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An Approximate Method for Solving a Class of Singular Perturbation Problems

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An approximate method for the numerical solution of a class of singularly perturbed two point boundary value problems is presented. The given region is divided into inner and outer regions. The original second-order problem is replaced by an asymptotically equivalent first-order problem and solved as an initial value problem in the inner region. A terminal boundary condition is then obtained from the solution of the inner region problem. In turn, an outer region problem is obtained, by replacing the second-order differential equation by an approximate first-order differential equation with a small deviating argument, and solved efficiently by employing the trapezoidal formula coupled with a discrete invariant imbedding algorithm. The proposed method is iterative on the terminal point of the inner region problem. Several numerical examples have been solved to demonstrate the applicability of the method. © 1988 Academic Press, Inc.

1. Introduction

The numerical solution of singular perturbation problems is currently a field in which active research is going on these days. Singular perturbation problems occur in a number of areas of applied mathematics, science, and engineering, among them fluid mechanics, elasticity, and quantum mechanics. A few notable examples are boundary layer problems, WKB problems, etc. For a detailed discussion on the analytic theory of singular perturbation problems, one may refer to O'Malley [11], Nayfeh [10], Kevorkian and Cole [8], Bender and Orszag [4], Eckhaus [5], Van Dyke [14], and Bellman [3]. Recently several authors have investigated solving singular perturbation problems by numerically constructing asymptotic solutions. A large collection of papers on the numerical solution of singular perturbation problems may be found in Hemker and Miller [7], Miller [9], and Axelsson *et al.* [2]. More recently, a nonasymptotic method also called the boundary value technique, has been introduced by Roberts [13] to solve certain classes of singular perturbation problems. The general

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motivation is to present simpler and more efficient computational techniques which avoid the principal problem of the conventional techniques, namely finding the appropriate asmptotic expansions in the inner and outer regions and matching the coefficients of the inner and outer expansions.

In this paper, we present an approximate method for the numerical solution of a class of linear singularly perturbed two point boundary value problems. It consists of the following steps: (1) The given region is divided into inner and outer regions. (2) The original second-order problem is replaced by an asymptotically equivalent first-order problem and solved as an initial value problem in the inner region. (3) A terminal boundary condition is then obtained from the solution of the inner region problem. (4) In turn, an outer region problem is obtained by replacing the second-order differential equation by an approximate first-order differential equation with a small deviating argument and solved efficiently by employing the trapezoidal formula coupled with a discrete invariant imbedding algorithm. (5) Finally, the solutions of inner and outer region problems are combined to obtain an approximate solution to the original problem. (6) The process is to be repeated for various choices of the terminal point of the inner region problem, until the solution profiles stabilize in both the regions. Several numerical examples have been solved to demonstrate the applicability of the method. Finally, the method is extended to a more general class of problems. Again one numerical example is solved in this general class.

2. Description of the Method

To fix the ideas, we consider a class of singular perturbation problems of the form:

$$\varepsilon y''(x) + [a(x)\, y(x)]' = h(x); \qquad 0 \leqslant x \leqslant 1 \tag{1}$$

with

$$y(0) = \alpha \qquad \text{and} \qquad y(1) = \beta, \tag{2}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$); α, β are given constants; $a(x)$ and $h(x)$ are assumed to be sufficiently continuously differentiable function in $[0, 1]$. Furthermore, we assume that $a(x) \geqslant M > 0$ throughout the interval $[0, 1]$, where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = 0$.

As mentioned the method consists of the following steps:

Step 1. Divide the original region into two regions, an inner region and an outer region. Let x_p be the terminal point or width or thickness of the inner region (boundary layer). Then the inner and outer regions are given by $0 \leqslant x \leqslant x_p$ and $x_p \leqslant x \leqslant 1$, respectively.

Step 2. Replace the original second-order problem by an asymptotically equivalent first-order problem as follows: By integrating Eq. (1), we obtain

$$\varepsilon y'(x) + a(x)\, y(x) = f(x) + K \tag{3}$$

where

$$f(x) = \int h(x)\, dx$$

and K is a constant to be determined.

In order to determine the constant K, we introduce the condition that the reduced equation of (3) should satisfy the boundary condition, $y(1) = \beta$; i.e.,

$$y(1) = \frac{1}{a(1)}\,[f(1) + K] = \beta$$

$$K = a(1)\,\beta - f(1). \tag{4}$$

Remark 1. This choice of K ensures that the solution of the reduced problem of Eq. (1) satisfies the reduced equation of the Eq. (3).

Now, we adjoin the condition (which we drop whenever we formulate the reduced problem of Eq. (1)) $y(0) = \alpha$ to Eq. (3) to obtain a first-order problem

$$\varepsilon y'(x) + a(x)\, y(x) = f(x) + K \tag{5}$$

$$y(0) = \alpha \tag{6}$$

where the constant K is given by Eq. (4). Thus in a manner of speaking, we have replaced the original second-order problem (1)–(2) with the asymptotically equivalent first-order problem (5)–(6).

We choose the transformation

$$t = x/\varepsilon \tag{7}$$

to create a new differential equation. Using (7), rescale Eq. (5) with

$$y(x) = y(t\varepsilon) = Y(t) \tag{8.1}$$

$$y'(x) = \frac{y'(t\varepsilon)}{\varepsilon} = \frac{Y'(t)}{\varepsilon} \tag{8.2}$$

$$a(x) = a(t\varepsilon) = A(t) \tag{8.3}$$

$$f(x) = f(t\varepsilon) = F(t) \tag{8.4}$$

to obtain the new differential equation

$$Y'(t) + A(t)\, Y(t) = F(t) + K. \tag{9}$$

The initial condition for the Eq. (9) is determined by (7), (8.1), and (6),

$$Y(0) = \alpha. \tag{10}$$

Theoretically the differential equation (9) can be solved over the entire interval $0 \leqslant t \leqslant 1/\varepsilon$ with the initial condition (10).

Practically, the interval $[0, 1/\varepsilon]$ becomes unreasonably large as $\varepsilon \to 0$ so we limit the range to $[0, t_p]$, where $t_p = x_p/\varepsilon \lll 1/\varepsilon$. Hence, it leads the inner region problem as an initial value problem:

$$Y'(t) + A(t)\, Y(t) = F(t) + K \tag{11}$$

for

$$0 \leqslant t \leqslant t_p \qquad \text{with} \quad Y(0) = \alpha. \tag{12}$$

We solve this inner region problem (11) with (12) to obtain the solutions over the interval $0 \leqslant t \leqslant t_p$.

The analytical solution of Eq. (11), using the initial condition (12), is given by

$$Y(t) = \left[\exp\left(-\int_0^t A(\xi)\, d\xi\right)\right] \times \left[\int_0^t (F(s) + K) \exp\left(\int_0^s A(\xi)\, d\xi\right) ds + \alpha\right]. \tag{13}$$

Step 3. Obtain the terminal boundary condition from Eq. (13) and denote

$$Y(t_p) = \bar{\alpha}. \tag{14}$$

Hence from Eqs. (7), (8.1), and (14) we get

$$y(x_p) = \bar{\alpha}. \tag{15}$$

Step 4. Obtain the outer region problem as follows: Let us denote $x_p = \delta$ (this is only for our convenience and notational simplicity) and then it is clear that $0 < \delta \ll 1$. By using Taylor series expansion in the neighbourhood of the point x, we have

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x) \tag{16}$$

and consequently, Eq. (1) is replaced by the first-order differential equation with a small deviating argument

$$\begin{aligned} &2\varepsilon y(x-\delta) - 2\varepsilon y(x) + 2\varepsilon\, \delta y'(x) + \delta^2 a(x)\, y'(x) \\ &\quad + \delta^2 a'(x)\, y(x) = \delta^2 h(x) \end{aligned} \tag{17}$$

for $\delta \leqslant x \leqslant 1$ with the boundary conditions

$$y(\delta) = \bar{\alpha} \qquad \text{and} \qquad y(1) = \beta. \tag{18}$$

Transition from Eq. (1) to Eq. (17) is admitted, because of the condition that δ is small ($0 < \delta \ll 1$). Further details on the validity of this transition can be found in Elsgolts [6, pp. 243, 244]. We rewrite Eq. (17) in the convenient form

$$y'(x) = p(x)\, y(x-\delta) + q(x)\, y(x) + r(x) \tag{19}$$

for $\delta \leqslant x \leqslant 1$, where

$$p(x) = \frac{-2\varepsilon}{2\varepsilon\delta + \delta^2 a(x)} \tag{20.1}$$

$$q(x) = \frac{2\varepsilon - \delta^2 a'}{2\varepsilon\delta + \delta^2 a(x)} \tag{20.2}$$

$$r(x) = \frac{\delta^2 h(x)}{2\varepsilon\delta + \delta^2 a(x)}. \tag{20.3}$$

Now, we describe the method for numerically solving the outer region problem given by Eq. (19) with the boundary conditions given by (18). We divide the interval $[\delta, 1]$ into N equal parts with mesh size h; i.e.,

$$h = \frac{1-\delta}{N} \qquad \text{and} \qquad x_i = \delta + ih \qquad \text{for} \quad i = 0, 1, 2, ..., N.$$

By integrating by parts Eq. (19) in $[x_i, x_{i+1}]$ ($i = 1, 2, ..., N-1$.) we get

$$y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} [p(x)\, y(x-\delta) + q(x)\, y(x) + r(x)]\, dx.$$

By making use of the trapezoidal formula for evaluating the integrals approximately, we obtain

$$y(x_{i+1}) - y(x_i) = \frac{h}{2}\left[p(x_{i+1})\, y(x_{i+1} - \delta) + p(x_i)\, y(x_i - \delta)\right] + \frac{h}{2}\left[q(x_{i+1})\, y(x_{i+1}) + q(x_i)\, y(x_i)\right] + \frac{h}{2}\left[r(x_{i+1}) + r(x_i)\right]. \tag{21}$$

Again, by means of Taylor series expansion, we have

$$y(x-\delta) \approx y(x) - \delta y'(x)$$

and, then by approximating $y'(x)$ by linear interpolation, we get

$$y(x_i - \delta) \approx y(x_i) - \delta\left(\frac{y(x_i) - y(x_{i-1})}{h}\right) = \left(1 - \frac{\delta}{h}\right) y(x_i) + \frac{\delta}{h} y(x_{i-1}). \tag{22}$$

Hence, by making use of Eq. (22) in Eq. (21) leads, after simple manipulation (rearrangement), to the final three-term recurrence relationship, namely

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \tag{23}$$

for $i = 1, 2, ..., N-1$, where

$$E_i = -\frac{\delta}{2} p_i \tag{24.1}$$

$$F_i = 1 + \frac{\delta}{2} p_{i+1} + \frac{h}{2}\left(1 - \frac{\delta}{h}\right) p_i + \frac{h}{2} q_i \tag{24.2}$$

$$G_i = 1 - \frac{h}{2}\left(1 - \frac{\delta}{h}\right) p_{i+1} - \frac{h}{2} q_{i+1} \tag{24.3}$$

$$H_i = \frac{h}{2}\left[r_{i+1} + r_i\right] \tag{24.4}$$

and $y_i = y(x_i)$, $p_i = p(x_i)$, $q_i = q(x_i)$, and $r_i = r(x_i)$. Equation (23) gives a system of $(N-1)$ equations with $(N+1)$ unknowns y_0 to y_N. The two given boundary conditions (18) together with these $(N-1)$ equations are then sufficient to solve the unknowns y_i's. The solution of the tridiagonal system (23) can easily be obtained by using an efficient algorithm called "discrete invariant imbedding" (Angel and Bellman [1]). In this algorithm we set a difference relation of the form

$$y_i = W_i y_{i+1} + T_i, \tag{25.1}$$

where W_i and T_i correspond to $W(x_i)$ and $T(x_i)$ are to be determined. From (25.1) we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1}. \tag{25.2}$$

Substituting (25.2) in (23),

$$\begin{aligned} E_i(W_{i-1} y_i + T_{i-1}) - F_i y_i + G_i y_{i+1} = H_i \\ y_i = \frac{G_i}{F_i - E_i W_{i-1}} y_{i+1} + \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}. \end{aligned} \tag{25.3}$$

By comparing Eq. (25.3) with (25.1), we get the recurrence relations

$$W_i = \frac{G_i}{F_i - E_i W_{i-1}} \tag{26.1}$$

and

$$T_i = \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}. \tag{26.2}$$

To solve these recurrence relations for $i = 1, 2, ..., N-1$, we need to know the initial conditions for W_0 and T_0. This can be done by considering the boundary condition $y(\delta) = \bar{\alpha}$, as follows

$$y_0 = \bar{\alpha} = w_0 y_1 + T_0.$$

If we choose $W_0 = 0$, then $T_0 = \bar{\alpha}$. Using these initial values, we first compute W_i and T_i for $i = 1, 2, ..., N-1$ from (26.1) and (26.2) in the forward process. Then we obtain the solutions y_i for $i = N-1, N-2, ..., 2, 1$; in the backward process from Eq. (25.1) using the remaining boundary condition $y_N = \beta$.

Step 5. Adjoin the solutions of inner and outer region problems to obtain an approximate solution to the original problem (1)–(2) over the interval $0 \leqslant x \leqslant 1$.

Step 6. Repeat the process for different choices of x_p (terminal point of the inner region), until the solution profiles do not differ materially from iteration to iteration. For a computational point of view, we use an absolute error criteria, namely

$$|y(x)^{m+1} - y(x)^m| \leqslant \sigma, \qquad x_p \leqslant x \leqslant 1, \tag{27}$$

where

$$y(x)^m = m\text{th iteration of the outer region solution},$$

and

$$\sigma = \text{prescribed tolerance bound}.$$

Remark 2. We have already mentioned that the differential equation (9) is valid over the entire interval $0 \leqslant t \leqslant 1/\varepsilon$. Hence, as an alternative to the solution of the outer region problem (19)–(18), we may use the solution of the initial value problem (9)–(10) over the interval $x_p \leqslant x \leqslant 1$.

3. Numerical Examples

To demonstrate the applicability of the method, we will discuss three examples: a homogeneous singular perturbation problem (SPP), a non-homogeneous SPP, and a SPP with variable coefficients. These examples have been chosen because either analytical or approximate solutions are available for comparison.

Example 1. Consider the following homogeneous SPP from Kevorkian and Cole [8, p. 33, Eqs. 2.3.26 and 2.3.27 with $\alpha = 0$],

$$\varepsilon y''(x) + y'(x) = 0 \tag{28.1}$$

$$\text{for} \quad 0 \leqslant x \leqslant 1 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y(1) = 1 \tag{28.2}$$

The exact solution is given by

$$y(x) = \frac{(1 - \exp(-x/\varepsilon))}{(1 - \exp(-1/\varepsilon))}. \tag{28.3}$$

By integrating Eq. (28.1), we get

$$\varepsilon y'(x) + y(x) = K. \tag{28.4}$$

The constant K is determined by using Eq. (4), as

$$K = y(1) = 1. \tag{28.5}$$

Then, by using the scaling $t = x/\varepsilon$, we get the inner region problem as an initial value problem

$$Y'(t) + Y(t) = 1 \tag{28.6}$$

$$\text{for} \quad 0 \leqslant t \leqslant t_p \quad \text{with} \quad Y(0) = 0. \tag{28.7}$$

The analytical solution of Eq. (28.6), using the initial condition (28.7), is given by

$$Y(t) = 1 - \exp(-t). \tag{28.8}$$

The terminal boundary condition is obtained from the Eq. (28.8) and denoted by

$$y(x_p) = Y(t_p) = \bar{\alpha}. \tag{28.9}$$

In turn, the outer region problem is obtained and solved by using Step 4.

The computational results are presented in Tables IA, IB, for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

TABLE IA

Numerical Results for Example 1, $\varepsilon = 10^{-3}$

x \ $t_p \rightarrow$	5 $y(x)$	10 $y(x)$	Exact solution
0.0	0.00000000	0.00000000	0.00000000
$5.0(10^{-4})$	0.39346933	0.39346933	0.39346933
$1.0(10^{-3})$	0.63212056	0.63212056	0.63212056
$2.5(10^{-3})$	0.91791500	0.91791500	0.91971500
$5.0(10^{-3})$	0.99326205	0.99326205	0.99326205
$7.5(10^{-3})$		0.99944691	0.99944691
$1.0(10^{-2})$		0.99995460	0.99995460
$2.0(10^{-2})$			1.00000000
0.1	1.00000000	1.00000000	1.00000000
0.2	1.00000000	1.00000000	1.00000000
0.3	1.00000000	1.00000000	1.00000000
0.4	1.00000000	1.00000000	1.00000000
0.5	1.00000000	1.00000000	1.00000000
0.6	1.00000000	1.00000000	1.00000000
0.7	1.00000000	1.00000000	1.00000000
0.8	1.00000000	1.00000000	1.00000000
0.9	1.00000000	1.00000000	1.00000000
1.0	1.00000000	1.00000000	1.00000000

TABLE IB

Numerical Results for Example 1, $\varepsilon = 10^{-4}$

x \ $t_p \rightarrow$	5 $y(x)$	10 $y(x)$	Exact solution
0.0	0.00000000	0.00000000	0.00000000
$5.0(10^{-5})$	0.39346933	0.39346933	0.39346933
$1.0(10^{-4})$	0.63212056	0.63212056	0.63212056
$2.5(10^{-4})$	0.91791500	0.91791500	0.91791500
$5.0(10^{-4})$	0.99326205	0.99326205	0.99326205
$7.5(10^{-4})$		0.99944691	0.99944691
$1.0(10^{-3})$		0.99995460	0.99995460
$2.0(10^{-3})$			1.00000000
0.1	1.00000000	1.00000000	1.00000000
0.2	1.00000000	1.00000000	1.00000000
0.3	1.00000000	1.00000000	1.00000000
0.4	1.00000000	1.00000000	1.00000000
0.5	1.00000000	1.00000000	1.00000000
0.6	1.00000000	1.00000000	1.00000000
0.7	1.00000000	1.00000000	1.00000000
0.8	1.00000000	1.00000000	1.00000000
0.9	1.00000000	1.00000000	1.00000000
1.0	1.00000000	1.00000000	1.00000000

EXAMPLE 2. Consider the following non-homogeneous SPP from fluid dynamics for fluid of small viscosity, Reinhardt [12, Example 2],

$$\varepsilon y''(x) + y'(x) = 1 + 2x \tag{29.1}$$

$$\text{for} \quad 0 \leqslant x \leqslant 1 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y(1) = 1. \tag{29.2}$$

The exact solution is given by

$$y(x) = x(x + 1 - 2\varepsilon) + (2\varepsilon - 1)\frac{(1 - \exp(-x/\varepsilon))}{(1 - \exp(-1/\varepsilon))}. \tag{29.3}$$

By integrating Eq. (29.1), we get

$$\varepsilon y'(x) + y(x) = x + x^2 + K. \tag{29.4}$$

The constant K is determined by using Eq. (4), as

$$K = y(1) - 1 - 1^2 = -1. \tag{29.5}$$

Then, by using the scaling $t = x/\varepsilon$, we get the inner region problem as an initial value problem

$$Y'(t) + Y(t) = t\varepsilon + t^2\varepsilon^2 - 1 \tag{29.6}$$

$$\text{for} \quad 0 \leqslant t \leqslant t_p \quad \text{with} \quad Y(0) = 0. \tag{29.7}$$

The analytical solution of Eq. (29.6), using the initial condition (29.7), is given by

$$Y(t) = t\varepsilon + t^2\varepsilon^2 - 1 - \varepsilon(2t\varepsilon + 1) + 2\varepsilon^2 + (1 + \varepsilon - 2\varepsilon^2)\, e^{-t}. \tag{29.8}$$

The terminal boundary condition is obtained from Eq. (29.8) and denoted by

$$y(x_p) = Y(t_p) = \bar{\alpha}. \tag{29.9}$$

In turn, the outer region problem is obtained and solved by using Step 4.

The computational results are presented in the Tables IIA and IIB, for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

EXAMPLE 3. Consider the following SPP with variable coefficients from Kevorkian and Cole [8, p. 33, Eqs. 2.3.26 and 2.3.27 with $\alpha = -\frac{1}{2}$,

$$\varepsilon y''(x) + \left(1 - \frac{x}{2}\right) y'(x) - \frac{1}{2} y(x) = 0 \tag{30.1}$$

$$\text{for} \quad 0 \leqslant x \leqslant 1 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y(1) = 1. \tag{30.2}$$

TABLE IIA

Numerical Results for Example 2, $\varepsilon = 10^{-3}$

x \ $t_p \rightarrow$	5 $y(x)$	10 $y(x)$	Exact solution
0.0	0.00000000	0.00000000	0.00000000
$5.0(10^{-4})$	−0.39336275	−0.39336275	−0.39218315
$1.0(10^{-3})$	−0.63175242	−0.63175242	−0.62985732
$2.5(10^{-3})$	−0.91632984	−0.91632984	−0.91357792
$5.0(10^{-3})$	−0.98923834	−0.98923834	−0.98626053
$7.5(10^{-3})$		−0.99290311	−0.98990677
$1.0(10^{-2})$		−0.99087255	−0.98787469
$2.0(10^{-2})$			−0.97764000
0.1	−0.88964006	−0.88820001	−0.88820001
0.2	−0.75968005	−0.75840001	−0.75840001
0.3	−0.60972004	−0.60860001	−0.60860001
0.4	−0.43976007	−0.43880000	−0.43880001
0.5	−0.24980013	−0.24899999	−0.24900001
0.6	−0.03984018	−0.03919998	−0.03920000
0.7	0.19011980	0.19060002	0.19060000
0.8	0.44007983	0.44040002	0.44039998
0.9	0.71003990	0.71020001	0.71019998
1.0	1.00000000	1.00000000	1.00000000

TABLE IIB

Numerical Results for Example 2, $\varepsilon = 10^{-4}$

x \ $t_p \rightarrow$	5 $y(x)$	10 $y(x)$	Exact solution
0.0	0.00000000	0.00000000	0.00000000
$5.0(10^{-5})$	−0.39345868	−0.39345868	−0.39334064
$1.0(10^{-4})$	−0.63208377	−0.63208377	−0.63189415
$2.5(10^{-4})$	−0.91775677	−0.91775677	−0.91748140
$5.0(10^{-4})$	−0.99286120	−0.99286120	−0.99256325
$7.5(10^{-4})$		−0.99879643	−0.99849661
$1.0(10^{-3})$		−0.99905378	−0.99875381
$2.0(10^{-3})$			−0.99779639
0.1	−0.88997293	−0.88982004	−0.88982000
0.2	−0.75997696	−0.75984003	−0.75984000
0.3	−0.60998099	−0.60986003	−0.60986000
0.4	−0.43998471	−0.43988004	−0.43988000
0.5	−0.24998811	−0.24990011	−0.24990000
0.6	−0.03999116	−0.03992017	−0.03991999
0.7	0.19000613	0.19005981	0.19006000
0.8	0.49000375	0.44003983	0.44003999
0.9	0.71000171	0.71001990	0.71002001
1.0	1.00000000	1.00000000	1.00000000

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [10, p. 148, Eq. 4.2.32] as our "exact" solution:

$$y(x) = \frac{1}{2-x} - \frac{1}{2} \exp\left(-\left(x - \frac{x}{4}\right)^2 \Big/ \varepsilon \right). \tag{30.3}$$

First we rewrite Eq. (30.1) in the form of Eq. (1), i.e., as

$$\varepsilon y''(x) + \left[\left(1 - \frac{x}{2}\right) y(x) \right]' = 0. \tag{30.4}$$

By integrating Eq. (30.4), we get

$$\varepsilon y'(x) + \left(1 - \frac{x}{2}\right) y(x) = K. \tag{30.5}$$

The constant K is determined by using Eq. (4), as

$$K = (1 - \tfrac{1}{2})\, y(1) = \tfrac{1}{2}. \tag{30.6}$$

Then, by using the scaling $t = x/\varepsilon$, we get the inner region problem as an initial value problem:

$$Y'(t) + \left(1 - \frac{t\varepsilon}{2}\right) Y(t) = \frac{1}{2} \tag{30.7}$$

$$\text{for} \quad 0 \leqslant t \leqslant t_p \quad \text{with} \quad Y(0) = 0. \tag{30.8}$$

Since it is difficult to obtain an analytical (closed form) solution of the problem (30.7)–(30.8), we have solved it numerically by the classical fourth-order Runge–Kutta method and obtained the terminal boundary condition, $y(x_p) = Y(t_p)$. In turn, the outer region problem is obtained and solved by using Step 4.

The computational results are presented in the Tables IIIA and IIIB, for $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

4. A More General Class of Problems

In this section, we extend our method to a more general class of problems of the form

$$\varepsilon y''(x) + [a(x)\, y(x)]' + b(x)\, y(x) = h(x) \tag{31}$$

$$\text{for} \quad 0 \leqslant x \leqslant 1 \quad \text{with} \quad y(0) = \alpha \quad \text{and} \quad y(1) = \beta, \tag{32}$$

TABLE IIIA

Numerical Results for Example 3, $\varepsilon = 10^{-3}$

x \ $t_p \rightarrow$	5 $y(x)$	10 $y(x)$	Exact solution
0.0	0.00000000	0.00000000	0.00000000
$5.0(10^{-4})$	0.19674221	0.19674221	0.19684075
$1.0(10^{-3})$	0.31610610	0.31610610	0.31626441
$2.5(10^{-3})$	0.45928897	0.45928897	0.45951910
$5.0(10^{-3})$	<u>0.49761321</u>	0.49761321	0.49786303
$7.5(10^{-3})$		0.50134929	0.50160159
$1.0(10^{-2})$		<u>0.50223590</u>	0.50248929
$2.0(10^{-2})$			0.50505050
0.1	0.52646507	0.52705440	0.52631579
0.2	0.55570632	0.55630083	0.55555556
0.3	0.58838637	0.58898142	0.58823530
0.4	0.62514970	0.62573835	0.62500000
0.5	0.66681240	0.66738427	0.66666666
0.6	0.71442340	0.71496310	0.71428571
0.7	0.76935448	0.76983835	0.76923077
0.8	0.83343384	0.83382540	0.83333333
0.9	0.90915328	0.90939540	0.90909090
1.0	1.00000000	1.00000000	1.00000000

TABLE IIIB

Numerical Results for Example 3, $\varepsilon = 10^{-4}$

x \ $t_p \rightarrow$	5 $y(x)$	10 $y(x)$	Exact solution
0.0	0.00000000	0.00000000	0.00000000
$5.0(10^{-5})$	0.19673530	0.19673530	0.19674528
$1.0(10^{-4})$	0.31606472	0.31606472	0.31608069
$2.5(10^{-4})$	0.45899055	0.45899055	0.45901360
$5.0(10^{-4})$	0.49672909	0.49672909	0.49675395
$7.5(10^{-4})$		0.49988562	0.49991064
$1.0(10^{-3})$		0.50020234	0.50022737
$2.0(10^{-3})$			0.50050050
0.1	0.52632324	0.52639090	0.52631579
0.2	0.55556350	0.55563144	0.55555556
0.3	0.58824352	0.58831131	0.58823530
0.4	0.62500860	0.62507524	0.62500000
0.5	0.66667552	0.66673988	0.66666666
0.6	0.71429463	0.71435488	0.71428571
0.7	0.76923909	0.76929286	0.76923077
0.8	0.83334073	0.83338355	0.83333333
0.9	0.90908986	0.90912206	0.90909090
1.0	1.00000000	1.00000000	1.00000000

where ε is a small positive parameter; α, β are given constants: $a(x)$, $b(x)$, and $h(x)$ are assumed to be sufficiently continuously differentiable functions in $[0, 1]$; and $a(x) \geqslant M > 0$ throughout the interval $[0, 1]$, where M is some positive constant.

It is clear that there will be difficulty in applying Step 2 (i.e., in the integration process) due to the presence of the term $b(x)\, y(x)$. To overcome this difficulty, we first modify Eq. (31) and then apply our method. For convenience, we shall term this extra step the "preliminary step."

Let y_0 be the solution of the reduced problem of (31)–(32); that is,

$$[a(x)\, y_0(x)]' + b(x)\, y_0(x) = h(x) \tag{33.1}$$

with

$$y_0(1) = \beta. \tag{33.2}$$

Preliminary step. Set up the approximate equation to the given equation (31) as

$$\varepsilon y''(x) + [a(x)\, y(x)]' + b(x)\, y_0(x) = h(x), \tag{34.1}$$

where we have simply replaced the $y(x)$-term by $y_0(x)$, the solution of the reduced problem (33.1)–(33.2). Then, we rewrite this equation (34.1) in the form of Eq. (1), i.e., as

$$\varepsilon y''(x) + [a(x)\, y(x)]' = H(x), \tag{34.2}$$

where $H(x) = h(x) - b(x)\, y_0(x)$.

Now, we can apply our method, Step 1 to Step 6, to the modified problem (34.2)–(32).

In order to verify this approach, we discuss one simple example in detail.

EXAMPLE 4. Consider the following singular perturbation problem from Bender and Orszag [4, p. 480, Problem 9.17 with $\alpha = 0$]:

$$\varepsilon y''(x) + y'(x) - y(x) = 0 \tag{35.1}$$

$$\text{for} \quad 0 \leqslant x \leqslant 1 \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y(1) = 1. \tag{35.2}$$

The exact solution is given by

$$y(x) = \frac{(e^{m_2} - 1)\, e^{m_1 x} + (1 - e^{m_1})\, e^{m_2 x}}{(e^{m_2} - e^{m_1})}, \tag{35.3}$$

where $m_1 = (-1 + \sqrt{1 + 4\varepsilon})/2\varepsilon$ and $m_2 = (-1 - \sqrt{1 + 4\varepsilon})/2\varepsilon$.

From the preliminary step, we get an approximate equation to (35.1) as

$$\varepsilon y''(x) + y'(x) - y_0(x) = 0, \tag{35.4}$$

where $y_0(x) = e^{x-1}$ is the solution of the reduced problem of (35.1)–(35.2); that is,

$$y_0'(x) - y_0(x) = 0 \tag{35.5}$$

with

$$y_0(1) = 1 \tag{35.6}$$

Then rewrite the Eq. (35.4) in the form of (1):

$$\varepsilon y''(x) + y'(x) = e^{x-1}. \tag{35.7}$$

By integrating Eq. (35.7), we get

$$\varepsilon y'(x) + y(x) = e^{x-1} + K. \tag{35.8}$$

The constant K is determined by using Eq. (4), as

$$K = y(1) - e^{1-1} = 0. \tag{35.9}$$

Then by using the scaling $t = x/\varepsilon$, we get the inner region problem as an initial value problem:

$$Y'(t) + Y(t) = e^{t\varepsilon - 1} \tag{36.1}$$

$$\text{for} \quad 0 \leqslant t \leqslant t_p \quad \text{with} \quad Y(0) = 1. \tag{36.2}$$

The analytical solution of Eq. (36.1), using the initial condition (36.2), is given by

$$Y(t) = \frac{e^{t\varepsilon - 1}}{(1+\varepsilon)} + \left[1 - \frac{e^{-1}}{(1+\varepsilon)}\right] e^{-t}. \tag{36.3}$$

The terminal boundary condition is obtained from Eq. (36.3) and denoted by

$$y(x_p) = Y(t_p) = \bar{\alpha}. \tag{36.4}$$

In turn, the outer region problem is obtained and solved by using Step 4.

The computational results are presented in the Tables IVA and IVB, $\varepsilon = 10^{-3}$ and 10^{-4}, respectively.

TABLE IVA

Numerical Results for Example 4, $\varepsilon = 10^{-3}$

x \ $t_p \rightarrow$	5 $y(x)$	10 $y(x)$	Exact solution
0.0	1.00000000	1.00000000	1.00000000
$5.0(10^{-4})$	0.75131915	0.75131915	0.75141688
$1.0(10^{-3})$	0.60055898	0.60055898	0.60079141
$2.5(10^{-3})$	0.42034964	0.42034964	0.42089517
$5.0(10^{-3})$	0.37361576	0.37361576	0.37432568
$7.5(10^{-3})$		0.37062845	0.37136243
$1.0(10^{-2})$		0.37123420	0.37197179
$2.0(10^{-2})$			0.37567774
0.1	0.40663699	0.40693049	0.40693440
0.2	0.44939510	0.44968342	0.44968726
0.3	0.49664925	0.49692805	0.49693177
0.4	0.54887223	0.54913630	0.54913982
0.5	0.60658645	0.60682965	0.60683289
0.6	0.67036937	0.67058437	0.67058726
0.7	0.74085910	0.74103729	0.74103969
0.8	0.81876083	0.81889214	0.81889392
0.9	0.90485402	0.90492660	0.90492758
1.0	1.00000000	1.00000000	1.00000000

TABLE IVB

Numerical Results for Example 4, $\varepsilon = 10^{-4}$

x \ t_p	5 $y(x)$	10 $y(x)$	Exact solution
0.0	1.00000000	1.00000000	1.00000000
$5.0(10^{-5})$	0.75128387	0.75128387	0.75128750
$1.0(10^{-4})$	0.60043713	0.60043713	0.60045052
$2.5(10^{-4})$	0.41982526	0.41982526	0.41986551
$5.0(10^{-4})$	<u>0.37228607</u>	0.37228607	0.37234153
$7.5(10^{-4})$		0.36846828	0.36852598
$1.0(10^{-3})$		<u>0.36823938</u>	0.36829735
$2.0(10^{-3})$			0.36863712
0.1	0.40651796	0.40660421	0.40659073
0.2	0.44927818	0.44936290	0.44934966
0.3	0.49653622	0.49661812	0.49660532
0.4	0.54876515	0.54884274	0.54883059
0.5	0.60648782	0.60655932	0.60654812
0.6	0.67028214	0.67034537	0.67035548
0.7	0.74078687	0.74083919	0.74083101
0.8	0.81870759	0.81874620	0.81874018
0.9	0.90482465	0.90484595	0.90484262
1.0	1.00000000	1.00000000	1.00000000

5. Discussion and Conclusions

We have described an approximate method for the numerical solution of a class of singular perturbation problems. As mentioned, the method is iterative on the terminal point of the inner region. The process is to be repeated for various choices of x_p (terminal point of the inner region), until the solution profiles stabilize in both the regions. The point x_p is not unique but can assume a wide number of values. To reduce the amount of computation, we choose the smallest value of x_p which gives the required accuracy. As an alternative of the solution of the outer region problem (19)–(18), we may use the solution of the initial value problem (9)–(10) over the interval $x_p \leqslant x \leqslant 1$. But for better accurate results we prefer to solve the outer region problem (19–(18) as it is.

We have implemented this method on three problems, a homogeneous SPP, a non-homogeneous SPP, and a SPP with variable coefficients, by taking different values of ε. Only one extra step, called the "preliminary step," is needed to apply the present method to a more general class of problems. We have verified this by solving a more general class of problem.

We have tabulated the numerical results obtained by the present method as well as the exact solution. In the tables, the underlined value indicates that it is a terminal boundary condition obtained by solving the inner region problem, and corresponding x is the terminal point x_p. Then $[0, x_p]$ is an inner region and $[x_p, 1]$ is an outer region. It can be observed from the tables that the present method approximates the exact solution very well. All the computations have been performed on the DEC-10 computer system at IIT Kanpur (India).

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