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Iterative Process with Errors to Nonlinear Φ-Strongly Accretive Operator Equations in Arbitrary Banach Spaces

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Abstract—Let X be an arbitrary Banach space and $T : D(T) \subset X \to X$ be a Lipschitz ϕ-strongly accretive operator with domain $D(T)$ and range $R(T)$. The Mann and Ishikawa type iterative sequences with errors which strongly converge to the unique solution of the equation $Tx = f$ under weaker conditions are given. The related results deal with the problems that the Mann and Ishikawa iterative sequences with errors strongly converge to the unique fixed point of Lipschitz ϕ-hemicontractive operators.

Keywords—Nonlinear ϕ-strongly accretive operator, ϕ-hemicontractive operator, Ishikawa type iterative sequence, Arbitrary Banach space.

1. INTRODUCTION

Let X be an arbitrary Banach space, X^* be its dual space, and $\langle x, f^* \rangle$ be the generalized duality pairing between $x \in X$ and $f^* \in X^*$. The mapping $J : X \to 2^{X^*}$ defined by

$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|f^*\| \, \|x\|, \ \|f^*\| = \|x\|\}$$

is called the normalized duality mapping. If X^* is strictly convex, then J is single-valued. We shall denote the single-valued duality mapping by j.

An operator $T : D(T) \subset X \to X$ with domain $D(T)$ and range $R(T)$ is called strongly accretive if, for each $x, y \in D(T)$, there exist a $j(x - y) \in J(x - y)$ and a constant $k > 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \geq k\|x - y\|^2. \tag{1.1}$$

T is called ϕ-strongly accretive if, for each $x, y \in D(T)$, there exist a $j(x - y) \in J(x - y)$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \geq \phi(\|x - y\|)\|x - y\|. \tag{1.2}$$

It is known (see, e.g., [1]) that the class of strongly accretive operators is a proper subset of the class of ϕ-strongly accretive operators. Closely related to the class of strongly accretive (respectively, ϕ-strongly accretive) operators is the class of strongly pseudocontractive (respectively, ϕ-strongly pseudocontractive) operators.

An operator $A : D(A) \subset X \to X$ with domain $D(A)$ and range $R(A)$ is called strongly pseudocontractive if, for each $x, y \in D(A)$, there exist a $j(x - y) \in J(x - y)$ and a constant $t > 1$ such

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that

$$\langle Ax - Ay, j(x-y)\rangle \le \frac{1}{t}\|x-y\|^2. \tag{1.3}$$

A is called ϕ-strongly pseudocontractive if, for each $x, y \in D(A)$ there exist a $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle Ax - Ay, j(x-y)\rangle \ge \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|. \tag{1.4}$$

Furthermore, A is said to be ϕ-hemicontractive if the fixed point set $F(A)$ of A is nonempty, and for each $x \in D(A)$ and $x^* \in F(A)$, there exist a $j(x-x^*) \in J(x-x^*)$ and a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle Ax - x^*, j(x-x^*)\rangle \le \|x-x^*\| - \phi(\|x-x^*\|)\|x-x^*\|. \tag{1.5}$$

It was shown in [1] that the class of strongly pseudocontractive operators is a proper subset of the class of ϕ-strongly pseudocontractive operators. The example in [2] shows that the class of ϕ-strongly pseudocontractive operators is a proper subset of the class of ϕ-hemicontractive operators. From the inequalities (1.1)–(1.4), it is easy to see that A is a strongly (respectively, ϕ-strongly) pseudocontractive operator if and only if $T = I - A$ is strongly (respectively, ϕ-strongly) accretive where I is the identity operator. The classes of strongly pseudocontractive operators and strongly accretive operators have been extensively studied by many authors. In particular, Deimling [3, Theorem 13.8] proved that if $T : X \to X$ is strongly accretive and semicontinuous (i.e., $x_n \to x$ implies that $Tx_n \rightharpoonup Tx$), then T maps X onto X; that is, for each $f \in X$, the equation $Tx = f$ has a solution in X.

Chidume [4] proved that the Mann iterative sequence [5] can be used to approximate the fixed point of the continuous strongly pseudocontractive operator $A : K \to K$, where K is a bounded closed convex subset of a uniformly smooth Banach space X. He pointed out that it is not known whether or not the Ishikawa iterative sequence converges for this class of nonlinear operators, see [4, p. 550]. Deng and Ding [6, Theorem 1] proved that the Ishikawa iterative sequence can be used to approximate the fixed point of the Lipschitz locally strongly pseudocontractive operator. They also pointed out that it is an open problem that the Lipschitz continuity of A can be dropped in Theorem 1. Ding [7] proved that the Mann and Ishikawa type iterative sequences with errors converge strongly to the unique fixed point of continuous locally strongly pseudocontractive operator $A : K \to K$ and to the unique solution of the equation $Tx = f$ involving a semicontinuous locally strongly accretive operator $T : X \to X$ under weaker assumptions, where K is a closed convex subset of a uniformly smooth Banach space X. These results answered positively the open problems mentioned by Chidume [4] and Deng-Ding [6].

Recently, Osilike [1,8] studied the classes of ϕ-strongly accretive operators and ϕ-hemicontractive operators and proved that the Ishikawa type iterative sequence strongly converges to the unique solution of the Lipschitz ϕ-strongly accretive operator equation $Tx = f$ and to the unique fixed point of the Lipschitz ϕ-hemicontractive operator A in q-uniformly smooth Banach spaces and in arbitrary Banach spaces, respectively.

The objective of this paper is to prove that the Mann and Ishikawa type iterative sequences with errors strongly converge to the unique solution of the Lipschitz ϕ-strongly accretive operator equation $Tx = f$ and to the unique fixed point of the Lipschitz ϕ-hemicontractive operator A in arbitrary Banach spaces under weaker assumptions. These results improve and generalize many corresponding results in recent literature.

2. PRELIMINARIES

Let us recall the following two iterative sequences due to Mann [5] and Ishikawa [9], respectively.

(I) The Mann iterative sequence is defined as follows: for a convex subset K of a Banach space X and an operator $T : K \to K$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_0 \in K, \qquad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \qquad n \ge 0, \tag{2.1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence satisfying $\alpha_0 = 1$, $0 < \alpha_n \le 1$ for all $n \ge 1$ and $\sum_n \alpha_n = \infty$ (or $\sum_n \alpha_n(1-\alpha_n) = \infty$).

(II) The Ishikawa iterative sequence is defined as follows: for an operator $T : K \to K$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\,x_n + \alpha_n T y_n, \qquad & x_0 \in K, \\ y_n &= (1-\beta_n)\,x_n + \beta_n T x_n, \qquad & n \ge 0, \end{aligned} \tag{2.2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers satisfying the conditions: $0 \le \alpha_n \le \beta_n \le 1$ for all $n \ge 0$, $\lim_n \beta_n = 0$ and $\sum_n \alpha_n\beta_n = \infty$.

The two iterative sequences have been extensively used and studied by many authors for approximating either fixed points of various nonlinear mappings or solutions of nonlinear operator equations in Banach spaces.

(III) The Ishikawa iterative sequence with errors is defined in [10] as follows: for a nonempty subset K of a Banach space X and an operator $T : K \to X$, the sequence $\{x_n\}$ in K is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\,x_n + \alpha_n T y_n + u_n, \qquad & x_0 \in K, \\ y_n &= (1-\beta_n)\,x_n + \beta_n T x_n + v_n, \qquad & n \ge 0, \end{aligned} \tag{2.3}$$

where $\{u_n\}$, $\{v_n\}$ are two sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying certain conditions.

It is clear that the Mann and Ishikawa iterative sequences are all special cases of the Ishikawa iterative sequence with errors.

In the proof of our main results, we shall need the following results.

LEMMA 2.1. *[11, p. 303] Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two nonnegative real sequences satisfying*

$$a_{n+1} \le a_n + b_n, \qquad \forall n \ge 0.$$

If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

LEMMA 2.2. *Let $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing function with $\phi(0) = 0$ and let $\{\lambda_n\}_{n=0}^{\infty}$, $\{\delta_n\}_{n=0}^{\infty}$, and $\{c_n\}$ be three nonnegative real sequences satisfying*

(i) $\sum_{n=0}^{\infty} \lambda_n = \infty$,
(ii) $\sum_{n=0}^{\infty} \delta_n < \infty$ *and* $\sum_{n=0}^{\infty} c_n < \infty$.

Suppose $\{a_n\}_{n=0}^{\infty}$ is a nonnegative real sequence satisfying

$$a_{n+1} \le (1+\delta_n)\,a_n - \lambda_n \frac{\phi(a_{n+1})}{1+\phi(a_{n+1})+a_{n+1}}\,a_n + c_n, \qquad \forall n \ge 0. \tag{2.4}$$

Then $\lim_{n\to\infty} a_n = 0$.

PROOF. By the inequality (2.4), we have

$$a_{n+1} \le (1+\delta_n)\,a_n + c_n, \qquad \forall n \ge 0. \tag{2.5}$$

It follows that

$$0 \le a_{n+1} \le a_k \prod_{j=k}^{n} (1+\delta_j) + \sum_{j=k}^{n} \left[\prod_{i=j+1}^{n} (1+\delta_i)\right] c_j, \qquad \forall n \ge 0. \tag{2.6}$$

Since $\sum_{n=0}^{\infty} \delta_n < \infty$, $\prod_{n=0}^{\infty}(1+\delta_n)$ converges to a finite number. Hence it follows from condition (ii) and inequality (2.6) that $\{a_n\}_{n=0}^{\infty}$ is bounded. Suppose $a_n \le D$, $\forall n \ge 0$. Then (2.5) implies that

$$a_{n+1} \le a_n + D\delta_n + c_n, \qquad \forall n \ge 0.$$

By using Lemma 2.1 with $b_n = D\delta_n + c_n$, $\lim_{n\to\infty} a_n$ exists. Let $\lim_{n\to\infty} a_n = \delta \geq 0$. We claim that $\delta = 0$. If $\delta > 0$, then there exists a nonnegative integer $N \geq 0$ such that $a_n > \delta/2$, $\forall n \geq N$. Since ϕ is strictly increasing, we have

$$\phi(a_{n+1}) \geq \phi\left(\frac{\delta}{2}\right), \qquad \forall n \geq N. \tag{2.7}$$

It follows from (2.4) and (2.7) that

$$\lambda_n \frac{\phi(\delta/2)\delta/2}{1+\phi(D)+D} \leq \lambda_n \frac{\phi(a_{n+1})}{1+\phi(a_{n+1})+a_{n+1}} a_n \leq a_n - a_{n+1} + D\delta_n + c_n, \tag{2.8}$$

and so

$$\begin{aligned} \frac{\phi(\delta/2)\delta/2}{1+\phi(D)+D} \sum_{j=N}^{n} \lambda_j &\leq a_N - a_{n+1} + \sum_{j=N}^{n} (D\delta_j + c_j) \\ &\leq a_N + \sum_{j=N}^{n} (D\delta_j + c_j). \end{aligned} \tag{2.9}$$

By condition (ii) and inequality (2.9), we have $\sum_{n=0}^{\infty} \lambda_n < \infty$, which contradicts condition (i). Hence, $\lim_{n\to\infty} a_n = \delta = 0$. This completes the proof.

REMARK 2.1. If $c_n = 0$ $\forall n \geq 0$, Lemma 2.2 reduces to Lemma of Osilike [8].

3. MAIN RESULTS

THEOREM 3.1. *Let X be an arbitrary Banach space and $T : D(T) \subset X \to X$ be a Lipschitz ϕ-strongly accretive operator with domain $D(T)$ and range $R(T)$. Suppose the equation $Tx = f$ has a solution. Let $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ be two sequences in X and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ be two real sequences in $[0,1]$ satisfying*

(i) $\sum_{n=0}^{\infty} \|v_n\| < \infty$ *and* $\sum_{n=0}^{\infty} \|u_n\| < \infty$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n)\beta_n < \infty$,
(iv) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

Suppose that, for some $x_0 \in D(T)$, the Ishikawa type iterative sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ with errors defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n(f + (I-T)x_n) + v_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n(f + (I-T)y_n) + u_n, \qquad \forall n \geq 0 \end{aligned} \tag{3.1}$$

are both contained in $D(T)$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of the equation $Tx = f$.

PROOF. By inequality (1.2), if the equation $Tx = f$ has a solution, then the solution is unique. Let $x^* \in D(T)$ be the unique solution and let L be the Lipschitz constant of T. Define an operator $S : D(T) \to X$ by

$$Sx = f + (I-T)x, \qquad \forall x \in D(T).$$

Then x^* is a fixed point of S, and S is also Lipschitz with constant $L_* = 1 + L$. Since T is ϕ-strongly accretive, we have that for all $x, y \in D(T)$,

$$\begin{aligned} \langle (I-S)x - (I-S)y, j(x-y)\rangle &= \langle Tx - Ty, j(x-y)\rangle \\ &\geq \phi(\|x-y\|)\|x-y\| \\ &\geq \frac{\phi(\|x-y\|)}{1+\phi(\|x-y\|)+\|x-y\|}\|x-y\|^2 \\ &= \sigma(x,y)\|x-y\|^2, \end{aligned}$$

where $\sigma(x,y) = \phi(\|x-y\|)/(1+\phi(\|x-y\|)+\|x-y\|) \in [0,1)$ for all $x, y \in X$. Hence,

$$\langle (I-S)x - \sigma(x,y)x - ((I-S)y - \sigma(x,y)y), j(x-y)\rangle \geq 0.$$

It follows from [12, Lemma 1.1] that for all $x, y \in D(T)$ and $r > 0$,

$$\|x-y\| \leq \|x-y+r[(I-S)x - \sigma(x,y)x - ((I-S)y - \sigma(x,y)y)]\|. \tag{3.2}$$

By (3.1), we obtain

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n S y_n - u_n \\ &= (1+\alpha_n)x_{n+1} + \alpha_n\left[(I-S)x_{n+1} - \sigma(x_{n+1},x^*)x_{n+1}\right] - (1-\sigma(x_{n+1},x^*))\alpha_n x_n \\ &\quad + (2-\sigma(x_{n+1},x^*))\alpha_n^2(x_n - Sy_n) + \alpha_n(Sx_{n+1} - Sy_n) + \alpha_n\sigma(x_{n+1},x^*)u_n - 2\alpha_n u_n - u_n. \end{aligned}$$

On the other hand, we have

$$x^* = (1+\alpha_n)x^* + \alpha_n\left[(I-S)x^* - \sigma(x_{n+1},x^*)x^*\right] - (1-\sigma(x_{n+1},x^*))\alpha_n x^*.$$

It follows that

$$\begin{aligned} x_n - x^* = (1+\alpha_n)&(x_{n+1}-x^*) \\ &+ \alpha_n\left[(I-S)x_{n+1} - \sigma(x_{n+1},x^*)x_{n+1} - ((I-S)x^* - \sigma(x_{n+1},x^*)x^*)\right] \\ &- (1-\sigma(x_{n+1},x^*))\alpha_n(x_n - x^*) + (2-\sigma(x_{n+1},x^*))\alpha_n^2(x_n - Sy_n) \\ &+ \alpha_n(Sx_{n+1} - Sy_n) - [1+\alpha_n(2-\sigma(x_{n+1},x^*))]u_n. \end{aligned}$$

Thus we have

$$\begin{aligned} \|x_n - x^*\| \geq (1+\alpha_n)&\left\| x_{n+1} - x^* + \frac{\alpha_n}{1+\alpha_n}\Big[(I-S)x_{n+1} \right. \\ &\left. - \sigma(x_{n+1},x^*)x_{n+1} - ((I-S)x^* - \sigma(x_{n+1},x^*)x^*)\Big]\right\| \\ &- (1-\sigma(x_{n+1},x^*))\alpha_n\|x_n - x^*\| - (2-\sigma(x_{n+1},x^*))\alpha_n^2\|x_n - Sy_n\| \\ &- \alpha_n\|Sx_{n+1} - Sy_n\| - 3\|u_n\| \\ \geq (1+\alpha_n)&\|x_{n+1} - x^*\| - (1-\sigma(x_{n+1},x^*))\alpha_n\|x_n - x^*\| \\ &- (2-\sigma(x_{n+1},x^*))\alpha_n^2\|x_n - Sy_n\| - \alpha_n\|Sx_{n+1} - Sy_n\| - 3\|u_n\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - x^*\| \leq &\frac{1+(1-\sigma(x_{n+1},x^*))\alpha_n}{1+\alpha_n}\|x_n - x^*\| \\ &+ 2\alpha_n^2\|x_n - Sy_n\| + \alpha_n\|Sx_{n+1} - Sy_n\| + 3\|u_n\|. \end{aligned} \tag{3.3}$$

By (3.1), we have

$$\begin{aligned} \|y_n - x^*\| &= \|(1-\beta_n)(x_n - x^*) + \beta_n(Sx_n - x^*) + v_n\| \\ &\leq (1+\beta_n(L_* - 1))\|x_n - x^*\| + \|v_n\| \\ &\leq L_*\|x_n - x^*\| + \|v_n\|, \end{aligned}$$

$$\|x_n - Sy_n\| \leq \|x_n - x^*\| + L_*\|y_n - x^*\| \leq \left(1+L_*^2\right)\|x_n - x^*\| + L_*\|v_n\|, \tag{3.4}$$

$$\begin{aligned} \|Sx_{n+1} - Sy_n\| &\leq L_*\|(1-\alpha_n)(x_n - y_n) + \alpha_n(Sy_n - y_n) + u_n\| \\ &= L_*\|(1-\alpha_n)\beta_n(x_n - Sx_n) - (1-\alpha_n)v_n + \alpha_n(Sy_n - y_n) + u_n\| \\ &\leq \left[L_*(1-\alpha_n)\beta_n(1+L_*) + \alpha_n L_*^2(1+L_*)\right]\|x_n - x^*\| \\ &\quad + L_*(1+\alpha_n L_*)\|v_n\| + L_*\|u_n\|. \end{aligned} \tag{3.5}$$

Using (3.4) and (3.5) in (3.3), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \frac{1 + (1 - \sigma(x_{n+1}, x^*))\alpha_n}{1 + \alpha_n} \|x_n - x^*\| \\
&\quad + \left[\alpha_n (1 - \alpha_n)\beta_n L_* (1 + L_*) + \left(L_*^3 + 3L_*^2 + 2\right)\alpha_n^2\right] \|x_n - x^*\| \\
&\quad + \alpha_n L_* (1 + 2\alpha_n + \alpha_n L_*) \|v_n\| + L_* \alpha_n \|u_n\| + 3 \|u_n\| \\
&\leq [1 + (1 - \sigma(x_{n+1}, x^*))\alpha_n] \left[1 - \alpha_n + \alpha_n^2\right] \|x_n - x^*\| \\
&\quad + \left[\alpha_n (1 - \alpha_n)\beta_n L_* (1 + L_*) + \left(L_n^3 + 3L_n^2 + 2\right)\alpha_n^2\right] \|x_n - x^*\| \\
&\quad + L_* (3 + L_*) \|v_n\| + (3 + L_*) \|u_n\| \\
&\leq \left[1 - \alpha_n \sigma(x_{n+1}, x^*) + \alpha_n^2\right] \|x_n - x^*\| \\
&\quad + \left[\alpha_n (1 - \alpha_n)\beta_n L_* (1 + L_*) + \left(L_*^3 + 3L_*^2 + 2\right)\alpha_n^2\right] \|x_n - x^*\| \\
&\quad + L_* (3 + L_*) \|v_n\| + (3 + L_*) \|u_n\| \\
&= [1 - \alpha_n \sigma(x_{n+1}, x^*)] \|x_n - x^*\| \\
&\quad + \left[\alpha_n (1 - \alpha_n)\beta_n L_* (1 + L_*) + \left(L_*^3 + 3\left(L_*^2 + 1\right)\right)\alpha_n^2\right] \|x_n - x^*\| \\
&\quad + L_* (3 + L_*) \|v_n\| + (3 + L_*) \|u_n\|.
\end{aligned} \tag{3.6}$$

Now let $a_n = \|x_n - x^*\|$, $\delta_n = \alpha_n(1 - \alpha_n)\beta_n L_*(1 + L_*) + (L_*^3 + 3(L_*^2 + 1))\alpha_n^2$, and $c_n = L_*(3 + L_*)\|v_n\| + (3 + L_*)\|u_n\|$. Then the inequality (3.6) reduces to

$$a_{n+1} \leq (1 + \delta_n) a_n - \alpha_n \frac{\phi(a_{n+1})}{1 + \phi(a_{n+1} + a_{n+1})} a_n + c_n.$$

By the assumptions (i)–(iv), we have $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \delta_n < \infty$, and $\sum_{n=0}^{\infty} c_n < \infty$. It follows from Lemma 2.2 that $\lim_{n\to\infty} a_n = 0$, so that $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^*. This completes the proof.

COROLLARY 3.1. *Let X be an arbitrary Banach space and $T : X \to X$ be a Lipschitz ϕ-strongly accretive operator. Suppose the equation $Tx = f$ has a solution. Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two sequences in X and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two real sequences in $[0, 1]$ such that the conditions (i)–(iv) in Theorem 3.1 hold. Then the Ishikawa type iterative sequence with errors defined by (3.1) for an arbitrary $x_0 \in X$ converges strongly to the unique solution of the equation $Tx = f$.*

PROOF. The conclusion follows from Theorem 3.1 with $D(T) = X$.

COROLLARY 3.2. *Let X be an arbitrary Banach space and $T : X \to X$ be a Lipschitz ϕ-strongly accretive operator. Suppose the equation $Tx = f$ has a solution. Let $\{u_n\}_{n=0}^{\infty}$ be a sequence in X and $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence in $[0, 1]$ such that*

(i) $\sum_{n=0}^{\infty} \|u_n\| < \infty$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

Then for any $x_0 \in X$, the Mann type iterative sequence with errors defined by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n (f + (I - T)x_n) + u_n, \qquad \forall n \geq 0$$

converges strongly to the unique solution of the equation $Tx = f$.

PROOF. The conclusion follows from Corollary 3.1 with $\beta_n = 0$ and $v_n = 0$ for all $n \geq 0$.

COROLLARY 3.3. *Let X be an arbitrary Banach space and $T : X \to X$ be a Lipschitz strongly accretive operator. Suppose that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are two sequences in X and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two real sequences in $[0, 1]$ such that the conditions (i)–(iv) in Theorem 3.1 hold.*

Then for any $f \in X$, the Ishikawa type iterative sequence $\{x_n\}_{n=0}^{\infty}$ with errors defined by (3.1) for an arbitrary $x_0 \in X$ converges strongly to the unique solution of the equation $Tx = f$.

PROOF. For any fixed $f \in X$, the existence of a solution of the equation $Tx = f$ follows from [13]. It follows from Corollary 3.1 that the conclusion holds.

THEOREM 3.2. *Let X be an arbitrary Banach space and $A : D(A) \subset X \to X$ be a Lipschitz ϕ-hemicontractive operator with domain $D(A)$. Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two sequences in X and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two real sequences in $[0,1]$ such that the conditions (i)–(iv) in Theorem 3.1 hold. Suppose that for some $x_0 \in D(A)$, the Ishikawa type iterative sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ with errors defined by*

$$\begin{aligned} y_n &= (1-\beta_n)\, x_n + \beta_n A x_n + v_n, \\ x_{n+1} &= (1-\alpha_n)\, x_n + \alpha_n A y_n + u_n, \qquad \forall\, n \ge 0 \end{aligned} \tag{3.7}$$

are both contained in $D(A)$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point x^* of A.

PROOF. The inequality (1.5) implies the fixed point set $F(A)$ of A is singleton. Let x^* be the unique fixed point of A. By (1.5), we have that for all $x \in D(A)$,

$$\langle (I-A)x - (I-A)x^*, j\,(x-x^*)\rangle \ge \phi\,(\|x-x^*\|)\,\|x-x^*\| \ge \sigma\,(x,x^*)\,\|x-x^*\|\,,$$

where $\sigma(x,x^*) = \phi(\|x-x^*\|)/(1+\phi(\|x-x^*\|)+\|x-x^*\|) \in [0,1)$, $\forall\, x \in D(A)$. Hence it follows from Lemma 1.1 of [12] that for all $x \in D(A)$ and $r > 0$,

$$\|x-x^*\| \le \|x-x^* + r\,[(I-A)x - \sigma\,(x,x^*)\,x - ((I-A)x^* - \sigma\,(x,x^*)\,x^*)]\|\,.$$

The rest of argument is now essentially same as in the proof of Theorem 3.1, and hence, is omitted.

COROLLARY 3.4. *Let K be a nonempty closed convex subset of an arbitrary Banach space X and $A : K \to K$ be a Lipschitz strongly pseudocontractive operator. Suppose $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two real sequences in $[0,1]$ such that*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n)\beta_n < \infty$,
(iii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

Then for any $x_0 \in K$, the Ishikawa iterative sequence $\{x_n\}$ defined by

$$\begin{aligned} y_n &= (1-\beta_n)\, x_n + \beta_n A x_n, \\ x_{n+1} &= (1-\alpha_n)\, x_n + \alpha_n A y_n, \qquad \forall\, n \ge 0, \end{aligned} \tag{3.8}$$

converges strongly to the unique fixed point x^* of A.

PROOF. The existence of the unique fixed point x^* of A follows from [14]. Since K is convex and A a selfmapping on K, by the definition of the Ishikawa iterative sequence, for any $x_0 \in K$, we must have $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty} \subset K$. The conclusion follows from Theorem 3.2 with $u_n = v_n = 0$, $\forall\, n \ge 0$.

REMARK 3.1. The results in Section 3 generalize the corresponding results of Osilike [1,8] to Ishikawa type iterative sequences with errors and to arbitrary Banach spaces which do not depend on any geometric structure of the underlying Banach space X, respectively. Hence, it is easy to see that our results also improve and generalize the corresponding results in [4,10,15–21] to the more general class of operators, to Ishikawa type iterative sequences with errors and to arbitrary Banach spaces which do not depend on any geometric structure of the underlying Banach space X.

REFERENCES

1. M.O. Osilike, Iterative solution of nonlinear equations of ϕ-strongly accretive type, *J. Math. Anal. Appl.* **200**, 259–271 (1996).
2. C.E. Chidume and M.O. Osilike, Fixed point iterations for strictly hemi-contractive maps in uniformly smooth Banach spaces, *Numer. Func. Anal. & Optim.* **15**, 779–790 (1994).
3. K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, (1985).
4. C.E. Chidume, Approximation of fixed points of strongly pseudocontractive mappings, *Proc. Amer. Math. Soc.* **120**, 545–551 (1994).
5. W.R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.* **4**, 506–510 (1953).
6. L. Deng and X.P. Ding, Iterative process for Lipschitz local strictly pseudocontractive mappings, *Appl. Math. and Mech.* **15**, 119–123 (1994).
7. X.P. Ding, Iterative process with errors to locally strictly pseudocontractive maps in Banach spaces, *Computers Math. Applic.* **32** (10), 91–97 (1996).
8. M.O. Osilike, Iterative solutions of nonlinear ϕ-strongly accretive operator equations in arbitrary Banach spaces, *J. Math. Anal. Appl.* (to appear).
9. S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.* **44**, 147–150 (1974).
10. L.S. Liu, Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.* **194**, 114–125 (1995).
11. K.K. Tan and H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.* **178**, 301–308 (1993).
12. T. Kato, Nonlinear semigroups and evolution equations, *J. Math. Soc. Japan* **19**, 508–520 (1964).
13. F.E. Browder, Nonlinear mappings of nonexpansine and accretive type in Banach spaces, *Bull. Amer. Math. Soc.* **73**, 875–882 (1967).
14. K. Deimling, Zeros of accretive operators, *Manuscripta Math.* **13**, 365–374 (1974).
15. C.E. Chidume, Iterative approximation of fixed points of Lipschitzian pseudocontractive mappings, *Proc. Amer. Math. Soc.* **99**, 283–288 (1987).
16. C.E. Chidume, An iterative process for nonlinear Lipschitzian strongly accretive mappings in L_p spaces, *J. Math. Anal. Appl.* **151**, 453–461 (1990).
17. L. Deng, Iteration processes for nonlinear Lipschitzian strongly accretive mappings in L_p spaces, *J. Math. Anal. Appl.* **188**, 128–140 (1994).
18. L. Deng and X.P. Ding, Iterative approximation of Lipschitz strictly pseudocontractive mappings in uniformly smooth Banach spaces, *Nonlinear Anal. TMA* **24** (7), 981–987 (1995).
19. C.E. Chidume, Iterative solution of nonlinear equations with strongly accretive operators, *J. Math. Anal. Appl.* **192**, 502–518 (1995).
20. J. Schu, On a theorem of C.E. Chidume concerning the iterative approximation of fixed points, *Math. Nachr.* **153**, 313–319 (1991).
21. K.K. Tan and H.K. Xu, Iterative solutions to nonlinear equations of strongly accretive operators in Banach spaces, *J. Math. Anal. Appl.* **178**, 9–21 (1993).