Classification of irreducible integrable representations for the full toroidal lie algebras

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Received 23 June 2004; received in revised form 27 November 2004
Available online 3 March 2005
Communicated by C. Kassel

\textbf{Abstract}

The purpose of this paper is to classify irreducible integrable modules of the full toroidal Lie-algebras, with finite-dimensional weight spaces and non-zero central charge.

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\textit{MSC:} 17B65; 17B67; 17B68

\section{1. Introduction}

It is well known that the representation theory of Virasoro algebra and affine Kac–Moody Lie-algebra plays an important role both in physics and in mathematics. For example see the book [8] and [10].

The Virasoro algebra acts on any (except when the level is negative of dual coxeter number) highest weight module of the affine Lie-algebra through the use of famous Sugawara operators. It is well known that affine Lie-algebras admit representation on the Fock space and hence admits representation of the Virasoro algebra. Thus the semidirect product of Virasoro algebra and affine Kac–Moody Lie-algebra with common center is an interesting object to study. The generalization of this classical object is the subject of the current paper.

\textsuperscript{*} This work is supported in part by NSF of China (No. 10271076).
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The affine Kac–Moody Lie-algebra is the universal central extension of algebraic maps from one-dimensional torus to finite-dimensional simple Lie-algebra \( \mathcal{G} \). The Virasoro Lie-algebra is the universal central extension of Lie algebra of diffeomorphisms of one-dimensional torus. Both algebras can be defined in an algebraic way as the universal central extension of \( \mathcal{G} \otimes \mathbb{C}[t, t^{-1}] \) for affine Lie-algebra and the universal central extension of \( \text{Der} \mathbb{C}[t, t^{-1}] \) for Virasoro algebra.

The generalization of the affine Kac–Moody Lie-algebra is the so-called toroidal Lie-algebra which can be defined as the universal central extension of \( \mathcal{G} \otimes \mathbb{C}[t_0^{\pm1}, t_1^{\pm1}, \ldots, t_v^{\pm1}] \). To generalize Virasoro algebra we have to first note that \( \text{Der} \mathbb{C}[t_0^{\pm1}, t_1^{\pm1}, \ldots, t_v^{\pm1}] \) is centrally closed for \( v \geq 1 \) [12]. Nevertheless \( \text{Der} \mathbb{C}[t_0^{\pm1}, t_1^{\pm1}, \ldots, t_v^{\pm1}] \) has an interesting abelian extension, which has emerged while generalizing the vertex operator construction on the Fock space in [16]. It is interesting to note that the abelian part is precisely the center of the toroidal Lie-algebra. Thus the semi-direct product of \( \text{Der} \mathbb{C}[t_0^{\pm1}, t_1^{\pm1}, \ldots, t_v^{\pm1}] \) and \( \mathcal{G} \otimes \mathbb{C}[t_0^{\pm1}, t_1^{\pm1}, \ldots, t_v^{\pm1}] \) with common extension which is denoted by \( \tau \) and called full toroidal Lie-algebra has emerged as an interesting mathematical object.

The first important question is to construct a representation for \( \tau \) through known methods. Several attempts have been made in [16,4,5]. Eventually in a remarkable paper [6] Billig succeeded in constructing a class representations for \( \tau \) using vertex operator algebras. These modules are irreducible integrable and have finite-dimensional weight spaces.

The purpose of this paper is to classify irreducible integrable modules for \( \tau \) with finite-dimensional weight spaces. The proof of our result depends heavily on [14,15,9]. In [9] the classification of irreducible integrable modules for a certain proper subalgebra has been attempted and the classification problem has been reduced to the classification of (\( A, \text{Der} A \)) modules where \( A = \mathbb{C}[t_0^{\pm1}, t_1^{\pm1}, \ldots, t_v^{\pm1}] \). In the present work using similar methods we have reduced the problem to the problem for \( A, \text{Der} A \) modules. The classification of irreducible (\( A, \text{Der} A \)) modules with finite-dimensional weight spaces is now available in [15]. Putting the above results together we have classified irreducible integrable modules for \( \tau \) in Theorem 3.3 for the non-zero center case. The zero center case has been given in Theorem 4.3.

2. Basic concepts and results

Let \( \hat{\mathfrak{g}} \) denote a finite-dimensional simple Lie-algebra over \( \mathbb{C} \), \( \hat{\mathfrak{h}} \) a Cartan subalgebra, \( \hat{\mathcal{A}} \) the root system of \( \hat{\mathfrak{g}} \), and \( \hat{\mathcal{A}}_+(\hat{\mathcal{A}}_-) \) the set of positive roots (negative roots). Then \( \hat{\mathfrak{g}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \hat{\mathcal{A}}_+} \hat{\mathfrak{g}}_\alpha \).

For \( \alpha \in \hat{\mathcal{A}} \), let \( \check{\alpha} \in \mathfrak{h} \) be such that \( \alpha(\check{\alpha}) = 2 \). Let \( e_\alpha, e_{-\alpha} \in \hat{\mathfrak{g}}_\alpha \) be such that \( [e_\alpha, e_{-\alpha}] = \check{\alpha}, [\check{\alpha}, e_\alpha] = 2e_\alpha, [\check{\alpha}, e_{-\alpha}] = -2e_{-\alpha} \). Let

\[
\hat{\mathfrak{g}}_+ = \bigoplus_{\alpha \in \hat{\mathcal{A}}_+} \hat{\mathfrak{g}}_\alpha, \quad \hat{\mathfrak{g}}_- = \bigoplus_{\alpha \in \hat{\mathcal{A}}_-} \hat{\mathfrak{g}}_\alpha, \quad \hat{\mathcal{Q}}_+ = \sum_{i=1}^l \mathbb{Z}_+ x_i, \quad \hat{\mathcal{Z}}_+ = \mathbb{N} \cup \{0\}.
\]

Let \( A = \mathbb{C}[t_0^{\pm1}, t_1^{\pm1}, \ldots, t_v^{\pm1}] \) be the ring of Laurent polynomials in commuting variables \( t_0, t_1, \ldots, t_v \). For \( \mathbf{n} = (n_1, n_2, \ldots, n_v) \in \mathbb{Z}^v \), \( n_0 \in \mathbb{Z} \), we denote \( t_0^{n_0} t_1^{n_1} \cdots t_v^{n_v} \) by
Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes A$ be the tensor product of $\mathfrak{g}$ and $A$ with the Lie bracket:

$$[x_1 \otimes f_1, x_2 \otimes f_2] = [x_1, x_2] \otimes f_1 f_2,$$

where $x_1, x_2 \in \mathfrak{g}$, $f_1, f_2 \in A$. Then $\tilde{\mathfrak{g}}$ can be viewed as the algebra of $\mathfrak{g}$-valued polynomial functions on a torus (see [13] for discussion of integrable modules of $\tilde{\mathfrak{g}}$). Let $\tilde{\mathcal{K}}$ be the free $A$-module with basis $\{k_0, k_1, \ldots, k_v\}$ and let $d \tilde{\mathcal{K}}$ be the subspace spanned by all elements of the form $\sum_{i=0}^v r_i t_i^0 r_i k_i$, for $(r_0, r) = (r_0, r_1, \ldots, r_v) \in \mathbb{Z}^{v+1}$. Let $\tilde{A} = \tilde{\mathcal{K}} / d \tilde{\mathcal{K}}$ and denote the image of $t_i^0 r_i k_i$ still by itself. Then $\tilde{A}$ is spanned by the elements $\{t_i^0 r_i k_i | p = 0, 1, \ldots, v, r_0 \in \mathbb{Z}, r \in \mathbb{Z}^v\}$ with the following relations:

$$\sum_{p=0}^v r_p t_i^0 r_i k_p = 0. \quad (2.1)$$

The toroidal Lie algebra associated to $\tilde{\mathfrak{g}}$ is

$$\tilde{\mathfrak{t}} = \tilde{\mathfrak{g}} \otimes A \oplus \tilde{\mathcal{K}}$$

with the bracket:

$$[g_1 \otimes f_1, g_2 \otimes f_2] = [g_1, g_2] \otimes f_1 f_2 + (g_1 | g_2) \sum_{p=0}^v (d_p(f_1) f_2) k_p \quad (2.2)$$

and

$$[\tilde{\mathfrak{t}}, \tilde{\mathcal{K}}] = 0. \quad (2.3)$$

$(\cdot | \cdot)$ is the normal invariant symmetric bilinear form on $\mathfrak{g}$ [11]. $d_p$ is the degree derivation of $A$, i.e.,

$$d_p = t_p \frac{d}{dt_p}, \quad p = 0, 1, \ldots, v.$$
It is known that the algebra $\mathcal{D}$ admits two non-trivial 2-cocyles with values in $\mathcal{K}$ (see [4]):

$$
\phi_1(t_0^{m_0} t_m d_a, t_0^{n_0} t_n b) = -n_a m_b \sum_{p=0}^v m_p t_0^{m_0+n_0} t_m^{m+n} k_p,
$$

$$
\phi_2(t_0^{m_0} t_m d_a, t_0^{n_0} t_n b) = m_a n_b \sum_{p=0}^v m_p t_0^{m_0+n_0} t_m^{m+n} k_p.
$$

Let $\phi$ be an arbitrary linear combination of $\phi_1$ and $\phi_2$. Then there is a corresponding Lie-algebra $\tau = \mathfrak{g} \otimes A \oplus \mathcal{K} \oplus \mathcal{D}$ with the Lie bracket (2.1)–(2.2) and the following:

$$
[t_0^{m_0} t_m d_a, t_0^{n_0} t_n k_b] = n_a t_0^{m_0+n_0} t_m^{m+n} k_b + \delta_{ab} \sum_{p=0}^v m_p t_0^{m_0+n_0} t_m^{m+n} k_p,
$$

(2.4)

$$
[t_0^{m_0} t_m d_a, t_0^{n_0} t_n d_b] = n_a t_0^{m_0+n_0} t_m^{m+n} d_b - m_b t_0^{m_0+n_0} t_m^{m+n} d_a
+ \phi(t_0^{m_0} t_m d_a, t_0^{n_0} t_n d_b),
$$

(2.5)

$$
[t_0^{m_0} t_m d_a, x \otimes t_0^{n_0} t_n] = n_a x \otimes t_0^{m_0+n_0} t_m^{m+n}.
$$

(2.6)

We call $\tau$ the full toroidal Lie-algebra associated to $\mathfrak{g}$ and $\phi$. Let

$$
\mathfrak{h} = \mathfrak{h} \oplus \left( \bigoplus_{i=0}^v \mathbb{C}k_i \right) \oplus \left( \bigoplus_{i=0}^v \mathbb{C}l_i \right).
$$

(2.7)

Then $\mathfrak{h}$ is an abelian Lie subalgebra of $\tau$. Let $\delta_i, A_i \in \mathfrak{h}^\ast (i = 0, 1, \ldots, v)$ be such that

$$
A_i(\mathfrak{h}) = 0, \quad A_i(k_j) = \delta_{ij}, \quad A_i(l_j) = 0, \quad i, j = 0, 1, \ldots, v,
$$

(2.8)

$$
\delta_i(\mathfrak{h}) = 0, \quad \delta_i(k_j) = 0, \quad \delta_i(l_j) = \delta_{ij}, \quad i, j = 0, 1, \ldots, v
$$

(2.9)

and denote $\sum_{i=1}^v m_i \delta_i$ by $\delta_{\mathfrak{m}}$. $\mathfrak{m} = (m_1, m_2, \ldots, m_v) \in \mathbb{Z}^v$. Then $\tau$ has the root space decomposition with respect to $\mathfrak{h}$ as follows:

$$
\tau = \mathfrak{h} \oplus \left( \bigoplus_{\beta \in \mathcal{A}} \tau_\beta \right).
$$

where $\mathcal{A} = \mathcal{A} \cup \{ x + m_0 \delta_0 + \delta_{\mathfrak{m}} | x \in \mathcal{A} \cup \{0\}, \mathfrak{m} \in \mathbb{Z}^v, m_0 \in \mathbb{Z}, (m_0, \mathfrak{m}) \neq (0, 0) \}$ and

$$
\tau_{x+m_0 \delta_0+\delta_{\mathfrak{m}}} = \mathfrak{h} \otimes t_0^{m_0} t_m,
$$

$$
\tau_{m_0 \delta_0+\delta_{\mathfrak{m}}} = \mathfrak{h} \otimes t_0^{m_0} t_m \oplus \left( \bigoplus_{i=0}^v \mathfrak{h} \otimes t_0^{m_0} t_m k_i \right) \oplus \left( \bigoplus_{i=0}^v \mathfrak{h} \otimes t_0^{m_0} t_m l_i \right).
$$
Let
\[ b = \mathcal{H} \oplus \mathcal{D} \]
and
\[ b_+ = \sum_{p=0}^{v} t_0 \mathbb{C}[t_0, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] k \oplus \sum_{p=0}^{v} t_0^{-1} \mathbb{C}[t_0^{-1}, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] d_p, \]
\[ b_- = \sum_{p=0}^{v} t_0^{-1} \mathbb{C}[t_0^{-1}, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] k \oplus \sum_{p=0}^{v} t_0^{-1} \mathbb{C}[t_0^{-1}, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] d_p, \]
\[ b_0 = \sum_{p=0}^{v} \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] k \oplus \sum_{p=0}^{v} \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] d_p, \]
\[ \tau_+ = \hat{g}_+ \otimes \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \oplus \hat{g} \otimes t_0 \mathbb{C}[t_0, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \oplus b_+, \]
\[ \tau_- = \hat{g}_- \otimes \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \oplus \hat{g} \otimes t_0^{-1} \mathbb{C}[t_0^{-1}, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \oplus b_-, \]
\[ \tau_0 = \hat{h} \otimes \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \oplus b_0. \]

Then
\[ b = b_+ \oplus b_0 \oplus b_-, \]
\[ \tau = \tau_+ \oplus \tau_0 \oplus \tau_. \]

Extend \( \alpha \in \hat{A} \) to the element in \( h^* \) by \( \alpha(k_i) = \alpha(d_j) = 0 \) (\( 0 \leq i \leq v \)) and the normal non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( h^* \) to a non-degenerate symmetric bilinear form on \( h^* \) by
\[ \langle \alpha_i | \delta_k \rangle = \langle \alpha_i | A_k \rangle = 0, \quad 1 \leq i, \ j \leq 1, \]
\[ \langle \delta_k | \delta_p \rangle = \langle A_k | A_p \rangle = 0, \quad \langle \delta_k | A_p \rangle = \delta_k p, \quad 0 \leq k, \ p \leq v. \]

For \( \gamma = \alpha + m_0 \delta_0 + \delta_m \in \hat{A}, \) \( \alpha \) is called a real root, if \( \langle \gamma | \gamma \rangle \neq 0. \) Denote the set of all real roots by \( \hat{A}^{\text{re}}. \) Define
\[ \gamma^\vee = \alpha^\vee + \frac{2}{\langle \alpha | \alpha \rangle} \sum_{i=0}^{v} m_i k. \]

Then
\[ \gamma(\gamma^\vee) = \alpha(\gamma^\vee) = 2. \]

Let \( \gamma \) be a real root. Define reflection on \( h^* \) by
\[ r_{\gamma}(\lambda) = \lambda - \lambda(\gamma^\vee) \gamma, \quad \lambda \in h^*. \]

Let \( \mathcal{W} \) be the Weyl group generated by \( \{ r_{\gamma} | \gamma \in A^{\text{re}} \}. \) Then \( \langle \cdot, \cdot \rangle \) defined above is \( \mathcal{W}^{\text{re}} \)-invariant. See [1,2] for some interesting results on Weyl groups in the context of Toroidal Lie-algebras (or more generally the Extended Affine Lie-algebras).
**Definition 2.1.** A module \( V \) of \( \tau \) is called integrable if

1. \( V \) admits a weight space decomposition, i.e.,
   \[
   V = \bigoplus_{\lambda \in h^*} V_{\lambda},
   \]
   where \( V_{\lambda} = \{ v \in V | h.v = \lambda(h)v, \forall h \in h \} \). Denote by \( P(V) \) the set of all weights.
2. For \( \alpha \in \Delta, m_0 \in \mathbb{Z}, m \in \mathbb{Z}^* \), \( e_\alpha \otimes t_0^{m_0}t^m \) is locally nilpotent on \( V \).

Let \( \theta_{\text{fin}} \) be the category of irreducible integrable \( \tau \)-modules with finite-dimensional weight spaces. Similar to the proof of Lemma 2.3 in [14], we can obtain the following results (one can also see [7]):

**Lemma 2.1.** Let \( V \in \theta_{\text{fin}} \). Then

1. \( P(V) \) is \( \mathcal{W} \)-invariant.
2. \( \dim V_{\lambda} = \dim V_{\alpha \lambda}, \omega \in \mathcal{W}, \lambda \in P(V) \).
3. For \( \alpha \in \Delta^{\text{re}}, \lambda \in P(V) \), we have \( \lambda(\alpha') \in \mathbb{Z} \).
4. Let \( \alpha \in \Delta^{\text{re}}, \lambda \in P(V) \). If \( \lambda(\alpha') > 0 \), then \( \lambda - \alpha \in P(V) \).
5. For \( \lambda \in P(V) \), \( \lambda(k_i) \) is a constant integer, \( i = 0, 1, \ldots, v \).

By Lemma 2.1, we can assume that

\[
\lambda(k_i) = c_i, \quad i = 0, 1, \ldots, v, \quad \forall \lambda \in P(V), c_i \in \mathbb{Z}. \tag{2.10}
\]

Throughout the paper, \( c_i(i = 0, 1, \ldots, v) \) are always defined by (2.10). For \( m = (m_0, m) \), denote \( t_0^{m_0}t^m \) by \( t^m \).

**Lemma 2.2.** Let \( A = (a_{ij})(0 \leq i, j \leq v) \) be a \( (v+1) \times (v+1) \)-matrix such that \( \det A = 1 \) and \( a_{ij} \in \mathbb{Z} \). Then there exists an automorphism \( \sigma \) of \( \tau \) such that

\[
\sigma(x \otimes t^m) = x \otimes t^mA^T, \\
\sigma(t^mk_j) = \sum_{p=0}^v a_{pj}t^mA^T k_p, \quad 0 \leq j \leq v, \\
\sigma(t^md_j) = \sum_{p=0}^v b_{jp}t^mA^T d_p, \quad 0 \leq j \leq v,
\]

where \( B = (b_{ij}) = A^{-1} \).

Let \( V \in \theta_{\text{fin}} \). If \( V \) has non-zero central charges, it follows from Lemma 2.2 that we can assume that \( c_0 \neq 0, c_1 = \cdots = c_v = 0 \). Similar to the proof of Theorem 2.1 in [9], we have
Theorem 2.1. Let $V \in \vartheta_{\text{fin}}$. Then

1. If $c_0 > 0$ and $c_1 = c_2 = \cdots = c_v = 0$, then
   \[ \{ v \in V | r_+ \cdot v = 0 \} \neq 0. \]

2. If $c_0 < 0$ and $c_1 = c_2 = \cdots = c_v = 0$, then
   \[ \{ v \in V | r_- \cdot v = 0 \} \neq 0. \]

3. If $c_0 = c_1 = \cdots = c_v = 0$, then there exist non-zero elements $v, w \in V$ such that
   \[ (\hat{\mathfrak{g}}_+ \otimes A) \cdot v = 0, \quad (\hat{\mathfrak{g}}_- \otimes A) \cdot w = 0. \]

3. Modules of $\vartheta$ in $\vartheta_{\text{fin}}$ with non-zero central charges

In this section, we discuss the structure of $V \in \vartheta_{\text{fin}}$ which has non-zero central charges. By Lemma 2.2, we can assume that $c_0 \neq 0, c_1 = c_2 = \cdots = c_v = 0$. Let

\[ T = \{ v \in V | r_+ \cdot v = 0 \} \text{ if } c_0 > 0 \quad \text{or} \quad T = \{ v \in V | r_- \cdot v = 0 \} \text{ if } c_0 < 0. \]

Obviously, $T$ is a $\tau_0$-module. Since $V$ is irreducible, it follows that

\[ V = U(\tau_-) \cdot T \quad \text{or} \quad V = U(\tau_+) \cdot T. \]

Therefore $T$ is irreducible as a $\tau_0$-module. Let

\[ T = \bigoplus_{m \in \mathbb{Z}^v} T_m, \quad (3.1) \]

where $T_m = \{ v \in T | d_i(v) = (\lambda_0(d_i) + m_i)v \}$, for a fixed $\lambda_0 \in P(V), m = (m_1, m_2, \ldots, m_v) \in \mathbb{Z}^v$. Then $T$ is $\mathbb{Z}^v$-graded. By Theorem 2.1 and the fact that $V$ has finite-dimensional weight spaces, $T_m$ is finite dimensional.

Let

\[ \mathfrak{g}_d = \hat{\mathfrak{g}} \otimes \mathbb{C}[t_0^{\pm 1}] \oplus \mathbb{C}k_0 \oplus \mathbb{C}d_0. \]

Then $\mathfrak{g}_d$ is an affine Lie subalgebra of $\tau$. Let

\[ \mathfrak{h}_d = \hat{\mathfrak{h}} \oplus \mathbb{C}k_0 \oplus \mathbb{C}d_0. \]

Similar to the proof in [9], we can deduce that

Lemma 3.1. For any $m \in \mathbb{Z}^v, v \in T \setminus \{0\}$, we have $\tau^m k_0 v \neq 0$, and

\[ \dim T_n = \dim T_m = n \]

for all $m, n \in \mathbb{Z}^v$. 
Let \( \{v_1, v_2, \ldots, v_n\} \) be a basis of \( T_0 \). Set
\[
v_i(m) = \frac{1}{c_0} t^m k_0 \cdot v_i, \quad i = 1, 2, \ldots, n. \tag{3.2}
\]
Then \( \{v_1(m), v_2(m), \ldots, v_n(m)\} \) is a basis of \( T_m \). Assume that
\[
\frac{1}{c_0} t^m k_0 (v_1(n), v_2(n), \ldots, v_n(n)) = (v_1(m + n), v_2(m + n), \ldots, v_n(m + n)) B_{m,n}. \tag{3.3}
\]
By Lemma 3.1, \( B_{m,n} \) is an \( n \times n \) invertible matrix and
\[
B_{m,n} B_{r,s} = B_{r,s} B_{m,n}, \quad B_{m,n} = B_{n,m}.
\]
We can assume that \( \{B_{m,n} \mid m, n \in \mathbb{Z}^v\} \) are all upper triangular matrices.

**Lemma 3.2.** For any \( m, n \in \mathbb{Z}^v \), \( B_{m,n} - I \) is a strictly upper triangular matrix.

**Proof.** It is similar to the proof of Lemmas 3.5–3.9. \( \square \)

For \( m = (m_1, \ldots, m_v), n = (n_1, \ldots, n_v) \in \mathbb{Z}^v \), assume that
\[
t^m d_a(v_1(n), v_2(n), \ldots, v_n(n)) = (v_1(m + n), v_2(m + n), \ldots, v_n(m + n)) A^{(a)}_{m,n},
\]
where \( A^{(a)}_{m,n} \in \mathbb{C}^{n \times n} \). Since \([t^m d_a, t^n k_0] = n_a t^{m+n} k_0, \quad 1 \leq a \leq v\), we have
\[
A^{(a)}_{m,n} = B_{m,n} A^{(a)}_{m,0} + n_a I, \quad 1 \leq a \leq v. \tag{3.4}
\]

**Theorem 3.1.** For all \( m, n \in \mathbb{Z}^v, h \in h_a \) and \( p = 1, 2, \ldots, v \), we have
\[
t^m k_p \cdot T = 0, \tag{3.5}
\]
\[
t^m k_0 t^n k_0 = c_0 t^{m+n} k_0, \quad B_{m,n} = I. \tag{3.6}
\]
\[
h \otimes t^m (v_1(n), v_2(n), \ldots, v_n(n)) = A(h)(v_1(m + n), v_2(m + n), \ldots, v_n(m + n)), \tag{3.7}
\]
\[
t^m d_0(v_1(n), v_2(n), \ldots, v_n(n)) = A(d_0)(v_1(m + n), v_2(m + n), \ldots, v_n(m + n)), \tag{3.8}
\]
where \( A \in P(V) \).

**Proof.** For \( m = (m_1, m_2, \ldots, m_v) \in \mathbb{Z}^v \). Assume that
\[
t^m k_p (v_1, v_2, \ldots, v_n) = (v_1(m), v_2(m), \ldots, v_n(m)) C_{m,p}.
\]
Then
\[
\begin{align*}
& t^n q m s(v_1, v_2, \ldots, v_n) \\
& = t^n q m s(v_1(m), v_2(m), \ldots, v_n(m)) C_{m,p} \\
& = \left. \frac{1}{c_0} t^m k_0 t^n q m s(v_1, v_2, \ldots, v_n) \right|_{C_{m,p}} \\
& = \left. \frac{1}{c_0} t^m k_0 (v_1(n), v_2(n), \ldots, v_n(n)) C_{n,q} C_{m,p} \right|_{C_{m,p}} \\
& = \left. \frac{1}{c_0} t^m k_0 (v_1, v_2, \ldots, v_n) \right|_{C_{m,p} C_{n,p}}.
\end{align*}
\]

Therefore
\[
C_{n,q} C_{m,p} = C_{m,p} C_{n,q}.
\]

This means that the Lie-algebra \( C \) spanned by \( \{ C_{m,p} | m \in \mathbb{Z}^v, 1 \leq p \leq v \} \) is a commutative Lie-algebra. Let \( m = (m_1, m_2, \ldots, m_v) \in \mathbb{Z}^v \) be such that \( m_a \neq 0 \), for some \( 1 \leq a \leq v \). Since
\[
[A(a) - m, 0 + m B \bigm{B}_{-m,m} \bigm{B}^{-1} \bigm{B}_{-m,m}] C_{m,p} = t^m k_p t^n q m s(v_1, v_2, \ldots, v_n)
\]

it follows that
\[
(A_{-m,0} + m B_{-m,m}) C_{m,p} = C_{m,p} A_{-m,0}.
\]

If \( C_{m,p} \) is invertible, then \( A(a) \) and \( A_{-m,0} + m B_{-m,m} \) are similar matrices, which is impossible, since \( m_a \) is non-zero and \( B_{-m,m} \) is an upper triangular matrix with elements on the diagonal line being all one. We deduce that \( t^m k_p \) is locally nilpotent. Note that
\[
(t^m k_p)^s(v_1, v_2, \ldots, v_n) = \left. \left( \frac{1}{c_0} t^m k_0 \right)^s(v_1, v_2, \ldots, v_n) \right|_{C_{m,p}}.
\]

so \( C_{m,p} \) is a nilpotent matrix. Since \( C \) is commutative, there exists a non-zero element \( v \) in \( T \) such that
\[
t^m k_p(v) = 0, \quad \forall m \in \mathbb{Z}^v, 1 \leq p \leq v.
\]

Let \( T_1 = \{ v \in T | t^m k_p(v) = 0, \forall m \in \mathbb{Z}^v, 1 \leq p \leq v \} \). Then \( T_1 \) is a non-zero submodule of \( \tau_0 \)-module \( T \) and (3.5) follows from the fact that \( T \) is irreducible. The proof of (3.6)–(3.8) is similar to the proof of Theorem 3.1 in [9]. □

By Theorem 3.1, \( T \) is an irreducible \( \mathfrak{b}^0 \)-module. Let \( A_\mathfrak{v} = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_\mathfrak{v}^{\pm 1}] \), and \( \text{Der} \ A \) the derivation algebra of \( A \) with the following Lie bracket:
\[
[t^m d_a, t^n d_b] = n_a t^{m+n} d_b - m_b t^{m+n} d_a.
\]
Extending the above Lie bracket to $A_v \oplus \text{Der } A_v$ by

$$[t^m, t^n] = 0, \quad [t^m d_{a}, t^n] = n_{d} t^{m+n}.$$ 

Then $A_v \oplus \text{Der } A_v$ is a Lie-algebra. Now define the action of $A_v$ on $T$ by

$$t^m (v_1(n), v_2(n), \ldots, v_n(n)) = (v_1(m+n), v_2(m+n), \ldots, v_n(m+n)), \quad \forall m, n \in \mathbb{Z}^r.$$

It is easy to see from (3.4) that $T$ is an irreducible $\mathbb{Z}^r$-graded $A_v \oplus \text{Der } A_v$-module with finite-dimensional weight spaces.

**Theorem 3.2** (Rao [15]). As $A_v \oplus \text{Der } A_v$-module, $T$ is isomorphic to $F^2(\psi, b)$, for some $(\alpha, \psi, b)$, where $\alpha = (\alpha_1, \ldots, \alpha_v) \in \mathbb{C}^v$, $b \in \mathbb{C}$, $F^2(\psi, b) = V(\psi, b) \otimes A_v$ is an irreducible $A_v \oplus \text{Der } A_v$-module such that $V(\psi, b)$ is a $n$-dimensional irreducible $\mathfrak{gl}_v(\mathbb{C})$-module, and

$$\psi(I) = b \text{id}_V(\psi, b),$$

$$t^r d_{r}(w \otimes t^m) = (m_r + \alpha_r)w \otimes t^{r+m} + \sum_{i=1}^{v} r_i \psi(E_{ip})w \otimes t^{r+m},$$

$$t^m (w \otimes t^n) = w \otimes t^{m+n},$$

where $w \in V(\psi, b)$. Therefore $T$ is isomorphic to $F^2(\psi, b)$, for some $(\alpha, \psi, b)$ as $\text{Der } A_v$-modules.

Let $F^2(\psi, b)$ be an irreducible module of $A_v \oplus \text{Der } A_v$ defined as in Theorem 3.2, and $A \in \mathfrak{h}_v$ be such that $A(k_0) = c_0 \neq 0$. Extend $F^2(\psi, b)$ to be a $\tau_0$-module as follows:

$$t^r k_0 (w \otimes t^m) = c_0 w \otimes t^{r+m}, \quad t^r k_p (w \otimes t^m) = 0, \quad 1 \leq p \leq v,$$

$$h \otimes t^r (w \otimes t^m) = \Lambda(h) w \otimes t^{r+m}, \quad t^r d_0 (w \otimes t^m) = \Lambda(d_0) w \otimes t^{r+m},$$

where $h \in \mathfrak{h}$. Denote $F^2(\psi, b)$ by $F^2(\psi, b, A)$ as the module of $\tau_0$. Let $\tau_+$ (if $c_0 > 0$) or $\tau_-$ (if $c_0 < 0$) act on $F^2(\psi, b, A)$ by zero and so we have the induced module for $\tau$:

$$F^2_{\tau}(\psi, b, A) = \text{Ind}_{\tau_0}^{\tau_0}(F^2(\psi, b, A)) \quad \text{if } c_0 > 0,$$

or

$$F^2_{\tau}(\psi, b, A) = \text{Ind}_{\tau_0}^{\tau_0}(F^2(\psi, b, A)) \quad \text{if } c_0 < 0.$$

**Theorem 3.3.** Let $F^2_{\tau}(\psi, b, A)(A(k_0) = c_0 \neq 0)$ be defined as above. Then

1. There is a maximal one among the submodules of $F^2_{\tau}(\psi, b, A)$ intersecting $F^2(\psi, b, A)$ trivially. We denote the maximal submodule by $F^2_{\tau}(\psi, b, A)'.$
(2) $\tilde{F}_c^x(\psi, b, A) = F_c^x(\psi, b, A) / F_c^x(\psi, b, A)^\prime$ is an irreducible $\tau$-module and $\tilde{F}_c^x(\psi, b, A)$ is integrable if and only if $A$ or $-A$ is a dominant integral weight of $\mathfrak{g}_0$.

(3) Let $V \in \mathfrak{g}_0$ be such that $c_0 \neq 0, c_1 = c_2 = \cdots = c_v = 0$. Then $V \cong \tilde{F}_c^x(\psi, b, A)$ for some $(x, A, \psi, b)$.

**Proof.** (1) is standard. (3) we have already seen this. (2) We will assume that $c_0 > 0$ as the other case is similar. Let $F = \tilde{F}_c^x(\psi, b, A)$. $F$ is integrable implies that $A$ is dominant integral is easy to see and very standard. If $A$ is dominant integral then $F$ is integrable is non-trivial and does not follow from our work. But fortunately $F$ has been constructed explicitly by Billig [6] through the use of vertex operator algebras. In particular it is easy to verify that $F$ is integrable if $A$ is dominant integral from [6] but one should be familiar with vertex operator algebras notation. □

4. The structure of $V$ with $c_0 = \cdots = c_v = 0$

In this section, we assume that $c_0 = \cdots = c_v = 0$. By Theorem 2.1,

$$T = \{ v \in V | (\hat{g}_+ \otimes A)v = 0 \} \neq \{0\}. \quad (4.1)$$

Obviously, $T$ is a $(b + \hat{h} \otimes A)$-module. Since $V$ is irreducible it follows that

$$h(w) = A(h)w, \quad \forall h \in \hat{h}, \forall w \in T$$

for some $A \in P(V)$. If $A|_{\hat{h}} = 0$, then $(\hat{g} \otimes A)V = 0$, and $V$ is an irreducible $b$-module. In the following discussion, we assume that $A|_{\hat{h}} \neq 0$, so there exists $h_0 \in \hat{h}$ such that $A(h_0) \neq 0$. Let

$$T = \bigoplus_{m \in \mathbb{Z}_{\geq 1}} T_m,$$

where $T_m = \{ v \in T | d_p(v) = (A(d_p) + m_p)v, 0 \leq p \leq v \}, m = (m_0, \ldots, m_v)$.

Considering $h_0 \otimes t^m (m \in \mathbb{Z}_{\geq 1})$ instead of $t^m k_0 (m \in \mathbb{Z})$, quite similar to the proof of Lemmas 3.1–3.3, we can deduce that

**Lemma 4.1.** For $h_0 \otimes t^m \in \tau$, if there exists a non-zero element $w$ in $T$ such that $h_0 \otimes t^m w = 0$, then $h_0 \otimes t^m$ is locally nilpotent on $T$ and $\dim T_n > \dim T_{n+m}$.

**Lemma 4.2.** If both $h_0 \otimes t^m$ and $h_0 \otimes t^n$ are locally nilpotent on $T$, then so is $h_0 \otimes t^{m+n}$. If $h_0 \otimes t^{m+n}$ is locally nilpotent, then $h_0 \otimes t^m$ or $h_0 \otimes t^n$ is locally nilpotent.

**Lemma 4.3.** For any $m \in \mathbb{Z}_{\geq 1}, v \in T \setminus \{0\}$, we have $h_0 \otimes t^m v \neq 0$, and

$$\dim T_n = \dim T_m = n$$

for all $m, n \in \mathbb{Z}_{\geq 1}$.
Proof. Suppose the lemma is false. By Lemma 4.1, there exists an element in \{h_0 \otimes t_i^{\pm 1} | 0 \leq i \leq \nu\} which is locally nilpotent. Assume that \(h_0 \otimes t_1\) is locally nilpotent, by Lemma 4.2, \(\{h_0 \otimes t_k^i | k \geq 1\}\) are all locally nilpotent too. Therefore

\[ T_{r+(k,0,...,0)} = 0 \quad (4.2) \]

for all \(k \in \mathbb{N}\), where \(r\) satisfies \(\dim T_r = \min(\dim T_n > 0 | n \in \mathbb{Z}^{\nu+1})\). Let \(v\) be a non-zero element in \(T_r\), and \(\{n_1, n_2, \ldots, n_s\} \subset \mathbb{N}\) such that \(n_i \neq n_j\), for \(i \neq j\). We say that \(\{(e_{-n} \otimes t_1^{n_i})(h_0 \otimes t_1^{-n_i})v | 1 \leq i \leq s\} (\alpha \in A_+, A(\alpha^{'}) \neq 0, 0 \neq e_{-n} \in \mathfrak{g}_2)\) are linearly independent. In fact, assume that

\[ \sum_{i=1}^{s} a_i (e_{-n} \otimes t_1^{n_i})(h_0 \otimes t_1^{-n_i})v = 0. \]

Then

\[ t_1^{n_j} d_1 e_{-n} \otimes t_1^{-n_j} \left( \sum_{i=1}^{s} a_i (e_{-n} \otimes t_1^{n_i})(h_0 \otimes t_1^{-n_i})v \right) = 0, \quad j = 1, 2, \ldots, s. \]

Therefore

\[ t_1^{n_j} d_1 \sum_{i=1}^{s} (e_{-n} \otimes t_1^{n_i})(e_{-n} \otimes t_1^{-n_i})(h_0 \otimes t_1^{-n_i})v \]

\[ + \sum_{i=1}^{s} a_i (-n_j + n_i) \alpha^{' \otimes t_1^{n_i}} h_0 \otimes t_1^{-n_i} v \]

\[ + \sum_{i=1}^{s} a_i \alpha^{' \otimes t_1^{-n_j + n_i}} (h_0 \otimes t_1^{n_j - n_i}) v \]

\[ + \sum_{i=1}^{s} a_i \alpha^{' \otimes t_1^{-n_j + n_i}} (h_0 \otimes t_1^{-n_i}) t_1^{n_j} d_1 v = 0. \]

By (4.1)–(4.2), we have

\[ a_j (-n_j) \alpha^{' \cdot} h_0 \cdot v = 0. \]

Therefore

\[ a_j = 0, \quad j = 1, 2, \ldots, s. \]

Since \(s\) can be any positive integer, \(V_\mu\) is infinite dimensional, where \(\mu |_{0^0} = A - \alpha, \mu(d_i) = (A(d_i) + r_i), i = 0, 1, \ldots, v\). This contradicts the assumption that \(V\) has finite-dimensional weight spaces. Therefore the lemma holds. \(\square\)

Let \(\{v_1, v_2, \ldots, v_m\}\) be a basis of \(T_0\). Set

\[ v_i(m) = \frac{1}{A(h_0)} h_0 \otimes t_1^{m} \cdot v_i, \quad i = 1, 2, \ldots, m. \]
Then \( \{v_1(m), v_2(m), \ldots, v_m(m)\} \) is a basis of \( T_m \). Assume that

\[
\frac{1}{A(h_0)} h_0 \otimes t^m(v_1(n), v_2(n), \ldots, v_m(n)) \nonumber \\
= (v_1(m + n), v_2(m + n), \ldots, v_m(m + n)) H_{m,n}.
\]

(4.4)

By Lemma 4.3, \( H_{m,n} \) is invertible. Since \( c_0 = c_1 = \cdots = c_v = 0 \), we have

\[
H_{-m,m} = H_{m,-m}.
\]

**Lemma 4.4.** (1) For \( m, n \in \mathbb{Z}^{r+1} \), there exist \( \lambda_{m,n} \in \mathbb{C} \) and a non-zero element \( v \in T_0 \) such that

\[
(h_0 \otimes t^m h_0 \otimes t^n - \lambda_{m,n} A(h_0) h_0 \otimes t^{m+n}) \cdot v = 0.
\]

(2) \( h_0 \otimes t^m h_0 \otimes t^{-m} - \lambda_{m,-m} A(h_0) h_0 \otimes t^m \) is locally nilpotent on \( T \).

(3) \( H_{-m,m} \) does not have different eigenvalues.

For \( m = (m_0, m_1, \ldots, m_v), n = (n_0, n_1, \ldots, n_v) \in \mathbb{Z}^{r+1} \), assume that

\[
t^m d_a (v_1(n), v_2(n), \ldots, v_m(n)) = (v_1(m + n), v_2(m + n), \ldots, v_m(m + n)) A^{(a)}_{m,n},
\]

where \( A^{(a)}_{m,n} \in \mathbb{C}^{m \times m} \). Since \( [t^m d_a, h_0 \otimes t^n] = n_a h_0 \otimes t^{m+n}, 0 \leq a \leq v \), we have

\[
A^{(a)}_{m,n} = H_{m,n} A^{(a)}_{m,0} + n_a I, \quad 0 \leq a \leq v.
\]

**Theorem 4.1.** \( (t^m k_p)T = 0, \forall m \in \mathbb{Z}^{r+1}, p = 0, 1, \ldots, v. \)

**Proof.** It is similar to the proof of (3.5). \( \square \)

**Theorem 4.2.** For all \( m, n \in \mathbb{Z}^{r+1} \) and \( h \in \mathfrak{h} \), we have

\[
h \otimes t^m (v_1(n), v_2(n), \ldots, v_m(n)) = A(h)(v_1(m + n), v_2(m + n), \ldots, v_m(m + n)).
\]

Therefore \( T \) is isomorphic to \( F^2(\psi, b) \), for some \( (\varphi, \psi, b) \), as \( A \oplus \text{Der } A \)-modules, where \( F^2(\psi, b) = V(\psi, b) \otimes A \) is an irreducible \( A \oplus \text{Der } A \)-module such that \( V(\psi, b) \) is a \( m \)-dimensional irreducible \( \mathfrak{gl}_{V+1}(\mathbb{C}) \)-module, and

\[
\psi(I) = b \text{id}_V(\psi, b),
\]

\[
t^r d_p (v \otimes t^m) = (m_p + \alpha_p) v \otimes t^{r+m} + \sum_{i=0}^v r_i \psi(E_{ip}) v \otimes t^{r+m}.
\]

Let \( A_0 \in \mathfrak{h}^\perp \) be such that \( A_0 \neq 0 \), \( (L(A_0), \pi) \) an irreducible highest weight module of \( \hat{\mathfrak{g}} \) with the highest weight \( A_0 \) and the associated highest weight vector \( v_{A_0} \), \( F^2(\psi, b) \) a
A ⊕ Der A-modules defined above. Let

\[ F^z(\psi, b, A_0) = L(A_0) \otimes F^z(\psi, b). \]

We define the action of \( \tau \) on \( F^z(\psi, b, A_0) \) by

\[ x \otimes \tau^r(w \otimes v(m)) = (\pi(x)w) \otimes v(m + r), \]

\[ \tau^r k_p(w \otimes v(m)) = 0, \]

\[ \tau^r d_p(w \otimes v(m)) = w \otimes \tau^r d_p v(m), \]

where \( x \in \hat{\mathfrak{g}}, w \in L(A_0), v(m) = v \otimes t^m \in V(\psi, b) \otimes A, r, m \in \mathbb{Z}^{v+1}, 0 \leq p \leq v. \) Then \( F^z(\psi, b, A_0) \) is an irreducible \( \tau \)-module.

**Theorem 4.3.** Let \( F^z(\psi, b, A_0) \) be an irreducible \( \tau \)-module defined above.

1. \( F^z(\psi, b, A_0) \) is integrable if and only if \( A_0 \) is a dominant integral weight of \( \hat{\mathfrak{g}}. \)
2. Let \( V \in \vartheta_{\text{fin}} \) be such that \( c_0 = c_1 = \cdots = c_v = 0 \) and \( (\hat{\mathfrak{g}} \otimes A)V \neq 0. \) Then \( V \cong F^z(\psi, b, A_0) \), for some \( (z, \psi, b, A_0). \)

**Acknowledgements**

We are grateful to the referee for invaluable comments and suggestions.

**References**