separator). In order to argue that in a grid after the removal of a set W the ordered chromatic number of a remaining connected component is high, we will rely heavily on Proposition 5 (monotonicity of the ordered chromatic number under taking minors or subgraphs) and make use of induction.

Before proceeding to the proof of the lower bounds, we should state some auxiliary results, related to separators in general, and also to the form of separators in grid-like graphs. The following lemma is from [9] and is similar to an exercise in [10].

Lemma 21. *A separator S of a connected graph G is inclusion minimal if and only if every vertex of S has a vertex adjacent in every connected component of $G - S$.*

We are now ready to state and prove some facts about the form of inclusion minimal separators in G_m. We remark that any inclusion minimal separator S of G_m for $m \geq 2$ has size $|S| < m^2$.

Lemma 22. *If S is an inclusion minimal separator of G_m, where $m \geq 2$, then $G_m - S$ consists of exactly two connected components.*

Proof. Assume for the sake of contradiction that $G_m - S$ consists of d connected components, with $d > 2$. In G_m, contract each of the above d components to a single vertex. By Lemma 21, the resulting graph is a supergraph of $K_{d,|S|}$ and is contained as a minor in G_m. Since G_m is planar, it contains no $K_{3,3}$ minor, and therefore $|S| \leq 2$. However, the only separator with $|S| \leq 2$ is (up to rotation) $\{(0, 1), (1, 0)\}$, for which $G_m - S$ has only two connected components, a contradiction. □

We say that two vertices (x_1, y_1), (x_2, y_2) of a separator of G_m are *neighboring* if $|x_1 - x_2| \leq 1$ and $|y_1 - y_2| \leq 1$. In other words, a vertex of a separator neighbors with any vertex directly up, left, down, right (we call these the *grid directions*), or directly up-left, down-left, down-right, up-right (we call these the *intermediate directions*). In order to avoid confusion, in this work, we reserve the term 'adjacent' for connection only along the grid directions and the term 'neighboring' for connection possibly also along the intermediate directions. The *boundary* of the grid consists of the four paths, each having m vertices, with $x = 0$, $x = m - 1$, $y = 0$, and $y = m - 1$, respectively.

Lemma 23. *The vertices of any inclusion minimal separator of G_m, for $m \geq 2$, can be put in a sequence such that no vertex is repeated, adjacent vertices in the sequence are neighboring, and either* (i) *the first and the last vertex of the sequence are also neighboring, or* (ii) *if the first and the last vertex of the sequence are not neighboring, then the first and the last vertex of the sequence are the only ones lying on the boundary of the grid.*

Proof. By Lemma 22, for any inclusion minimal separator S of G_m, $G_m - S$ consists of two connected components, say A and B, and by Lemma 21, every vertex of the separator is adjacent with a vertex from both connected components. Consider a vertex of the separator which is not on the sides of the grid. This vertex cannot have less than two neighboring vertices in the separator, because then all its adjacent vertices in the grid that are not in the separator are in the same connected component.

It is also not possible that a vertex v of the inclusion minimal separator has two neighbors such that both of these neighbors are in the grid directions. Assume, without loss of generality, that vertex $v = (x, y)$ has neighbors $u = (x, y - 1)$ and $w = (x + 1, y)$ in the separator. Then, by Lemma 21, since $a = (x, y + 1)$ and $b = (x - 1, y)$ must be in different components, vertex $z = (x - 1, y + 1)$ must also be in the separator (see Fig. 6). But then, vertex v has three neighbors in the inclusion minimal separator and this is a case that we will prove impossible immediately in the following.

Fig. 7. Four possible neighboring cases for an inclusion minimal separator vertex.

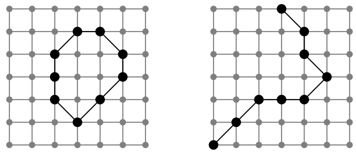

Fig. 8. Inclusion minimal separators of types (i) and (ii).

We are now going to prove that a vertex v of an inclusion minimal separator S cannot have more than two neighbors in S. Assume for the sake of contradiction that v has three distinct neighbors u, w, z in the separator. Define the graph G' to be G_m with additional edges vu, vw, and vz (if they do not exist in G_m). G' is planar, because one can add edges vu, vw, and vz as straight line segments to the standard drawing of G_m without introducing crossings. Remember that $G_m - S$ has two connected components A and B. Now, in G', contract the vertices of A to a single vertex v_A and the vertices of B to a single vertex v_B. In the resulting graph, each one of v, v_A, v_B has an edge to each one of u, w, z, that is, $G' \succcurlyeq K_{3,3}$, which is a contradiction, because G' is planar.

The four possible neighboring cases of a vertex in the inclusion minimal separator which does not lie on the sides of the grid, ignoring rotations, are shown in Fig. 7.

We also cannot have, for an inclusion minimal separator S, a vertex $v \in S$ on the boundary of the grid with only one neighbor $w \in S$, such that w is also on the boundary of the grid, because then v has only neighbors in one connected component of $G_m - S$.

Now, we are going to build a special sequence of vertices of an inclusion minimal separator S.

If there is no vertex with only one neighbor in the separator (case i), then we choose any vertex v as the initial vertex of the sequence. Vertex v has two neighbors in S, say v' and v''. We choose any of them, say v', to be the next vertex in the sequence. Then, we extend the sequence by choosing the next element to be the neighbor of the current element not already in the sequence, until we reach the neighbor v'' of v (this always happens because the separator S has a finite number of elements). We claim that the built sequence includes all vertices of separator S. Consider, the closed polygonal line K with vertices in the order of the sequence built. By the Jordan curve theorem, K divides the plane in a region inside the curve and an unbounded region outside the curve. Both regions contain vertices of the original graph, as can be seen in Fig. 7. Every embedding of a path in G_m connecting two vertices in different regions has to touch closed curve K, and in particular one of its vertices. Thus, the vertices of the built sequence are indeed a separator, and since S is inclusion minimal the built sequence includes all vertices of S.

If there is a vertex v with only one neighbor in S (case ii), then we choose v as the initial vertex of the sequence (this vertex has to lie on the boundary of the grid). Then, we extend the sequence by choosing the next element to be the neighbor of the current element not already in the sequence, until we reach an element w with only one neighbor in S (this always happens because the separator S has a finite number of elements). We claim that the built sequence includes all vertices of separator S. Consider, the polygonal line K with vertices in the order of the sequence built. With the help of the Jordan curve theorem, one can prove that K divides the square containing the embedding of the grid in two regions. Both regions contain vertices of the original graph, as can be seen in Fig. 7. Every embedding of a path in G_m connecting two vertices in different regions has to touch closed curve K, and in particular one of its vertices. Thus, the vertices of the built sequence are indeed a separator, and since S is inclusion minimal the built sequence includes all vertices of S.

As a result, inclusion minimal separators are of the following two types, according to their aforementioned built sequence:

(i) the first and the last vertex of the sequence are neighboring,
(ii) if the first and the last vertex of the sequence are not neighboring, then the first and the last vertex of the sequence are the only ones lying on the boundary of the grid.

We remark that a separator of type (i) surrounds one of the connected components of $G_m - S$. □

Examples of the two types of inclusion minimal separators are shown in Fig. 8.

Proposition 24. *For $m \geq 2$, $\chi_o(G_m) \geq 3m/2$.*

Proof. The lower bound is true for $m = 2$ and $m = 3$, because $\chi_o(G_2) = 3$ and $\chi_o(G_3) = 5$. We consider any optimal ordered coloring C of G_m. The set of vertices in G_m with uniquely occurring colors in C contains an inclusion minimal separator of G. We will analyze all possible cases of an inclusion minimal separator S and will prove that in every case either we have $|S| + \chi_o(G_m - S) \geq 3m/2$, or for some set W of vertices with unique colors (our choice of W depends on S) we have

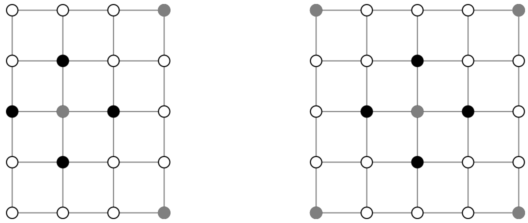

Fig. 9. Diamonds (in black) and their frontiers (in white).

$|W| + \chi_o(G_m - W) \geq 3m/2$. Then, by Lemma 20, $\chi_o(G_m) \geq 3m/2$. Since we want to prove a $3m/2$ lower bound, it is enough to consider only separators of size $|S| < 3m/2$.

Separator of type (i). If the inclusion minimal separator S is of type (i), then it surrounds a connected component, say A of $G_m - S$. This connected component does not include any vertex of the boundary of the grid. Assume that this connected component is exactly contained in a $G_{w,h}$ subgraph of G_m and without loss of generality $w \geq h$. This connected component has a vertex in every one of the w columns it spans and since this vertex must be separated by vertices above and below that are not in the connected component, there must be at least two vertices of the separator in every one of these w columns. Moreover, a vertex in the leftmost of the w columns must be separated by vertices to the left of it and a vertex in the rightmost of the w columns must be separated by vertices to the right of it, i.e., there are two more vertices in the separator that we have not counted before. In total, separator S has at least $2w + 2$ vertices. Moreover, every vertex of the separator touches a vertex of A and is thus contained in a $G_{w+2,h+2}$ grid subgraph of G_m. This grid subgraph $G_{w+2,h+2}$ is contained in a G_{w+2} subgraph of G_m. If, from G_m, we remove this G_{w+2}, then remove edges completely in the $w + 2$ rows and the $w + 2$ columns that G_{w+2} occupied, and then smooth the remaining vertices of the $w + 2$ rows and $w + 2$ columns that G_{w+2} occupied, then we end up with a G_{m-w-2} graph. Thus $G_m - S$ contains a G_{m-w-2} topological minor, with $G_{m-w-2} \supseteq G_2$, because an inclusion minimal separator of type (i) cannot touch both a boundary row and a boundary column of G_m. As a result, $|S| + \chi_o(G_m - S) \geq |S| + \chi_o(G_{m-w-2}) \geq 2w + 2 + \frac{3}{2}(m - w - 2) = \frac{3}{2}m + \frac{w}{2} - 1$.

Case i.1. If $w > 1$, then the $3m/2$ lower bound is true.

Case i.2. If $w = 1$, the above bounding method does not apply and in the following we resort to a different method in order to prove a $3m/2$ lower bound: We observe that connected component A consists of a single vertex $v = (x, y)$, because $w \geq h$. We call the inclusion minimal separator

$$\{(x - 1, y), (x, y - 1), (x + 1, y), (x, y + 1)\}$$

a *diamond* with *center* $v = (x, y)$ (see Fig. 9). If under the optimal coloring C, G_m contains a set of uniquely colored vertices which form a different inclusion minimal separator S' than a diamond, then we focus on S' and argue as in the other cases of this proof. Otherwise, the uniquely colored vertices by C in G_m form only diamond inclusion minimal separators. We can assume that in C the centers of each diamond are colored with color 1, because we can assume that no uniquely colored vertex is colored with 1 (if it is we can switch color 1 with any other color that is repeating in the coloring and still have an optimal ordered coloring).

We define the *frontier* of a diamond to be the set of vertices not including the center or vertices of the diamond that neighbor with at least one of the vertices of the diamond in one of the eight directions: up, down, left, right, up-left, up-right, down-left, down-right (see Fig. 9). If a vertex of the diamond is on the boundary of G_m, then the frontier consists of 13 vertices (left part of Fig. 9), otherwise the frontier consists of 16 vertices (right part of Fig. 9). The diamond cannot have its center at any of $(1, 1)$, $(1, m - 2)$, $(m - 2, 1)$, $(m - 2, m - 2)$, because then it is not inclusion minimal. We call a diamond *free* if there is no vertex with unique color in its frontier (i.e., no other diamond touches its frontier).

If there is a non-free diamond, then we consider it together with a diamond that touches its frontier. We have the following five cases, ignoring symmetries: if the diamonds share some vertex, then either (a) the diamonds have two vertices in common, or (b) the diamonds have one vertex in common; if the diamonds do not share some vertex, then the diamonds either (c) share two common frontier vertices, or they share one common frontier vertex and (d) the two diamonds are completely contained in three rows or columns, or (e) the two diamonds are not completely contained in three rows or columns. These five cases are shown in Fig. 10. In all five case, we locate a set U of vertices with unique colors contained in a G_q grid subgraph. If, from G_m, we remove this G_q subgraph, then remove all edges that lie completely in the q rows and q columns that G_q occupied, and then smooth all vertices in the q rows and q columns that G_q occupied, what remains is a G_{m-q} graph. Therefore, $G_m - U$ contains G_{m-q} as a topological minor, which by induction has ordered chromatic number $\chi_o(G_{m-q}) \geq \frac{3}{2}(m - q)$, because $m - q \geq 2$ (otherwise there are non-diamond inclusion minimal separators induced by some uniquely colored vertices in the coloring of G_m). Using Lemma 20, $\chi_o(G_m)$ is at least $|U| + \frac{3}{2}(m - q)$. The grid G_q is shown with a gray background in every one of the five cases in Fig. 10. We check indeed that the lower bound applies in every case.

(a) $|U| = 6$, $q = 4$ and the lower bound is $6 + \frac{3}{2}(m - 4) = 3m/2$;
(b) $|U| = 5$, $q = 3$ and the lower bound is $5 + \frac{3}{2}(m - 3) > 3m/2$;

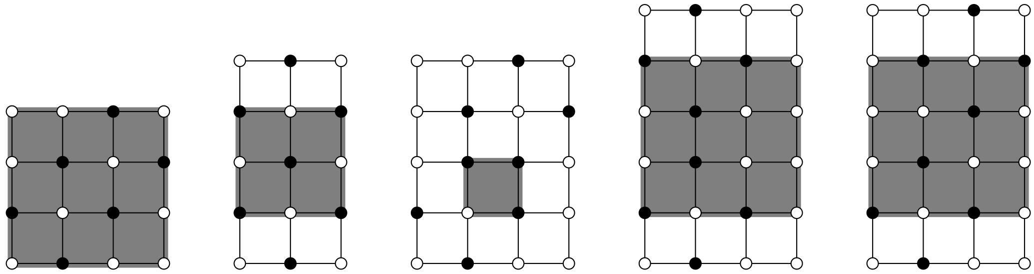

Fig. 10. Non-free diamonds.

(c) $|U| = 3, q = 2$ and the lower bound is $3 + \frac{3}{2}(m-2) = 3m/2$;
(d) $|U| = 6, q = 4$ and the lower bound is $6 + \frac{3}{2}(m-4) = 3m/2$;
(e) $|U| = 6, q = 4$ and the lower bound is $6 + \frac{3}{2}(m-4) = 3m/2$.

Now, we prove that it is not possible that every diamond is free, because then the optimal coloring C is not ordered. Assume that we remove all diamonds and their centers from the graph and we consider the coloring restricted to the remaining graph G'. Since $G' \subseteq G_m$ the coloring restricted to G' is ordered. Graph G' contains a connected set with more than one vertex (for example the frontier of some removed diamond) and thus its coloring uses more than one color. In this coloring, every color occurs more than once with the possible exception of color 1 (because we have removed all diamonds that had the uniquely colored vertices). Thus, the maximum color in the restricted coloring of G' is occurring in two different vertices v and u of G'. If G' is connected, then in a simple path of G' from u to v, the maximum color occurring in the path is not unique. So, it is enough to prove that G' is connected in order to have a contradiction. The proof is by induction on the number of free diamonds. If there is no free diamond, then we have G_m which is connected. By the inductive hypothesis, graph G_m after the removal of k free diamonds and their centers (we denote the resulting graph with H_k) is connected. Consider the subgraph H_{k+1} of H_k, where we remove one more free diamond d. The frontier of d is still contained in H_{k+1}. Thus, every simple path p in H_k between two vertices of H_{k+1} that is using an internal vertex in d can be transformed to a path with the same endvertices that avoids d, as follows: Starting from one endvertex, follow path p until you reach the first occurrence in p of a vertex in the frontier of d, then continue with a path in the frontier of d to the last occurrence in p of a vertex in the frontier of d, and then follow path p until you reach the other endvertex. Thus, H_{k+1} is connected.

Separator of type (ii). If the inclusion minimal separator S is of type (ii), then we consider two cases according to whether $|S| < m$ or $|S| \geq m$.

Case ii.1. If the separator has size $s = |S| < m$, then it can neither span all rows nor span all columns of G_m. Moreover, since, according to Lemma 23, elements of S can be put in a sequence such that consecutive elements in the sequence are neighboring, the following is true: If there is a vertex of S in row (or column) i and a vertex of S in row (or column) j, then there is a vertex of S in every row (or column) between i and j. Thus, there is at least one row and one column in the boundary of G_m that contains no vertex of S. Assume without loss of generality that S has neither a vertex in column $x = m-1$ nor in row $y = m-1$. Consider the grid subgraph $G_{m-\lceil s/2\rceil}$ of G_m that contains vertex $(m-1, m-1)$. We claim that S cannot contain some vertex of this $G_{m-\lceil s/2\rceil}$ subgraph. Assume for the sake of contradiction that it does. A sequence of S from Lemma 23 starts at a vertex in column $x = 0$ or row $y = 0$, then goes on for at least $\lceil s/2\rceil$ vertices before reaching a vertex in S, and then it goes on for at least $\lceil s/2\rceil$ vertices before reaching a vertex in column $x = 0$ or row $y = 0$. Therefore, the sequence has length at least $\lceil s/2\rceil + 1 + \lceil s/2\rceil > s$, which is a contradiction because $|S| = s$. As a result, $G - S \supseteq G_{m-\lceil s/2\rceil}$ (with $m - \lceil s/2\rceil \geq 2$, because $m > 3$ and $s < m$) and by the inductive hypothesis and Lemma 20, we have $\chi_o(G_m) \geq s + \frac{3}{2}(m - \lceil s/2\rceil) \geq 3m/2$, whenever $s \geq 2$ (which is true because no single vertex forms a separator).

Case ii.2. If the separator has size $s = |S| \geq m$, then we study two subcases depending on whether $s \geq m+2$, or $s \leq m+1$.

Subcase ii.2.1. If $s \geq m + 2$, we consider the four grid $G_{\lfloor m/2-s/6\rfloor}$ subgraphs of G_m, each containing one of the four corner vertices $(0, 0)$, $(0, m-1)$, $(m-1, 0)$, $(m-1, m-1)$ of G_m. We claim that S cannot contain a vertex in every one of the four $G_{\lfloor m/2-s/6\rfloor}$ subgraphs. Assume for the sake of contradiction that it does. Consider a sequence of S from Lemma 23. In this sequence, choose four vertices, one from each of the four $G_{\lfloor m/2-s/6\rfloor}$ subgraphs. Between any two subsequent of the above four vertices there are at least $m - 2\lfloor \frac{m}{2} - \frac{s}{6}\rfloor$ vertices in the sequence not contained in any of the four $G_{\lfloor m/2-s/6\rfloor}$ subgraphs. Therefore, the sequence has length at least

$$4 + 3\left(m - 2\left\lfloor \frac{m}{2} - \frac{s}{6}\right\rfloor\right) \geq 4 + 3\left(m - 2\left(\frac{m}{2} - \frac{s}{6}\right)\right) = s + 4 > s,$$

which is a contradiction because $|S| = s$. As a result, $G-S \supseteq G_{\lfloor m/2-s/6\rfloor}$. We check that since $s < 3m/2$, $\lfloor m/2-s/6\rfloor \geq \lfloor m/4\rfloor$, which is greater than 1 for $m \geq 8$. For $3 < m < 8$ and $m+2 \leq s < 3m/2$, we have that $\lfloor m/2 - s/6\rfloor = 1$ only for the pairs:

$$(m = 5, s = 7), (m = 6, s = 8), (m = 7, s = 10).$$

For all three pairs the lower bound that we would like to prove, $\chi_o(G_5) \geq 8$, $\chi_o(G_6) \geq 9$, $\chi_o(G_7) \geq 11$, respectively, is one more than the size s of the separator S and since $G - S$ contains at least one vertex, we can prove it. For the rest of the cases, by the inductive hypothesis and Lemma 20, we have

$$\chi_o(G_m) \geq s + \frac{3}{2}\left\lfloor \frac{m}{2} - \frac{s}{6} \right\rfloor \geq s + \frac{3}{2}\left(\frac{m}{2} - \frac{s}{6} - 1\right) = \frac{3}{4}(m + s - 2) \geq 3m/2,$$

because $s \geq m + 2$.

Subcase ii.2.2. If $s = m$ or $s = m + 1$, we consider the four grid $G_{\lceil m/2 - s/6 \rceil}$ subgraphs of G_m, each containing one of the four corner vertices $(0, 0)$, $(0, m - 1)$, $(m - 1, 0)$, $(m - 1, m - 1)$ of G_m. Consider a sequence of S from Lemma 23. If S has a vertex common with each of the four $G_{\lceil m/2 - s/6 \rceil}$ subgraphs, then, in this sequence, choose four vertices, one from each of the four $G_{\lceil m/2 - s/6 \rceil}$ subgraphs. Between any two subsequent of the above four vertices there are at least $m - 2\lceil \frac{m}{2} - \frac{s}{6} \rceil$ vertices in the sequence not contained in any of the four $G_{\lceil m/2 - s/6 \rceil}$ subgraphs. Therefore, the sequence has length at least $L = 4 + 3(m - 2\lceil \frac{m}{2} - \frac{s}{6} \rceil)$.

If m is of the form $3k$ or $3k + 2$, where k is an integer, then $L > s$, as shown in the following.

- If $m = 3k$ and $s = m$, then $L = 4 + 3(3k - 2(\frac{3}{2}k - \frac{3}{6}k)) = 3k + 4 = s + 4$.
- If $m = 3k$ and $s = m + 1$, then $L = 4 + 3(3k - 2\lceil k - \frac{1}{6} \rceil) = 4 + 3(3k - 2k) = 3k + 4 = s + 3$.
- If $m = 3k + 2$ and $s = m$, then $L = 4 + 3(3k + 2 - 2\lceil k + \frac{1}{3} \rceil) = 3k + 4 = s + 2$.
- If $m = 3k + 2$ and $s = m + 1$, then $L = 4 + 3(3k + 2 - 2\lceil k + \frac{1}{6} \rceil) = 3k + 4 = s + 1$.

Therefore, S cannot contain a vertex in every one of the four $G_{\lceil m/2 - s/6 \rceil}$ subgraphs. As a result, $G - S \supseteq G_{\lceil m/2 - s/6 \rceil}$ and by the inductive hypothesis and Lemma 20, we have $\chi_o(G_m) \geq s + \frac{3}{2}(\lceil m/2 - s/6 \rceil) \geq 3m/4 + 3s/4 \geq 3m/2$, because $s \geq m$.

If however m is of the form $3k + 1$ (and $s = m$ or $s = m + 1$), then S might have a vertex in every one of four G_{k+1} subgraphs at the corners of the grid (for both $s = m$ and $s = m + 1$, $\lceil m/2 - s/6 \rceil = k + 1$). We may assume without loss of generality that the sequence of S from Lemma 23 starts (without loss of generality) from column $x = 0$ and ends at column $x = m - 1$, with the exception of $m = 4$ and $s = 5$. If not, i.e., if the sequence of S starts from column $x = 0$ and ends at row $y = 0$ (without loss of generality), then consider the G_{k+1} subgraph containing the corner vertex $(m - 1, m - 1)$. The sequence starts from column $x = 0$ continues for at least $m - (k + 1)$ vertices before it reaches a vertex in the above subgraph and then continues for at least $m - (k + 1)$ vertices before it reaches row $y = 0$, i.e., it has length at least

$$m - (k + 1) + 1 + m - (k + 1) = 2m - 2k - 1 = 2(3k + 1) - 2k - 1 = 4k + 1,$$

which is always greater than s, if $s = m = 3k + 1$ or if $s = m + 1 = 3k + 2$ and $k > 1$, i.e., when neither $m = 4$ nor $s = 5$.

If the sequence of S from Lemma 23 starts from column $x = 0$ and ends at column $x = m - 1$, then, as we mentioned before, it contains vertices in every column. Moreover, if $s = m$ it contains exactly one vertex in every column, whereas is $s = m + 1$, it contains one vertex in every column with the exception of one column (which is not on the boundary of the grid). In the $s = m + 1$ case, the two vertices in the same column $x = i$ must be neighboring (if they are not neighboring, the separator has another column next to column i with more than one vertices). We can assume without loss of generality that there is no column with two vertices among the first $k + 1$ columns (from $x = 0$ to $x = k$). Then, since S must have a vertex in both the G_{k+1} at corner $(0, 0)$ and the G_{k+1} in corner $(0, m - 1)$, we can assume that the restriction of the separator in the first $k + 1$ columns is exactly the set $\{(i, k + i) \mid 0 \leq i \leq k\}$. We then consider the grid G_{k+1} subgraph with corners at $(1, 0)$, $(k + 1, 0)$, $(k + 1, k)$, $(1, k)$. We call this subgraph X.

We may assume that S also has a vertex v in X, because otherwise we can already prove a lower bound of $3m/2$ as before with the help of X. This vertex v must be in column $x = k + 1$. If $k > 1$, then vertex $(k, 2k)$ of the separator cannot neighbor (even along a diagonal) with vertex v, because v has y-coordinate at most k. Therefore, there must be one more vertex w of the separator different from v in column $k+1$, which implies that $s = m+1$, i.e., $s = m$ is not possible. As described above, w has to neighbor with v. This vertex has also to neighbor with $(k, 2k)$ and thus the only possibility is that $w = (k+1, 2k-1)$. Moreover, v and w are the only vertices in column $x = k+1$. If $k > 2$, v and w are not neighboring, which is a contradiction. If $k = 2$, i.e., $m = 7$, then necessarily $w = (3, 3)$ and $v = (3, 2)$ (see Fig. 11).

By a similar argument as above, since every one of the last 3 columns ($x = 5, 6, 7$) has exactly one vertex of S, then the separator restricted to these last columns is either:

$$l_1 = \{(4, 4), (5, 3), (6, 2)\} \text{ or } l_2 = \{(4, 2), (5, 3), (6, 4)\}.$$

If the restriction is l_1, then w has three neighbors in the separator, $(3, 2)$, $(2, 4)$, and $(4, 4)$, and thus S is not inclusion minimal, which is a contradiction. If the restriction is l_2, then S has no vertex in the G_3 grid subgraph with corners at $(3, 4)$, $(5, 4)$, $(5, 6)$, $(3, 6)$, that we call Y (see Fig. 11), and we can prove a $3m/2$ lower bound as before with the help of Y.

We have not argued about the case when $k = 1$ ($m = 4$). For $m = 4$, if $s = m = 4$ the sequence of S goes (without loss of generality) from row $x = 0$ to row $x = m - 1$ and thus contains no vertex in column $y = 0$. Column $y = 0$ is a path graph P_4 with $\chi_o(P_4) = 3$. Thus, we have a lower bound of $4 + \chi_o(G - S) \geq 4 + 3 = 7 > 6$. If $s = m + 1 = 5$, then $G - S$ contains at least some (non-empty) component and thus we have a lower bound of $5 + \chi_o(G - S) \geq 5 + 1 = 6$. □

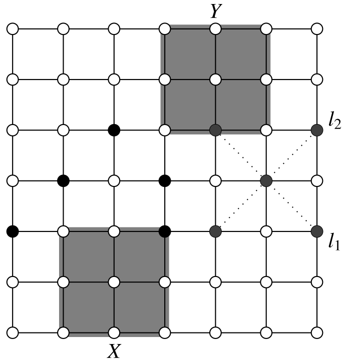

Fig. 11. The case $m = 7$ when $s = 7$ or $s = 8$.

Since $\widehat{G}_m$ contains G_m as a subgraph, because of monotonicity of the ordered chromatic number under subgraphs, we immediately have the following corollary.

Corollary 25. *For $m \geq 2$, $\chi_o(\widehat{G}_m) \geq 3m/2$.*

4. Discussion and open problems

The most important problem still left open is the exact value of $\chi_o(G_m)$. For small values of m the correct answer seems to be $2m - 1$, but maybe this is just an exception for small values of m, and asymptotics could be different and closer to $2.5m$. At the moment, in our lower bound proof for the square grid, after the removal of a separator, we only consider square grids that remain (either as a subgraph or a minor). It would also be interesting to study lower bounds for the rhombus and the triangle subgraphs, or possibly for non-square grids, and then combine them to improve the lower bound for the square grid.

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