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Periodic solutions for nonlinear nth order differential equations with delays ☆

Bingwen Liu [a,*], Lihong Huang [b]

[a] *Department of Mathematics, Hunan University of Arts and Science, Changde 415000, PR China*
[b] *College of Mathematics and Econometrics, Hunan University, Changsha 410082, PR China*

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Abstract

By applying the continuation theorem of coincidence degree theory, we establish the existence of 2π-periodic solutions for a class of nonlinear nth order differential equations with delays.

Keywords: nth order differential equation; Periodic solution; Existence; Coincidence degree; Delay

1. Introduction

In this paper, we study the existence of 2π-periodic solutions of the nonlinear nth order delay differential equation

$$x^{(n)} + \sum_{j=2}^{n-1} a_j x^{(j)} + \sum_{i=1}^{m} h_i(x)|x'|^{2\beta_i} + f_1(x)|x'|^2 + f_2\big(x(t-\delta)\big)x'(t-\delta) + g\big(t, x\big(t-\tau(t)\big)\big) = e(t), \tag{1.1}$$

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* Corresponding author.
E-mail address: liubw007@yahoo.com.cn (B. Liu).

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where h_i $(i = 1, 2, \dots, m)$, $f_1, f_2, \tau, e : R \to R$ and $g : R \times R \to R$ are continuous functions, $\tau(t)$ and $e(t)$ are 2π-periodic with respect to t, g is 2π-periodic in the first variable, δ, β_i $(i = 1, 2, \dots, m)$ and a_j $(j = 2, 3, \dots, n-1)$ are constants, and $\beta_i \geqslant 0$.

During the past twenty years, there has been a great amount of work in periodic solutions for the high-order Duffing equation

$$x^{(2k)} + \sum_{j=1}^{k-1} a_j x^{(2j)} + (-1)^{(k+1)} g(t, x) = 0, \tag{1.2}$$

or

$$x^{(2k+1)} + \sum_{j=1}^{k-1} a_j x^{(2j+1)} + g(t, x) = 0. \tag{1.3}$$

Many of these results can be found in [1,3,4,9–11,13] and references cited therein. Among the known results, we find that the assumption

$(\widetilde{H_0})$ $g(t, x)$ is continuous and there are positive constants m_0 and M_0 such that

$$m_0 \leqslant \left|g_x(t, x)\right| \leqslant M_0 \quad \text{for all } (t, x), \tag{1.4}$$

is employed, and it plays an important role in the proofs of these known results (see, for example, [1,3,4,9–11]). It is easy to see that (1.1) includes (1.2) and (1.3) as special cases. Moreover, when $n = 2$, $h_i(x) = 0$ $(i = 1, 2, \dots, m)$, $\delta = 0$, $f_1(x) \equiv 0$, $f_2(x) = f(x)$ and $g(t, x(t - \tau(t))) = g(x(t - \tau(t)))$, Eq. (1.1) reduces to

$$x'' + f(x)x' + g\big(x\big(t - \tau(t)\big)\big) = e(t) \tag{1.5}$$

which has been known as the delayed Liénard equation. Therefore, we consider Eq. (1.1) as a high-order delayed Liénard equation. Arising from problems in applied sciences, it is well known that the existence of periodic solutions of Eq. (1.5) has been extensively studied over the past fifty years (see, for example, [2,6,15–18]). However, when $n \geqslant 2$, $\delta \neq 0$, $\tau(t) \not\equiv 0$, $g(t, x) \neq g(x)$, $h_i(x) \not\equiv 0$ $(i = 1, 2, \dots, m)$, and $f_1(x) \not\equiv 0$, the study of Eq. (1.1) is rare.

Thus, it is worth while to study the existence of the periodic solutions of Eq. (1.1). In this paper, using the continuation theorem of coincidence degree theory, we will give some results on the existence of the 2π-periodic solution to Eq. (1.1) when condition $(\widetilde{H_0})$ is avoided.

If n is even, let $n = 2k$, then Eq. (1.1) becomes

$$\begin{aligned} x^{(2k)} &+ \sum_{j=2}^{2k-1} a_j x^{(j)} + \sum_{i=1}^{m} h_i(x)|x'|^{2\beta_i} + f_1(x)|x'|^2 + f_2\big(x(t-\delta)\big)x'(t-\delta) \\ &+ g\big(t, x\big(t - \tau(t)\big)\big) = e(t), \end{aligned} \tag{1.6}$$

If n is odd, let $n = 2k + 1$, then Eq. (1.1) becomes

$$x^{(2k+1)} + \sum_{j=2}^{2k} a_j x^{(j)} + \sum_{i=1}^{m} h_i(x)|x'|^{2\beta_i} + f_1(x)|x'|^2 + f_2\big(x(t-\delta)\big)x'(t-\delta)$$
$$+ g\big(t, x\big(t-\tau(t)\big)\big) = e(t). \tag{1.7}$$

For ease of exposition, throughout this paper we will adopt the following notations:

$$|x|_p = \left(\int_0^{2\pi} \big|x(t)\big|^p\,dt\right)^{1/p}, \quad |x|_\infty = \max_{t\in[0,2\pi]} \big|x(t)\big|, \quad a^+ = \max\{0, a\},$$
$$\|x\| = \sum_{j=0}^{n-1} |x^{(j)}|_\infty, \quad x^{(0)} = x, \quad f_1'(x) = \frac{df_1(x)}{dx},$$
$$h_i'(x) = \frac{dh_i(x)}{dx} \quad (i = 1, 2, \ldots, m),$$
$$A_1 = 1 - a^+_{2(k-1)} - |a_{2(k-2)}| - a^+_{2(k-3)} - \cdots - |a_4| - a_2^+,$$
$$A_2 = 1 - a^+_{2(k-1)} - |a_{2(k-2)}| - a^+_{2(k-3)} - \cdots - a_4^+ - |a_2|,$$
$$A_3 = a_{2k} - a^+_{2(k-1)} - |a_{2(k-2)}| - a^+_{2(k-3)} - \cdots - |a_4| - a_2^+,$$
$$\bar{A}_3 = 1 - a^+_{2k-1} - |a_{2k-3}| - a^+_{2k-5} - \cdots - |a_5| - a_3^+,$$
$$A_4 = a_{2k} - a^+_{2(k-1)} - |a_{2(k-2)}| - a^+_{2(k-3)} - \cdots - a_4^+ - |a_2|,$$
$$\bar{A}_4 = 1 - a^+_{2k-1} - |a_{2k-3}| - a^+_{2k-5} - \cdots - a_5^+ - |a_3|.$$

It is convenient to introduce the following assumptions:

(H_0) There exists a constant B such that $(-1)^k x h_i(x) \geqslant B$, $i = 1, 2, \ldots, m$.
(H_1) There exists a constant L such that $L \geqslant 0$, and $(-1)^k x f_1(x) \geqslant -L$ for all $x \in R$.
(H_2) $\int_0^{2\pi} e(t)\,dt = 0$.
(H_3) $f_1'(x), h_i'(x) \in C(R, R)$, $(-1)^k h_i'(x) \geqslant 0$ $(i = 1, 2, \ldots, m)$ and $(-1)^k f_1'(x) \geqslant 0$ for all $x \in R$.
(H_4) There exists a constant $d_1 > 0$ such that $(-1)^k x g(t, x) > 0$, $(-1)^k x h_i(x) \geqslant 0$ $(i = 1, 2, \ldots, m)$ and $(-1)^k x f_1(x) \geqslant 0$ for all $t \in R$, $|x| \geqslant d_1$.
(H_5) There exists a constant $d_2 > 0$ such that $(-1)^k x g(t, x) < 0$, $(-1)^k x h_i(x) \leqslant 0$ $(i = 1, 2, \ldots, m)$ and $(-1)^k x f_1(x) \leqslant 0$ for all $t \in R$, $|x| \geqslant d_2$.
(H_6) There exist constants b and A such that

$$\limsup_{|x|\to+\infty} \big|x^{-1} g(t, x)\big| = b < \frac{A-L}{4\pi}.$$

(H_7) There exist constants b and A such that

$$\limsup_{|x|\to+\infty} \big|x^{-1} g(t, x)\big| = b < \frac{A-L}{4}.$$

2. Several lemmas

Let us introduce the auxiliary equation

$$x^{(n)} + \lambda\Bigg[\sum_{j=2}^{n-1} a_j x^{(j)} + \sum_{i=1}^{m} h_i(x)|x'|^{2\beta_i} + f_1(x)|x'|^2 + f_2\big(x(t-\delta)\big)x'(t-\delta) + g\big(t, x\big(t-\tau(t)\big)\big)\Bigg] = \lambda e(t), \quad \lambda \in (0,1). \tag{2.1$_\lambda$}$$

Let

$$X = \big\{x \mid x \in C^{n-1}(R,R),\ x(t+2\pi) = x(t),\ \text{for all } t \in R\big\}$$

and

$$Y = \big\{x \mid x \in C(R,R),\ x(t+2\pi) = x(t),\ \text{for all } t \in R\big\}$$

be two Banach spaces with the norm

$$\|x\|_X = \|x\| = \sum_{j=0}^{n-1} |x^{(j)}|_\infty \quad \text{and} \quad \|x\|_Y = |x|_\infty = \max_{t\in[0,2\pi]} \big|x(t)\big|.$$

Define a linear operator $L : D(L) \subset X \to Y$ by setting

$$D(L) = \big\{x \mid x \in X,\ x^{(n)} \in C(R,R)\big\}$$

and for $x \in D(L)$,

$$Lx = x^{(n)}. \tag{2.2}$$

We also define a nonlinear operator $N : X \to Y$ by setting

$$Nx = -\lambda\Bigg[\sum_{j=2}^{n-1} a_j x^{(j)} + \sum_{i=1}^{m} h_i(x)|x'|^{2\beta_i} + f_1(x)|x'|^2 + f_2\big(x\big(t-\delta\big)\big)x'(t-\delta) + g\big(t, x\big(t-\tau(t)\big)\big)\Bigg] + \lambda e(t). \tag{2.2$'$}$$

It is easy to see that

$$\operatorname{Ker} L = R \quad \text{and} \quad \operatorname{Im} L = \left\{x \mid x \in Y,\ \int_0^{2\pi} x(s)\,ds = 0\right\}.$$

Thus the operator L is a Fredholm operator with index zero.

Define the continuous projectors $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y/\operatorname{Im} L$ by setting

$$Px(t) = \frac{1}{2\pi}\int_0^{2\pi} x(s)\,ds$$

and

$$Qx(t) = \frac{1}{2\pi}\int_0^{2\pi} x(s)\,ds.$$

Hence, $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Denoting by $L_P^{-1}: \operatorname{Im} L \to D(L) \cap \operatorname{Ker} P$ the inverse of $L|_{D(L)\cap \operatorname{Ker} P}$, one can observe that L_P^{-1} is a compact operator. Therefore, N is L-compact on $\bar{\Omega}$, where Ω is an open bounded subset of X.

In view of (2.2) and (2.2)′, the operator equation

$$Lx = \lambda N x, \quad \lambda \in (0,1),$$

is equivalent to the auxiliary equation $(2.1)_\lambda$.

For convenience of use, we introduce the continuation theorem [5] as follows.

Lemma 2.1. *Let X and Y be two Banach spaces. Suppose that $L: D(L) \subset X \to Y$ is a Fredholm operator with index zero, and $N: \bar{\Omega} \to Y$ is L-compact on $\bar{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that the following conditions are satisfied*:

(1) $Lx \neq \lambda N x$, $\forall x \partial\Omega \cap D(L)$, $\lambda \in (0,1)$;
(2) $Nx \notin \operatorname{Im} L$, $\forall x \partial\Omega \cap \operatorname{Ker} L$;
(3) *The Brower degree*

$$\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.$$

Then equation $Lx = Nx$ has a solution on $\bar{\Omega} \cap D(L)$.

The following lemmas will be useful to prove our main results in Section 3.

Lemma 2.2. *If $x \in C^2(R,R)$, $x(t+2\pi) = x(t)$, then*

$$\left|x'(t)\right|_2^2 \leqslant \left|x''(t)\right|_2^2. \tag{2.3}$$

Lemma 2.2 is known as Wirtinger inequality, for the proof of which, see [16,17].

Lemma 2.3. *Let (H_4) (or (H_5)) hold. If $x(t)$ is a 2π-periodic solution of $(2.1)_\lambda$, then there exists a constant $d = \max\{d_1, d_2\}$ such that*

$$|x|_\infty \leqslant d + \sqrt{2\pi}\,|x'|_2. \tag{2.4}$$

Proof. Let $x(t)$ be a 2π-periodic solution of Eq. $(2.1)_\lambda$. Integrating $(2.1)_\lambda$ from 0 to 2π, we see that

$$\int_0^{2\pi}\left[\sum_{i=1}^m h_i(x)|x'|^{2\beta_i} + f_1(x)|x'|^2 + g\big(t, x\big(t-\tau(t)\big)\big)\right]dt = \int_0^{2\pi} e(t)\,dt = 0. \tag{2.5}$$

Thus, there exists a $\xi \in [0, 2\pi]$ such that

$$\sum_{i=1}^{m} h_i\big(x(\xi)\big)\big|x'(\xi)\big|^{2\beta_i} + f_1\big(x(\xi)\big)\big|x'(\xi)\big|^2 + g\big(\xi, x\big(\xi - \tau(\xi)\big)\big) = 0.$$

If $|x(\xi)| \leqslant d = \max\{d_1, d_2\}$, then, and using the Schwarz inequality and the following relation:

$$\big|x(t)\big| = \left|x(\xi) + \int_{\xi}^{t} x'(s)\,ds\right| \leqslant d + \int_{0}^{2\pi} \big|x'(s)\big|\,ds, \quad t \in [0, 2\pi], \tag{2.6}$$

we have

$$|x|_\infty = \max_{t\in[0,2\pi]} \big|x(t)\big| \leqslant d + \sqrt{2\pi}\,|x'|_2, \tag{2.7}$$

which implies that (2.4) is satisfied.

Let $|x(\xi)| > d = \max\{d_1, d_2\}$. In view of (H_4) and (H_5), we shall consider two cases as follows.

Case (i). If (H_4) holds, then, using (2.5) and (H_4), we obtain

$$x(\xi) > d_1, \qquad x\big(\xi - \tau(\xi)\big) \leqslant d_1, \tag{2.8}$$

or

$$x(\xi) < -d_1, \qquad x\big(\xi - \tau(\xi)\big) \geqslant -d_1. \tag{2.9}$$

Since $x(t)$ is a continuous function on R, it follows that there exists a constant $\xi_0 \in R$ such that

$$\big|x(\xi_0)\big| \leqslant d_1 \leqslant d.$$

Let $\xi_0 = 2m\pi + \bar{\xi}$, where $\bar{\xi} \in [0, 2\pi]$ and m is an integer. Then,

$$\big|x(\bar{\xi})\big| = \big|x(\xi_0)\big| \leqslant d_1 \leqslant d,$$

which, together with (2.6) and (2.7), implies that (2.4) is true.

Case (ii). If (H_5) holds, then by a similar argument as in the proof of case (i), we see that (2.4) holds true. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Let* (H_4) (*or* (H_5)) *hold, if* $x(t)$ *is a* 2π*-periodic solution of* $(2.1)_\lambda$, *then, there exists a constant* $d = \max\{d_1, d_2\}$ *such that*

$$|x|_2 \leqslant 2|x'|_2 + \sqrt{2\pi}\,d. \tag{2.10}$$

Proof. Let $x(t)$ be a 2π-periodic solution of Eq. $(2.1)_\lambda$. From the proof of Lemma 2.3, one can observe that there exists a constant $t_0 \in [0, 2\pi]$ such that

$$\big|x(t_0)\big| \leqslant d.$$

Let

$$y(t)=\begin{cases} x(t+t_0-2\pi)-x(t_0), & 2\pi-t_0\leqslant t\leqslant 2\pi,\\ x(t+t_0)-x(t_0), & 0\leqslant t<2\pi-t_0.\end{cases}$$

Then $y(0)=y(2\pi)=0$ and $y'(t)=x'(t+t_0)$, which, together with the following inequality (see [7, Theorem 225]):

$$|y|_2\leqslant 2|y'|_2=2\left|x'(t+t_0)\right|_2=2|x'|_2,$$

imply that

$$\begin{aligned}|x|_2^2&=\int_{t_0}^{2\pi}\left|x(t)\right|^2dt+\int_0^{t_0}\left|x(t)\right|^2dt\\ &=\int_0^{2\pi-t_0}\left|x(t-t_0)\right|^2dt+\int_{2\pi-t_0}^{2\pi}\left|x(t+t_0-2\pi)\right|^2dt\\ &=\left|y(t)+x(t_0)\right|_2^2\leqslant\left(|y|_2+\left|x(t_0)\right|_2\right)^2\leqslant 4|y'|_2^2+4\sqrt{2\pi}\,d|y'|_2+2\pi d^2\\ &=4|x'|_2^2+4\sqrt{2\pi}\,d|x'|_2+2\pi d^2=\left(2|x'|_2+\sqrt{2\pi}\,d\right)^2.\end{aligned}$$

This completes the proof of Lemma 2.4. □

3. Main results

Theorem 3.1. *Let* (H_0)*–*(H_3)*,* (H_6) *and* (H_4) *(or* (H_5)*) hold. Assume that* k *is even,* $\beta_i<1$ $(i=1,2,\dots,m)$ *and* $A=A_1$*. Then Eq.* (1.6) *has at least one* 2π*-periodic solution.*

Proof. We shall seek to apply Lemma 2.1. To do this, it suffices to prove that the set of all possible 2π-periodic solutions of Eq. $(2.1)_\lambda$ are bounded.

Let $x(t)$ be a 2π-periodic solution of Eq. $(2.1)_\lambda$. Multiplying $x(t)$ and Eq. $(2.1)_\lambda$ and integrating from 0 to 2π, we have

$$\begin{aligned}&\int_0^{2\pi}|x^{(k)}|^2dt+\lambda\int_0^{2\pi}\bigl[-a_{2(k-1)}|x^{(k-1)}|^2+a_{2(k-2)}|x^{(k-2)}|^2+\cdots\\ &\qquad+a_4|x''|^2-a_2|x'|^2+xf_1(x)|x'|^2\bigr]dt\\ &\quad=-\lambda\int_0^{2\pi}\sum_{i=1}^m xh_i(x)|x'|^{2\beta_i}dt-\lambda\int_0^{2\pi}g\bigl(t,x\bigl(t-\tau(t)\bigr)\bigr)x\,dt+\lambda\int_0^{2\pi}e(t)x\,dt. \qquad (3.1)\end{aligned}$$

From (H_6), for $\varepsilon=\frac{1}{2}[\frac{A_1-L}{4\pi}-b]$, there exists a constant $N_1>d$ such that

$$\begin{aligned}&\left|g\bigl(t,x\bigl(t-\tau(t)\bigr)\bigr)\right|<(b+\varepsilon)\left|x\bigl(t-\tau(t)\bigr)\right|\leqslant(b+\varepsilon)|x|_\infty\\ &\quad\text{for all } t\in R,\ \left|x\bigl(t-\tau(t)\bigr)\right|>N_1. \qquad (3.2)\end{aligned}$$

Set

$$E_1 = \{t \mid t \in [0, 2\pi],\ |x(t - \tau(t))| \leqslant N_1\},$$
$$E_2 = \{t \mid t \in [0, 2\pi],\ |x(t - \tau(t))| > N_1\},$$

and

$$B_1 = \sup\{|g(t, x)|:\ t \in R,\ |x| \leqslant N_1\}.$$

Then

$$\left\{\int_{E_2} |g(t, x(t - \tau(t)))|^2\, dt\right\}^{1/2} \leqslant \sqrt{2\pi}(b + \varepsilon)|x|_\infty \leqslant \sqrt{2\pi}(b + \varepsilon)\big(d + \sqrt{2\pi}|x'|_2\big). \tag{3.3}$$

From (H_1), (2.4), (2.10), (3.1) and the Schwarz inequality, we have

$$\begin{aligned}
(A_1 - L)|x'|_2^2 &\leqslant (A_1 - L)|x^{(k)}|_2^2 \\
&\leqslant |x^{(k)}|_2^2 - a_{2(k-1)}^+|x^{(k-1)}|_2^2 - |a_{2(k-2)}||x^{(k-2)}|_2^2 - \cdots - |a_4||x''|_2^2 \\
&\quad - a_2^+|x'|_2^2 - \int_0^{2\pi} L|x'|^2\, dt \\
&\leqslant \int_0^{2\pi} |x^{(k)}|^2\, dt + \lambda \int_0^{2\pi} \big[-a_{2(k-1)}|x^{(k-1)}|^2 + a_{2(k-2)}|x^{(k-2)}|^2 + \cdots \\
&\qquad + a_4|x''|^2 - a_2|x'|^2 + xf_1(x)|x'|^2\big]\, dt \\
&= -\lambda \int_0^{2\pi} \sum_{i=1}^m xh_i(x)|x'|^{2\beta_i}\, dt - \lambda \int_0^{2\pi} g(t, x(t - \tau(t)))x\, dt \\
&\quad + \lambda \int_0^{2\pi} e(t)x\, dt \\
&\leqslant |B| \sum_{i=1}^m \int_0^{2\pi} |x'|^{2\beta_i}\, dt + |e|_1|x|_\infty + \int_{E_1} |g(t, x(t - \tau(t)))| \cdot |x|\, dt \\
&\quad + \int_{E_2} |g(t, x(t - \tau(t)))| \cdot |x|\, dt \\
&\leqslant |B| \sum_{i=1}^m \int_0^{2\pi} |x'|^{2\beta_i}\, dt + |e|_1 d + \big[B_1 + (b + \varepsilon)d\big]2\pi d \\
&\quad + \big[|e|_1 + 2B_1 + 2(b + \varepsilon)(\pi + 1)d\big]\sqrt{2\pi}|x'|_2 \\
&\quad + 4\pi(b + \varepsilon)|x'|_2^2.
\end{aligned} \tag{3.4}$$

Combining this and the following inequality:

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|x'|^r\,dt\right)^{1/r} \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi}|x'|^2\,dt\right)^{1/2} \quad \text{for } 0\leqslant r\leqslant 2, \tag{3.5}$$

we have

$$\begin{aligned}(A_1-L)|x'|_2^2 &\leqslant |B|\sum_{i=1}^m\int_0^{2\pi}|x'|^{2\beta_i}\,dt+|e|_1|x|_\infty+\int_{E_1}\bigl|g\bigl(t,x\bigl(t-\tau(t)\bigr)\bigr)\bigr|\cdot|x|\,dt\\ &\quad+\int_{E_2}\bigl|g\bigl(t,x\bigl(t-\tau(t)\bigr)\bigr)\bigr|\cdot|x|\,dt\\ &\leqslant |B|\sum_{i=1}^m\left[\left(\frac{1}{2\pi}\right)^{\beta_i-1}|x'|_2^{2\beta_i}\right]+|e|_1 d+\bigl[B_1+(b+\varepsilon)d\bigr]2\pi d\\ &\quad+\bigl[|e|_1+2B_1+2(b+\varepsilon)(\pi+1)d\bigr]\sqrt{2\pi}\,|x'|_2\\ &\quad+4\pi(b+\varepsilon)|x'|_2^2.\end{aligned} \tag{3.6}$$

In view of (H_6) and $\beta_i<1$ $(i=1,2,\ldots,m)$, (3.6) implies that there exist positive constants C_1 and C_2 such that

$$|x|_\infty<C_1 \quad\text{and}\quad |x'|_2<C_2. \tag{3.7}$$

To estimate $x'(t)$, multiplying $-x''(t)$ and Eq. $(2.1)_\lambda$ and integrating from 0 to 2π, together with (H_3), (2.3) and Schwarz inequality, we have

$$\begin{aligned}A_1|x''|_2^2 &\leqslant A_1|x^{(k+1)}|_2^2\\ &\leqslant \int_0^{2\pi}|x^{(k+1)}|^2\,dt+\lambda\int_0^{2\pi}\bigl[-a_{2(k-1)}|x^{(k)}|^2+a_{2(k-2)}|x^{(k-1)}|^2+\cdots\\ &\qquad +a_4|x^{(3)}|^2-a_2|x''|^2\bigr]\,dt\\ &\quad+\lambda\int_0^{2\pi}\left[\sum_{i=1}^n\frac{1}{2\beta_i+1}h_i'(x)|x'|^{2\beta_i+2}\right]dt+\frac{\lambda}{3}\int_0^{2\pi}f_1'(x)|x'|^4\,dt\\ &=-\lambda\int_0^{2\pi}e(t)x''\,dt+\lambda\int_0^{2\pi}g\bigl(t,x\bigl(t-\tau(t)\bigr)\bigr)x''\,dt\\ &\quad+\lambda\int_0^{2\pi}f_2\bigl(x(t-\delta)\bigr)x'(t-\delta)x''\,dt\\ &\leqslant \bigl(|e|_2+\sqrt{2\pi}\,C_3\bigr)|x''|_2+\bar{C}_3|x'|_2\cdot|x''|_2\\ &\leqslant \bigl(|e|_2+\sqrt{2\pi}\,C_3+\bar{C}_3C_2\bigr)|x''|_2,\end{aligned} \tag{3.8}$$

where $C_3 = \max_{t\in R,\,|x|\leqslant C_1} |g(t,x)|$ and $\bar{C}_3 = \max_{|x|\leqslant C_1} |f_2(x)|$. Thus,

$$|x''|_2 \leqslant \frac{1}{A_1}\big(|e|_2 + \sqrt{2\pi}\, C_3 + \bar{C}_3 C_2\big) := C_4. \tag{3.9}$$

Since $x(t)$ is 2π-periodic, there exists a $T_0 \in (0, 2\pi)$ such that $x'(T_0) = 0$. Therefore,

$$\big|x'(t)\big| = \left|\int_{T_0}^{t} x''(s)\,ds\right| \leqslant \sqrt{2\pi}\cdot\left(\int_0^{2\pi}\big|x''(s)\big|^2\,ds\right)^{1/2} \leqslant \sqrt{2\pi}\, C_4 := C_5. \tag{3.10}$$

Now, we shall estimate $x^{(j)}$ $(j = 2, \ldots, 2k-1)$, multiplying $x^{(2k)}$ and Eq. $(2.1)_\lambda$ and integrating from 0 to 2π, we have

$$\begin{aligned}
A_1|x^{(2k)}|_2^2 &= A_1\int_0^{2\pi} |x^{(2k)}|^2\,dt \\
&\leqslant \int_0^{2\pi} |x^{(2k)}|^2\,dt + \lambda\int_0^{2\pi}\big[-a_{2(k-1)}^{+}|x^{(2k-1)}|^2 - |a_{2(k-2)}||x^{(2k-2)}|^2 - \cdots \\
&\qquad - a_4|x^{(k+2)}|^2 - a_2^{+}|x^{(k+1)}|^2\big]\,dt \\
&\leqslant \int_0^{2\pi} |x^{(2k)}|^2\,dt + \lambda\int_0^{2\pi}\left(\sum_{j=2}^{2k-1} a_j x^{(j)}\right)x^{(2k)}\,dt \\
&= \lambda\int_0^{2\pi}\Bigg[e(t) - \sum_{i=1}^{m} h_i(x)|x'|^{2\beta_i} - f_1(x)|x'|^2 - f_2\big(x(t-\delta)\big)x'(t-\delta) \\
&\qquad - g\big(t, x\big(t-\tau(t)\big)\big)\Bigg]x^{(2k)}\,dt \\
&\leqslant \sqrt{2\pi}\big(|e|_\infty + D_1 + D_2 + D_3\big)|x^{(2k)}|_2,
\end{aligned}$$

where

$$D_1 = \max\left\{\sum_{i=1}^{m}\big|h_i(x)\big||x'|^{2\beta_i} + \big|f_1(x)\big||x'|^2\colon\ |x| \leqslant C_1,\ |x'| \leqslant C_5\right\},$$

$$D_2 = \max\big\{\big|f_2(x)x'\big|\colon\ |x| \leqslant C_1,\ |x'| \leqslant C_5\big\},$$

and

$$D_3 = \max\big\{\big|g(t,x)\big|\colon\ t \in R,\ |x| \leqslant C_1\big\}.$$

Thus, we obtain

$$|x^{(2k)}|_2 \leqslant \frac{1}{A_1}\sqrt{2\pi}\big(|e|_\infty + D_1 + D_2 + D_3\big) := C_6. \tag{3.11}$$

Consequently,

$$|x^{(2k-1)}| \leqslant \sqrt{2\pi}\, C_6$$

and

$$\left|x^{(j)}(t)\right| \leqslant \sqrt{2\pi}\, C_6(\sqrt{2\pi}\,)^{(2k-1-j)}, \quad t \in R,\ j = 2, 3, \ldots, 2k-1. \tag{3.12}$$

Therefore, for all possible 2π-periodic solutions $x(t)$ of $(2.1)_\lambda$, there exists a constant M_1 such that

$$\|x\| = \sum_{j=0}^{2k-1} |x^{(j)}|_\infty < M_1, \tag{3.13}$$

with $M_1 > 0$ independent of λ.

If $x \in \Omega_1 = \{x \mid x \in \operatorname{Ker} L \cap X \text{ and } Nx \in \operatorname{Im} L\}$, then there exists a constant M_2 such that

$$x(t) \equiv M_2 \quad \text{and} \quad \int_0^{2\pi} \left[g(t, M_2) - e(t)\right] dt = \int_0^{2\pi} g(t, M_2)\, dt = 0. \tag{3.14}$$

Thus,

$$\left|x(t)\right| \equiv |M_2| < d \quad \text{for all } x(t) \in \Omega_1. \tag{3.15}$$

Let $M = M_1 + d$. Set

$$\Omega = \left\{ x \mid x \in X,\ \|x\| = \sum_{j=0}^{2k-1} |x^{(j)}|_\infty < M \right\}.$$

Since N is L-compact on $\bar{\Omega}$, it is easy to see from (3.14) and (3.15) that the conditions (1) and (2) in Lemma 2.1 hold.

Furthermore, define the continuous functions $\Psi_1(x, \mu)$ and $\Psi_2(x, \mu)$ by setting

$$\Psi_1(x, \mu) = -(1-\mu)x - \mu \cdot \frac{1}{2\pi} \int_0^{2\pi} \left[g(t, x) - e(t)\right] dt, \quad \mu \in [0, 1],$$

$$\Psi_2(x, \mu) = (1-\mu)x - \mu \cdot \frac{1}{2\pi} \int_0^{2\pi} \left[g(t, x) - e(t)\right] dt, \quad \mu \in [0, 1].$$

If (H_4) holds, then

$$x\Psi_1(x, \mu) \neq 0 \quad \text{for all } x \in \partial\Omega \cap \operatorname{Ker} L.$$

Hence, using the homotopy invariance theorem, we have

$$\begin{aligned} \deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\left\{ -\frac{1}{2\pi} \int_0^{2\pi} \left[g(t, x) - e(t)\right] dt, \Omega \cap \operatorname{Ker} L, 0 \right\} \\ &= \deg\{-x, \Omega \cap \operatorname{Ker} L, 0\} \neq 0. \end{aligned}$$

If (H_5) holds, then

$$x\Psi_2(x,\mu)\neq 0 \quad \text{for all } x\in\partial\Omega\cap\operatorname{Ker}L.$$

Hence, using the homotopy invariance theorem, we obtain

$$\begin{aligned}\deg\{QN,\Omega\cap\operatorname{Ker}L,0\}&=\deg\left\{-\frac{1}{T}\int_0^T\big[g(t,x)-e(t)\big]\,dt,\Omega\cap\operatorname{Ker}L,0\right\}\\&=\deg\{x,\Omega\cap\operatorname{Ker}L,0\}\neq 0.\end{aligned}$$

In view of all the above discussions, we conclude from Lemma 2.1 that Theorem 3.1 is proved. □

Theorem 3.2. *Let (H_0)–(H_3), (H_7) and (H_4) (or (H_5)) hold. Assume that k is even, $\beta_i<1$ $(i=1,2,\dots,m)$, $\tau(t)\equiv\tau$ is a constant, and $A=A_1$. Then Eq.* (1.6) *has at least one 2π-periodic solution.*

Proof. Let $x(t)$ be a 2π-periodic solution of Eq. $(2.1)_\lambda$. From (H_7), for $\varepsilon=\frac{1}{2}[\frac{A_1-L}{4}-b]$, there exists a constant $N_1>d$ such that

$$\big|g\big(t,x(t-\tau)\big)\big|<(b+\varepsilon)\big|x(t-\tau)\big| \quad \text{for all } t\in R,\ \big|x(t-\tau)\big|>N_1. \tag{3.16}$$

This, together with the definitions of B_1, E_1 and E_2, implies that

$$\left\{\int_{E_2}\big|g\big(t,x(t-\tau)\big)\big|^2\,dt\right\}^{1/2}\leqslant(b+\varepsilon)\big|x(t-\tau)\big|_2=(b+\varepsilon)|x|_2. \tag{3.17}$$

In view of (3.4), (3.16) and (3.17), we obtain

$$\begin{aligned}(A_1-L)|x'|_2^2&\leqslant(A_1-L)|x^{(k)}|_2^2\\&\leqslant|B|\sum_{i=1}^m\int_0^{2\pi}|x'|^{2\beta_i}\,dt+|e|_1|x|_\infty+\int_{E_1}\big|g\big(t,x(t-\tau)\big)\big|\cdot|x|\,dt\\&\quad+\int_{E_2}\big|g\big(t,x(t-\tau)\big)\big|\cdot|x|\,dt\\&\leqslant|B|\sum_{i=1}^m\left[\left(\frac{1}{2\pi}\right)^{\beta_i-1}|x'|_2^{2\beta_i}\right]+|e|_1d+\big[B_1+(b+\varepsilon)d\big]2\pi d\\&\quad+\big[|e|_1+2B_1+4(b+\varepsilon)d\big]\sqrt{2\pi}|x'|_2+4(b+\varepsilon)|x'|_2^2.\end{aligned} \tag{3.18}$$

Combining (H_7), (3.18) implies that (3.7) holds.

Now the proof proceeds in the same way as in Theorem 3.1. □

Theorem 3.3. *Let (H_2), (H_3) and (H_4) (or (H_5)) hold. Suppose k is even, and $A=A_1$. Moreover, assume that one of the following conditions holds*:

$(\widetilde{H}_6)$ *There exist constants* F_2*, b and A such that*

$$F_2 = \sup_{x\in R}\left|f_2(x)\right| \quad \textit{and} \quad \limsup_{|x|+\infty}\left|x^{-1}g(t,x)\right| = b < \frac{A-F_2}{2\pi}.$$

$(\widetilde{H}_7)$ $\tau(t)\equiv\tau$ *is a constant, and there exist constants* F_2*, b and A such that*

$$F_2 = \sup_{x\in R}\left|f_2(x)\right| \quad \textit{and} \quad \limsup_{|x|+\infty}\left|x^{-1}g(t,x)\right| = b < \frac{A-F_2}{2}.$$

Then Eq. (1.6) *has at least one* 2π*-periodic solution.*

Proof. Let $x(t)$ be a 2π-periodic solution of Eq. $(2.1)_\lambda$. Multiplying $-x''(t)$ and Eq. $(2.1)_\lambda$ and integrating from 0 to 2π, we can show that (3.7) holds true. In view of $(\widetilde{H}_6)$ and $(\widetilde{H}_7)$, we shall consider two cases as follows.

Case (i). If $(\widetilde{H}_6)$ holds, then from (H_3), (2.3) and the Schwarz inequality, we have

$$\begin{aligned}
A_1|x''|_2^2 &\leqslant A_1|x^{(k+1)}|_2^2 \\
&\leqslant \int_0^{2\pi}|x^{(k+1)}|^2\,dt + \lambda\int_0^{2\pi}\bigl[-a_{2(k-1)}|x^{(k)}|^2 + a_{2(k-2)}|x^{(k-1)}|^2 + \cdots \\
&\qquad + a_4|x^{(3)}|^2 - a_2|x''|^2\bigr]\,dt \\
&\quad + \lambda\int_0^{2\pi}\left[\sum_{i=1}^n \frac{1}{2\beta_i+1}h_i'(x)|x'|^{2\beta_i+2}\right]dt + \frac{\lambda}{3}\int_0^{2\pi} f_1'(x)|x'|^4\,dt \\
&= -\lambda\int_0^{2\pi} e(t)x''\,dt + \lambda\int_0^{2\pi} g\bigl(t,x\bigl(t-\tau(t)\bigr)\bigr)x''\,dt \\
&\quad + \lambda\int_0^{2\pi} f_2\bigl(x(t-\delta)\bigr)x'(t-\delta)x''\,dt \\
&\leqslant \int_0^{2\pi}\left|f_2\bigl(x(t-\delta)\bigr)\right|\cdot\left|x'(t-\delta)\right|\cdot|x''|\,dt + \int_0^{2\pi}\left|g\bigl(t,x\bigl(t-\tau(t)\bigr)\bigr)\right|\cdot|x''|\,dt \\
&\quad + \int_0^{2\pi}\left|e(t)\right|\cdot|x''|\,dt.
\end{aligned} \tag{3.19}$$

For $\varepsilon = \frac{1}{2}[\frac{A_1-F_2}{2} - b]$, from $(\widetilde{H}_6)$, there exists a constant $\bar{N}_1$ $(\bar{N}_1 > d)$ such that

$$\left|g\bigl(t,x\bigl(t-\tau(t)\bigr)\bigr)\right| < (b+\varepsilon)\left|x\bigl(t-\tau(t)\bigr)\right| \quad \text{for all } t\in R,\ \left|x\bigl(t-\tau(t)\bigr)\right| > \bar{N}_1. \tag{3.20}$$

Set

$$\tilde{E}_1 = \{t \mid t \in [0, 2\pi],\ |x(t-\tau(t))| \leqslant \bar{N}_1\},$$
$$\tilde{E}_2 = \{t \mid t \in [0, 2\pi],\ |x(t-\tau(t))| > \bar{N}_1\},$$

and

$$B_2 = \sup\{|g(t,x)|:\ t \in R,\ |x| \leqslant \bar{N}_1\}.$$

In view of (3.3), (2.10), (3.19), (3.20) and the inequalities of Schwarz and Wirtinger, we obtain

$$\begin{aligned} A_1|x''|_2^2 &\leqslant \int_0^{2\pi} |f_2(x(t-\delta))| \cdot |x'(t-\delta)| \cdot |x''|\,dt + \int_{\tilde{E}_1} |g(t, x(t-\tau(t)))| \cdot |x''|\,dt \\ &\quad + \int_{\tilde{E}_2} |g(t, x(t-\tau(t)))| |x''|\,dt + \int_0^{2\pi} |e(t)| \cdot |x''|\,dt \\ &\leqslant |e|_2|x''|_2 + F_2|x'|_2|x''|_2 + B_2\sqrt{2\pi}|x''|_2 \\ &\quad + (b+\varepsilon)(d + \sqrt{2\pi}|x'|_2)\sqrt{2\pi}|x''|_2 \\ &\leqslant [F_2 + 2\pi(b+\varepsilon)]|x''|_2^2 + [|e|_2 + (B_2 + (b+\varepsilon)d)\sqrt{2\pi}]|x''|_2, \end{aligned} \tag{3.21}$$

which, together with $(\widetilde{H}_6)$, implies that there exist positive constants D_1 and D_2 such that

$$|x''|_2 < D_1 \tag{3.22}$$

and

$$|x'|_2 < D_2, \qquad |x|_\infty < D_2. \tag{3.23}$$

Thus, (3.7) holds.

Case (ii). If $(\widetilde{H}_7)$ holds, using a similar fashion, we can show that (3.7) also holds true. Now the proof proceeds in the same way as in Theorem 3.1. □

Similarly to the proofs of Theorems 3.1–3.3, one can prove the following results.

Theorem 3.4. *Suppose that* (H_0)–(H_3), *and* (H_4) (*or* (H_5)) *hold. Assume that one of the following conditions holds*:

(1) (H_6) *is true, k is odd,* $\beta_i < 1$ $(i = 1, 2, \ldots, m)$ *and* $A = A_2$.
(2) (H_7) *is true, k is odd,* $\tau(t) \equiv \tau$ *is a constant,* $\beta_i < 1$ $(i = 1, 2, \ldots, m)$ *and* $A = A_2$.

Then Eq. (1.6) *has at least one* 2π*-periodic solution.*

Theorem 3.5. *Let* (H_2), (H_3), *and* (H_4) (*or* (H_5)) *hold. Assume that one of the following conditions holds*:

(1) $(\widetilde{H_6})$ *is true, k is odd, and* $A = A_2$.
(2) $(\widetilde{H_7})$ *is true, k is odd, $\tau(t) \equiv \tau$ is a constant, and* $A = A_2$.

Then Eq. (1.6) *has at least one 2π-periodic solution.*

We are now in a position to establish the existence of 2π-periodic solutions of Eq. (1.7).

Theorem 3.6. *Let* (H_0)–(H_3), (H_6) *and* (H_4) (*or* (H_5)) *hold. Assume that k is even, $\beta_i < 1$ $(i = 1, 2, \ldots, m)$, $A = A_3$ and $\bar{A}_3 > 0$. Then Eq.* (1.7) *has at least one 2π-periodic solution.*

Proof. Let $x(t)$ be a 2π-periodic solution of Eq. $(2.1)_\lambda$. Similarly to the proof of Theorem 3.1, first multiplying $x^{(j)}(t)$ $(j = 0, 2, 2k + 1)$ and Eq. $(2.1)_\lambda$ and integrating from 0 to 2π, together with (H_0)–(H_3) and (H_4^*) and (or (H_5^*)), we have

$$\|x(t)\| = \sum_{j=0}^{2k} |x^{(j)}|_\infty \leqslant M_1$$

with M_1 independent of λ. Therefore, using a similar argument to the one of the proof of Theorem 3.1, we can show that all of the conditions needed in Lemma 2.1 are satisfied. Thus, Eq. (1.7) has at least one 2π-periodic solution. □

A similar argument leads to

Theorem 3.7. *Let* (H_0)–(H_3), (H_7) *and* (H_4) (*or* (H_5)) *hold. Assume that k is even, $\tau(t) \equiv \tau$ is a constant, $\beta_i < 1$ $(i = 1, 2, \ldots, m)$, $A = A_3$ and $\bar{A}_3 > 0$. Then Eq.* (1.7) *has at least one 2π-periodic solution.*

Theorem 3.8. *Let* (H_2), (H_3), *and* (H_4) (*or* (H_5)) *hold. Assume that one of the following conditions holds*:

(1) *Let* (H_0), (H_1) *and* (H_6) *hold, k is odd, $\beta_i < 1$ $(i = 1, 2, \ldots, m)$, $A_4 > 0$ and $\bar{A}_4 > 0$.*
(2) *Let* (H_0), (H_1) *and* (H_7) *hold, k is odd, $\tau(t) \equiv \tau$ is a constant, $\beta_i < 1$ $(i = 1, 2, \ldots, m)$, $A_4 > 0$ and $\bar{A}_4 > 0$.*
(3) *Let* $(\widetilde{H_6})$ *hold, k is even, $A = A_3$ and $\bar{A}_3 > 0$.*
(4) *Let* $(\widetilde{H_7})$ *hold, k is even, $\tau(t) \equiv \tau$ is a constant, $A = A_3$ and $\bar{A}_3 > 0$.*
(5) *Let* $(\widetilde{H_6})$ *hold, k is odd, $A_4 > 0$ and $\bar{A}_4 > 0$.*
(6) *Let* $(\widetilde{H_7})$ *hold, k is odd, $\tau(t) \equiv \tau$ is a constant, $A_4 > 0$ and $\bar{A}_4 > 0$.*

Then Eq. (1.7) *has at least one 2π-periodic solution.*

4. Examples and remarks

Example 4.1. Let $f_1(x) = 1 - x$, $f_2(x) = \frac{1}{2}\sin x$, $g(t, x(t - \tau(t))) = -x^{1/3}(t - 30)e^{\sin t}$, and $e(t) = 2\cos t$. Then, the following delayed Liénard equation:

$$x'' + f_1(x)|x'|^2 + f_2(x)x' + g\big(t, x\big(t - \tau(t)\big)\big) = e(t) \tag{4.1}$$

has at least one 2π-periodic solution.

Proof. For Eq. (4.1), we have $f_1(x) = 1 - x$, $f_2(x) = \frac{1}{2}\sin x$, $g = -x^{1/3}(t - 30)e^{\sin t}$, $xf_1(x) < \frac{1}{4}$ and $e(t) = 2\cos t$. It is obvious that the assumptions (H_2)–(H_4) and $(\widetilde{H}_6)$ hold. Hence, by Theorem 3.5, Eq. (4.1) has at least one 2π-periodic solution. □

Remark 4.1. In view of T.A. Burton [2], we can see that Eq. (4.1) is a Liénard-type equation with delay $\tau(t) = 30$. Since $f_1(x) \not\equiv 0$ and $|g_x(t, x)|$ is unbounded, the results obtained in [1–4,6,8–18] are invalid for Eq. (4.1). On the other hand, to our best knowledge, existence of 2π-periodic solutions of (1.1) with $n = 2$, $f_1(x) \not\equiv 0$ and $g(t, x) \neq g(x)$ has not been studied in previous works. Hence, the results of this paper are essentially new.

Example 4.2. The equation

$$\begin{aligned} &x^{(8)} + 10x^{(7)} - 20x^{(6)} + \frac{1}{2}x^{(4)} - 10x^{(2)} + (x - 4)|x'|^{8/5} + \frac{1}{4}(x - 2)|x'|^2 \\ &\quad + x^3(t - 1)x'(t - 1) + \frac{1}{17}x(t - 12)e^{-\sin^2 t} = \sin t \end{aligned} \tag{4.2}$$

has at least one 2π-periodic solution.

Proof. It is straightforward to check that all assumptions needed in Theorem 3.1 are satisfied. Therefore, Eq. (4.2) has at least one 2π-periodic solution. □

Remark 4.2. As in [1–4,9–11], the papers [13,14] study the nth order ordinary differential equation only with one nonlinear term $g(t, x)$. Therefore, all the results in [1–4,6,8] are invalid for Eq. (4.2). Moreover it is easy to find that all the results obtained in [9–18] also fail for Eq. (4.2).

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References

[1] K.J. Brown, S.S. Lin, Periodically perturbed conservative systems and a global inverse function theorem, Nonlinear Anal. 4 (1979) 193–201.

[2] T.A. Burton, Stability and Periodic Solution of Ordinary and Functional Differential Equations, Academic Press, Orlando, FL, 1985.

[3] F. Cong, Periodic solutions for $2k$th with nonresonance, Nonlinear Anal. 32 (1997) 787–793.

[4] F. Cong, Q. Hunang, S. Shi, Existence and uniqueness of periodic solutions for $(2n + 1)$th order differential equations, J. Math. Anal. Appl. 241 (2000) 1–9.

[5] R.E. Gaines, J. Mawhin, Coincidence Degree and Nonlinear Differential Equations, Lecture Notes in Math., vol. 568, Springer-Verlag, 1977.

[6] R. Grafton, Periodic solutions certain Liénard equations with Delay, J. Differential Equations 11 (1972) 519–527.
[7] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, Cambridge Univ. Press, London, 1964.
[8] R. Iannacci, M.N. Nkashama, On periodic solutions of forced second order differential equations with deviating arguments, in: Lecture Notes in Math., vol. 1151, Springer-Verlag, 1984, pp. 224–232.
[9] A.C. Lazer, Application of a lemma on bilinear forms to a problem in nonlinear oscillations, Proc. Amer. Math. Soc. 33 (1972) 89–94.
[10] W. Li, Periodic solutions for $2k$th order ordinary differential equation with resonance, J. Math. Anal. Appl. 259 (2001) 157–167.
[11] Y. Li, H. Wang, Periodic solutions of high order Duffing equation, Appl. Math. J. Chinese Univ. 6 (1991) 407–412 (in Chinese).
[12] B. Liu, J. Yu, On the existence of harmonic solution for the n-dimensional Liénard-type equations with delay, Acta Math. Sci. Ser. A 22 (2002) 323–331 (in Chinese).
[13] W. Liu, Y. Li, The existence of periodic solutions for high order duffing equations, Acta Math. Sinica 46 (2003) 49–56 (in Chinese).
[14] Z. Liu, Periodic solutions for nonlinear nth order ordinary differential equations, J. Math. Anal. Appl. 204 (1996) 46–64.
[15] S. Lu, W. Ge, Periodic solutions for a kind of Liénard equations with deviating arguments, J. Math. Anal. Appl. 249 (2004) 231–243.
[16] J. Mawhin, Degré topologique et solutions périodiques des systèmes différentiels nonlineares, Bull. Soc. Roy. Sci. Liège 38 (1969) 308–398.
[17] J. Mawhin, An extension of a theorem of A.C. Lazer on forced nonlinear oscillations, J. Math. Anal. Appl. 40 (1972) 20–29.
[18] P. Omari, G. Villari, F. Zanolin, Periodic solutions of the Liénard equation with one-sided growth restrictions, J. Differential Equations 67 (1987) 278–293.