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Journal of Algebra 319 (2008) 5039–5059

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Serial coalgebras and their valued Gabriel quivers ☆

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Received 2 July 2007

Available online 24 April 2008

Communicated by Kent R. Fuller

Abstract

We study serial coalgebras by means of their valued Gabriel quivers. In particular, Hom-computable and representation-directed serial coalgebras are characterized. The Auslander–Reiten quiver of a serial coalgebra is described. Finally, a version of Eisenbud–Griffith Theorem is proved, namely, every subcoalgebra of a prime, hereditary and strictly quasi-finite coalgebra is serial.

Keywords: Serial coalgebras; Valued Gabriel quiver; Representation-directed coalgebras; Hereditary coalgebras; Prime coalgebras; Strictly quasi-finite coalgebras

1. Introduction

A systematic study of serial coalgebras was initiated in [5], where, in particular, it was shown that any serial indecomposable coalgebra over an algebraically closed field is Morita–Takeuchi equivalent to a subcoalgebra of a path coalgebra of a quiver which is either a cycle or a chain (finite or infinite) [5, Theorem 2.10]. In this paper, we take advantage of the valued Gabriel quivers associated to a coalgebra to characterize indecomposable serial coalgebras over any field (Theorem 2.5). In conjunction with localization techniques (see Section 3), this combinatorial tool allows to complete the study made in [5] in more remarkable aspects. Thus, in Section 4, we characterize Hom-computable serial coalgebras in the sense of [29] (Proposition 4.3), and representation-directed coalgebras (Proposition 4.4). Section 5 is devoted to describe the

☆ Research supported by Spanish MEC project MTM2004-01406, and FEDER.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.03.020

Auslander–Reiten quiver of the category of finite-dimensional (right) comodules of a serial coalgebra.

It was observed in [5] that a consequence of [9, Corollary 3.2] is that the finite dual coalgebra of a hereditary noetherian prime algebra over a field is serial. In Section 6 we reconsider this result of Eisenbud and Griffith from the coalgebraic point of view: we prove, using the results developed in the previous sections, that any subcoalgebra of a prime, hereditary and strictly quasi-finite coalgebra is serial (Corollary 6.3).

Throughout we fix a field K and we assume C is a K-*coalgebra*. We refer the reader to the books [1,19] and [31] for notions and notations about coalgebras. Unless otherwise stated, all C-comodules are right C-comodules. It is well known that C has a decomposition, as right C-comodule,

$$C_C = \bigoplus_{i \in I_C} E_i^{t_i},$$

where $\{E_i\}_{i \in I_C}$ is a complete set of pairwise non-isomorphic *indecomposable injective* right C-comodules and t_i is a positive integer for any $i \in I_C$. This produces a decomposition of the *socle* of C (the sum of all its simple subcomodules), $\operatorname{soc} C$, as follows:

$$\operatorname{soc} C = \bigoplus_{i \in I_C} S_i^{t_i},$$

where $\{S_i\}_{i \in I_C}$ is a complete set of pairwise non-isomorphic *simple* right C-comodules. It is easy to prove that

$$t_i = \frac{\dim_K S_i}{\dim_K(\operatorname{End}_C(S_i))}$$

for any $i \in I_C$, see [27].

For any right C-comodule M, we denote by $\operatorname{soc} M$ the socle of M and by $E(M)$ its *injective envelope*. We assume that $\operatorname{soc} E_i = S_i$, for each $i \in I_C$, and consequently, $E(S_i) = E_i$.

Throughout we denote by G_i the division K-algebra of endomorphism $\operatorname{End}_C(S_i)$ for each $i \in I_C$. The coalgebra C is said to be *basic* if $t_i = 1$ for any $i \in I_C$, or, equivalently, if $\dim_K S_i = \dim_K G_i$ for any $i \in I_C$, or, equivalently, if S_i is simple subcoalgebra of C for any $i \in I_C$, see for instance [30]. In particular, C is called *pointed* if $\dim_K S_i = 1$ for any $i \in I_C$.

If the field K is algebraically closed then C is pointed if and only if C is basic (cf. [27, Corollary 2.7]).

Since every coalgebra is Morita–Takeuchi equivalent (that is, their categories of comodules are equivalent) to a basic one (cf. [8]), throughout we assume that C is basic and there are decompositions

$$C = \bigoplus_{i \in I_C} E_i \quad \text{and} \quad \operatorname{soc} C = \bigoplus_{i \in I_C} S_i, \tag{1}$$

where $E_i \ncong E_j$ and $S_i \ncong S_j$ for $i \neq j$. Symmetrically, there exists the left-hand version of all the facts explained above. In particular, C admits a decomposition as left C-comodule

$${}_C C = \bigoplus_{i \in I_C} F_i \quad \text{and} \quad \operatorname{soc} C = \bigoplus_{i \in I_C} S_i. \tag{2}$$

We recall from [5] that a right C-comodule M is said to be uniserial if its lattice of subcomodules is a chain. This property can be characterized through the socle filtration, namely, M has a filtration

$$0 \subset \operatorname{soc} M \subset \operatorname{soc}^2 M \subset \cdots \subset M$$

called the *Loewy series*, where, for $n > 1$, $\operatorname{soc}^n M$ is the unique subcomodule of M satisfying that $\operatorname{soc}^{n-1} M \subset \operatorname{soc}^n M$ and

$$\frac{\operatorname{soc}^n M}{\operatorname{soc}^{n-1} M} = \operatorname{soc}\left(\frac{M}{\operatorname{soc}^{n-1} M}\right),$$

see [12] and [21] for some properties of the Loewy series.

Lemma 1.1. *(See [5].) The following statements are equivalent*:

(a) *M is uniserial.*
(b) *The Loewy series is a composition series.*
(c) *Each finite-dimensional subcomodule of M is uniserial.*

The coalgebra C is said to be right (left) serial if any indecomposable injective right (left) C-comodule is uniserial. C is called serial if it is both right and left serial.

Throughout we denote by $\mathcal{M}_f^C$, $\mathcal{M}_{qf}^C$ and $\mathcal{M}^C$ the category of finite-dimensional, quasi-finite and all right C-comodules, respectively. Dually, ${}^C\mathcal{M}_f$, ${}^C\mathcal{M}_{qf}$ and ${}^C\mathcal{M}$ denote the corresponding categories of left C-comodules.

A full subcategory $\mathcal{T}$ of $\mathcal{M}^C$ is said to be *dense* (or a *Serre class*) if each exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

in $\mathcal{M}^C$ satisfies that M belongs to $\mathcal{T}$ if and only if M_1 and M_2 belong to $\mathcal{T}$. Following [10] and [23], for any dense subcategory $\mathcal{T}$ of $\mathcal{M}^C$, there exists an abelian category $\mathcal{M}^C/\mathcal{T}$ and an exact functor $T : \mathcal{M}^C \to \mathcal{M}^C/\mathcal{T}$, such that $T(M) = 0$ for each $M \in \mathcal{T}$, satisfying the following universal property: for any exact functor $F : \mathcal{M}^C \to \mathcal{C}$ such that $F(M) = 0$ for each $M \in \mathcal{T}$, there exists a unique functor $\overline{F} : \mathcal{M}^C/\mathcal{T} \to \mathcal{C}$ verifying that $F = \overline{F}T$. The category $\mathcal{M}^C/\mathcal{T}$ is called the *quotient category* of $\mathcal{M}^C$ with respect to $\mathcal{T}$, and T is known as the *quotient functor*.

Let now $\mathcal{T}$ be a dense subcategory of the category $\mathcal{M}^C$, $\mathcal{T}$ is said to be *localizing* (cf. [10]) if the quotient functor $T : \mathcal{M}^C \to \mathcal{M}^C/\mathcal{T}$ has a right adjoint functor S, called the *section functor*. If the section functor is exact, $\mathcal{T}$ is called *perfect localizing*. Let us list some properties of the localizing functors (cf. [10, Chapter III]).

Lemma 1.2. *Let $\mathcal{T}$ be a dense subcategory of the category of right comodules $\mathcal{M}^C$ over a coalgebra C. The following statements hold*:

(a) *T is exact.*
(b) *If $\mathcal{T}$ is localizing, then the section functor S is left exact and the equivalence $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ holds.*

From the general theory of localization in Grothendieck categories [10], it is well known that there exists a one-to-one correspondence between localizing subcategories of $\mathcal{M}^C$ and sets of indecomposable injective right C-comodules, and, as a consequence, sets of simple right C-comodules. More precisely, a localizing subcategory is determined by an injective right C-comodule $E = \bigoplus_{j\in J} E_j$, where $J \subseteq I_C$ (therefore the associated set of indecomposable injective comodules is $\{E_j\}_{j\in J}$). Then $\mathcal{M}^C/\mathcal{T} \simeq \mathcal{M}^D$, where D is the coalgebra of coendomorphism $\mathrm{Cohom}_C(E, E)$ (cf. [32] for definitions), and the quotient and section functors are $\mathrm{Cohom}_C(E, -)$ and $-\Box_D E$, respectively.

In [4,15] and [33], localizing subcategories are described by means of idempotents in the dual algebra C^*. In particular, it is proved that the quotient category $\mathcal{M}^C/\mathcal{T}$ is the category of right comodules over the coalgebra eCe, where $e \in C^*$ is an idempotent associated to the localizing subcategory $\mathcal{T}$ (that is, $E = Ce$, where E is the injective right C-comodule associated to the localizing subcategory $\mathcal{T}$). The coalgebra structure of eCe (cf. [24]) is given by

$$\Delta_{eCe}(exe) = \sum_{(x)} ex_{(1)}e \otimes ex_{(2)}e \quad \text{and} \quad \epsilon_{eCe}(exe) = \epsilon_C(x)$$

for any $x \in C$, where $\Delta_C(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ using the sigma-notation of [31]. Throughout we denote by $\mathcal{T}_e$ the localizing subcategory associated to the idempotent e. For completeness, we recall from [4] (see also [15]) the following description of the localizing functors. We recall that, given an idempotent $e \in C^*$, for each right C-comodule M, the vector space eM is endowed with a structure of right eCe-comodule given by

$$\rho_{eM}(ex) = \sum_{(x)} ex_{(1)} \otimes ex_{(0)}e$$

where $\rho_M(x) = \sum_{(x)} x_{(1)} \otimes x_{(0)}$ using the sigma-notation of [31].

Lemma 1.3. *Let C be a coalgebra and e be an idempotent in C^*. Then the following statements hold*:

(a) *The quotient functor $T : \mathcal{M}^C \to \mathcal{M}^{eCe}$ is naturally equivalent to the functor $e(-)$. T is also naturally equivalent to the cotensor functor $-\Box_C eC$ and the* Cohom *functor $T_e = \mathrm{Cohom}_C(Ce, -)$.*
(b) *The section functor $S : \mathcal{M}^{eCe} \to \mathcal{M}^C$ is naturally equivalent to the cotensor functor $S_e = -\Box_{eCe} Ce$.*
(c) *$\mathcal{T}_e$ is perfect localizing if and only if Ce is injective as right eCe-comodule.*

We refer the reader to [13,14] and [15] for basic definitions, notations and properties about quivers and path coalgebras. The localization in categories of comodules over path coalgebras is described in detail in [15].

2. The valued Gabriel quiver

Associating a graphical structure to a certain mathematical object is a very common strategy. Sometimes, it provides us a nice method for replacing the object with a simpler one and improving our intuition about its properties. In our case, when dealing with representation theory of coalgebras, the quivers associated to a coalgebra play a prominent rôle in order to study their structure in depth. This section is devoted to analyze the shape of the so-called valued Gabriel quiver of a serial coalgebra carrying on with the results obtained in [5] which generalize the well-known ones for finite-dimensional serial algebras. Throughout we assume that C is a basic coalgebra with decompositions (1) and (2). Following [17], let us recall the notion of right *valued Gabriel quiver* (Q_C, d_C) of the coalgebra C as follows: the set of vertices of (Q_C, d_C) is the set of simple right C-comodules $\{S_i\}_{i\in I_C}$, and there exists a unique valued arrow

$$S_i \xrightarrow{(d'_{ij},d''_{ij})} S_j$$

if and only if $\mathrm{Ext}^1_C(S_i, S_j) \neq 0$ and,

$$d'_{ij} = \dim_{G_i} \mathrm{Ext}^1_C(S_i, S_j) \quad \text{and} \quad d''_{ij} = \dim_{G_j} \mathrm{Ext}^1_C(S_i, S_j),$$

as a right G_i-module and as a left G_j-module, respectively.

The (non-valued) Gabriel quiver of C is obtained by taking the same set of vertices and the number of arrows from a vertex S_i to a vertex S_j is given by the integer $\dim_{G_i} \mathrm{Ext}^1_C(S_i, S_j)$, where $\mathrm{Ext}^1_C(S_i, S_j)$ is viewed as right G_i-module. If C is pointed (or K is algebraically closed) then it is isomorphic to the one used by Montgomery and Chin in [8] and Woodcock in [33] in order to prove that C is a subcoalgebra of the path coalgebra of its (non-valued) Gabriel quiver.

In [28], the valued Gabriel quiver of C is described through the notion of irreducible morphisms between indecomposable injective right C-comodules. Let us denote by inj^C (respectively ${}^C\mathrm{inj}$) the full subcategory of $\mathcal{M}^C$ (respectively ${}^C\mathcal{M}$) formed by socle-finite (i.e., comodules whose socle is finite-dimensional) injective right (respectively left) C-comodules. Let E and E' be two comodules in inj^C. A morphism $f : E \to E'$ is said to be irreducible if f is not an isomorphism and given a factorization

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & E' \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & Z & \end{array}$$

of f, where Z is in inj^C, g is a section, or h is a retraction. Analogously to the case of finite-dimensional algebras, there it is proven that the set of irreducible morphism $\mathrm{Irr}_C(E_i, E_j)$ between two indecomposable injective right C-comodules E_i and E_j is isomorphic, as G_j-G_i-bimodule, to the quotient $\mathrm{rad}_C(E_i, E_j)/\mathrm{rad}^2_C(E_i, E_j)$. We recall that, for each two indecomposable injective right C-comodules E_i and E_j, the *radical* of $\mathrm{Hom}_C(E_i, E_j)$ is the K-subspace $\mathrm{rad}_C(E_i, E_j)$ of $\mathrm{Hom}_C(E_i, E_j)$ generated by all non-isomorphisms. Observe that if $i \neq j$, then $\mathrm{rad}_C(E_i, E_j) = \mathrm{Hom}_C(E_i, E_j)$. The square of $\mathrm{rad}_C(E_i, E_j)$ is defined to be the K-subspace

$$\mathrm{rad}^2_C(E_i, E_j) \subseteq \mathrm{rad}_C(E_i, E_j) \subseteq \mathrm{Hom}_C(E_i, E_j)$$

generated by all composite homomorphisms of the form

$$E_i \xrightarrow{f} E_k \xrightarrow{g} E_j,$$

where $f \in \operatorname{rad}_C(E_i, E_k)$ and $g \in \operatorname{rad}_C(E_k, E_j)$. The mth power $\operatorname{rad}_C^m(E_i, E_j)$ of $\operatorname{rad}_C(E_i, E_j)$ is defined analogously, for each $m > 2$.

Lemma 2.1. *(See [28, Theorem 2.3(a)].) Let C be a basic coalgebra and set $G_i = \operatorname{End}_C(S_i)$ for each $i \in I_C$. There is an arrow*

$$S_i \xrightarrow{(d'_{ij}, d''_{ij})} S_j$$

in the right valued Gabriel quiver (Q_C, d_C) of C if and only if $\operatorname{Irr}_C(E_j, E_i) \neq 0$ and

$$d'_{ij} = \dim_{G_j} \operatorname{Irr}_C(E_j, E_i) \quad \textit{and} \quad d''_{ij} = \dim_{G_i} \operatorname{Irr}_C(E_j, E_i),$$

as right G_j-module and as left G_i-module, respectively.

Let us see that right serial coalgebras are easy to distinguish from their valued Gabriel quivers. This is the natural generalization of the result for finite-dimensional serial algebras. Firstly, we remind from [5, Proposition 1.7] the following lemma.

Lemma 2.2. *A basic coalgebra C is right serial if and only if the right C-comodule $\operatorname{soc}^2 E/\operatorname{soc} E$ is zero or simple for each indecomposable injective right C-comodule E.*

Proposition 2.3. *A basic coalgebra C is right serial if and only if each vertex S_i of the right valued Gabriel quiver (Q_C, d_C) is at most the sink of one arrow and, if such an arrow exists, it is of the form*

$$S_j \xrightarrow{(1,d)} S_i,$$

for some vertex S_j and some positive integer d.

Proof. Recall that, for any simple right C-comodule S_i,

$$\operatorname{Ext}^1_C(S_j, S_i) \cong \operatorname{Hom}_C(S_j, E_i/S_i)$$

as right G_j-modules for all simple right C-comodule S_j, see for instance [17] and [21, Lemma 1.2].

Assume now that C is right serial and $E_i/S_i \neq 0$ (otherwise there is no arrow ending at S_i) then E_i/S_i is a subcomodule of an indecomposable injective right comodule E_j, and then

$$\operatorname{Ext}^1_C(S, S_i) \cong \operatorname{Hom}_C(S, E_i/S_i) \cong \begin{cases} G_j, & \text{if } S_j = S; \\ 0, & \text{otherwise,} \end{cases}$$

as right $\mathrm{End}_C(S)$-modules. Hence, there is a unique arrow ending at S_i of the form

$$S_j \xrightarrow{(1,d)} S_i.$$

Conversely, the immediate predecessors of S_i correspond to the simple comodules contained in $\mathrm{soc}(E_i/S_i)$. Since, by hypothesis, there is only one arrow ending at S_i, $\mathrm{soc}(E_i/S_i) = (S_j)^t$ for some simple right comodule S_j and some positive integer t. Now, since t is the first component of the label of the arrow, $\mathrm{soc}^2 E_i/\mathrm{soc}\, E_i = \mathrm{soc}(E_i/S_i) = S_j$ is a simple comodule. By the previous lemma, C is right serial. □

Symmetrically, we prove that C is left serial if and only if each vertex S of the left valued Gabriel quiver $({}_C Q, {}_C d)$ is at most the sink of one arrow and, if such an arrow exists, it is of the form

$$S' \xrightarrow{(1,d)} S,$$

for some vertex S' and some positive integer d.

The following simple result is very useful, see also [18, Corollary 2.26].

Proposition 2.4. *Let C be a basic coalgebra. The right valued Gabriel quiver (Q_C, d_C) of C is the opposite valued quiver of the left valued Gabriel quiver $({}_C Q, {}_C d)$ of C.*

Proof. The result follows directly from the duality $\mathrm{Hom}_K(-, K)$ on the category of finite-dimensional comodules. □

Let us now prove the main result of this section that generalizes [5, Theorem 2.10]. In what follows we denote the labeled arrows $\circ \xrightarrow{(1,1)} \circ$ simply by $\circ \longrightarrow \circ$. As well we denote a valued quiver (Q, d) simply by Q if $(d^1_{ij}, d^2_{ij}) = (1, 1)$ for any i and j.

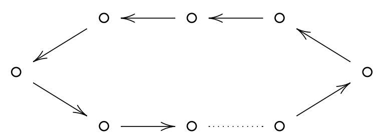
Theorem 2.5. *Let C be an indecomposable basic coalgebra over an arbitrary field K. Then C is serial if and only if the right (and then also the left) valued Gabriel quiver of C is one of the following valued quivers:*

(a) ${}_\infty\mathbb{A}_\infty$: $\cdots\cdots \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \cdots\cdots.$

(b) $\mathbb{A}_\infty$: $\circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \cdots\cdots.$

(c) ${}_\infty\mathbb{A}$: $\cdots\cdots \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ.$

(d) $\mathbb{A}_n$: $\circ \longrightarrow \circ \longrightarrow \circ \cdots\cdots \circ \longrightarrow \circ \longrightarrow \circ$ *n vertices, $n \geqslant 1$.*

(e) $\widetilde{\mathbb{A}}_n$:  *n vertices, $n \geqslant 1$.*

Proof. Assume that C is serial and let S be a simple right (and left) C-comodule. Since C is right serial, there exists at most one arrow in (Q_C, d_C) ending at S. Analogously, since C is left serial there exists at most one arrow in $({}_CQ, {}_Cd)$ ending at S and then, by Proposition 2.4, we may deduce that there is at most one arrow in (Q_C, d_C) starting at S. Also, by Proposition 2.3 and its left-hand version and Proposition 2.4, any of these possible arrows are labeled by $(1, 1)$.

Taking into account the above discussion, let S_0 be a simple right comodule. If there is no arrow neither ending nor starting at S_0, i.e., S_0 is an isolated vertex, since C is indecomposable (and therefore Q_C is connected, cf. [28]) then $Q_C = \mathbb{A}_1$. Similarly, if there is a loop at S_0, then $Q_C = \widetilde{\mathbb{A}}_1$. Therefore we may assume that there is no loop in Q_C and then S_0 is inside a (maybe infinite) path

$$\cdots\cdots \longrightarrow S_{-n} \longrightarrow \cdots\cdots \longrightarrow S_{-1} \longrightarrow S_0 \longrightarrow S_1 \longrightarrow \cdots\cdots \longrightarrow S_m \longrightarrow \cdots\cdots.$$

If there exist two non-negative integers n and m such that $S_{-n} = S_m$, then Q_C must be a crown, i.e., $Q_C = \widetilde{\mathbb{A}}_p$ for some integer p. If not, Q_C must be a line, that is, it is a quiver as showed in (a), (b), (c) or (d) depending on the finiteness of the two branches. Clearly, the converse holds. □

Corollary 2.6. *A basic coalgebra C is serial if and only if each of the connected component of its right (or left) valued Gabriel quiver is either ${}_\infty\mathbb{A}_\infty$, or $\mathbb{A}_\infty$, or ${}_\infty\mathbb{A}$; or $\mathbb{A}_n$ or $\widetilde{\mathbb{A}}_n$ for some $n \geqslant 1$.*

3. Localization in serial coalgebras

Let us now apply the localization techniques developed in [14,15,21] and [30] to (right) serial coalgebras. In particular, we give a characterization of serial coalgebras by means of its "local structure," that is, by means of its localized coalgebras.

The following proposition shows that the localization process preserves the uniseriality of comodules and the seriality of coalgebras. For each idempotent $e \in C^*$ we denote by T_e the quotient functor $\mathrm{Cohom}_C(Ce, -) : \mathcal{M}^C \to \mathcal{M}^{eCe}$.

Proposition 3.1. *Let $E = Ce$ be a quasi-finite injective right C-comodule and M a uniserial right C-comodule. Then $T_e(M) = \mathrm{Cohom}_C(E, M) = eM$ is a uniserial right eCe-comodule.*

Proof. Let us consider the (composition) Loewy series of M as right C-comodule,

$$\operatorname{soc} M = S_1 \subset \operatorname{soc}^2 M \subset \operatorname{soc}^3 M \subset \cdots \subset M$$

whose composition factors are S_1 and $S_k = M[k] = \operatorname{soc}^k M / \operatorname{soc}^{k-1} M$ for $k \geqslant 2$. Since a simple C-comodule is either torsion or torsion-free, let us suppose that $S_{i_1}, S_{i_2}, S_{i_3}, \ldots$ are the torsion-free composition factors of M, where $i_1 < i_2 < i_3 < \cdots$.

For each $k < i_1$, $T_e(M[k]) = T_e(S_k) = 0$ and then $T_e(\operatorname{soc}^k M) = T_e(S_1) = 0$. As a consequence, by [15, Remark 2.3], $T_e(\operatorname{soc}^{i_1} M) = T_e(S_{i_1}) = S_{i_1}$. Moreover, since $M[i_1] = \operatorname{soc}(M/\operatorname{soc}^{i_1-1} M) = S_{i_1}$, then $M/\operatorname{soc}^{i_1-1} M$ is torsion-free and, by [21, Proposition 3.2(c)],

$$S_{i_1} = T_e(S_{i_1}) = T_e\left(\operatorname{soc}\left(\frac{M}{\operatorname{soc}^{i_1-1} M}\right)\right) = \operatorname{soc}\left(T_e\left(\frac{M}{\operatorname{soc}^{i_1-1} M}\right)\right) = \operatorname{soc} T_e(M).$$

Applying the same arguments, we may obtain that $T_e(\operatorname{soc}^k M) = S_{i_1}$ for each $i_1 \leqslant k < i_2$, and $M/\operatorname{soc}^{i_2-1} M$ is a torsion-free right C-comodule. Then

$$T_e(M)[2] = \frac{\operatorname{soc}^2 T_e(M)}{\operatorname{soc} T_e(M)} = \operatorname{soc}\left(\frac{T_e(M)}{\operatorname{soc} T_e(M)}\right) = \operatorname{soc}\left(\frac{T_e(M)}{T_e(\operatorname{soc}^{i_1} M)}\right)$$
$$= \operatorname{soc}\left(\frac{T_e(M)}{T_e(\operatorname{soc}^{i_2-1} M)}\right) = T_e\left(\operatorname{soc}\left(\frac{M}{\operatorname{soc}^{i_2-1} M}\right)\right) = S_{i_2}.$$

Thus $\operatorname{soc}^2 T_e(M) = T_e(\operatorname{soc}^{i_2} M)$. If we continue in this fashion, we may prove that

$$T_e\big(\operatorname{soc}^{i_1} M\big) \subset T_e\big(\operatorname{soc}^{i_2} M\big) \subset T_e\big(\operatorname{soc}^{i_3} M\big) \subset \cdots \subset T_e(M)$$

is the Loewy series of $T_e(M)$. Hence $T_e(M)$ is uniserial as a right eCe-comodule. □

Corollary 3.2. *Let C be a right (left) serial coalgebra and $e \in C^*$ an idempotent. Then the localized coalgebra eCe is right (left) serial.*

Proof. Let $\overline{E}_i$ be an indecomposable injective right eCe-comodule. By [21, Proposition 3.2], $T_e(E_i) = \overline{E}_i$, where E_i is the indecomposable injective right C-comodule such that $\operatorname{soc} E_i = \operatorname{soc} \overline{E}_i$. Since E_i is uniserial, by Proposition 3.1, so is $\overline{E}_i$. □

Lemma 3.3. *Let C be a coalgebra. If the localized coalgebra eCe is right (left) serial for each idempotent $e \in C^*$ associated to a subset of simple comodules with cardinal less or equal than three, then C is right (left) serial.*

Proof. Let us suppose that C is not right serial. By Lemma 2.2 there exists an indecomposable injective right C-comodule E such that $S_1 \oplus S_2 \subseteq \operatorname{soc}(E/S)$, where S_1 and S_2 are simple comodules. Consider the idempotent $e \in C^*$ associated to the set $\{S, S_1, S_2\}$. Then, by [21, Lemma 2.1],

$$T_e(S_1 \oplus S_2) = S_1 \oplus S_2 \subseteq T_e\big(\operatorname{soc}(E/S)\big) \subseteq \operatorname{soc}\big(T(E/S)\big) = \operatorname{soc}(\overline{E}/S),$$

where $T_e(E) = \overline{E}$ is an indecomposable injective eCe-comodule. Thus eCe is not right serial. □

Proposition 3.4. *Let C be a coalgebra. C is right (left) serial if and only if each socle-finite localized coalgebra of C is right (left) serial.*

Proof. Apply Corollary 3.2 and Lemma 3.3. □

The former results are quite surprising since, as the following proposition shows, the localization process increases the label of an arrow (if exists) between two torsion-free vertices.

Proposition 3.5. *Let C be a coalgebra and $e \in C^*$ idempotent. Let S_1 and S_2 be two torsion-free simple C-comodules in the torsion theory associated to the localizing subcategory $\mathcal{T}_e$. If there exists an arrow $S_1 \to S_2$ in (Q_C, d_C) labeled by (d'_{12}, d''_{12}), then there exists an arrow $S_1 \to S_2$ in (Q_{eCe}, d_{eCe}) labeled by (t'_{12}, t''_{12}), where $t'_{12} \geqslant d'_{12}$ and $t''_{12} \geqslant d''_{12}$.*

Proof. Let us suppose that $\operatorname{soc}(E_2/S_2) = \bigoplus_{i\in I_C} S_i^{n_i}$ for some non-negative integers n_i. Then there exists an arrow $S_i \to S_2$ if and only if $n_i \neq 0$ and, furthermore, in such a case, it is labeled by (n_i, m_i) for some positive integer m_i. Now, if there exists an arrow $S_1 \to S_2$ in (Q_C, d_C) labeled by (d'_{12}, d''_{12}), then

$$S_1^{n_1} \subseteq \bigoplus_{i\in I_e} S_i^{n_i} = T_e\bigl(\operatorname{soc}(E_2/S_2)\bigr) \subseteq \operatorname{soc} T(E_2/S_2) \subseteq \overline{E}_2/S_2,$$

since S_1 and S_2 are torsion-free. Hence there is an arrow

$$S_1 \xrightarrow{(t'_{12}, t''_{12})} S_2$$

in (Q_{eCe}, d_{eCe}) and $t'_{12} = \dim_{G_1} \operatorname{Hom}_{eCe}(S_1, \overline{E}_2/S_2) \geqslant n_1 = d'_{12}$. By the left-hand version of this reasoning and Proposition 2.4, also $t''_{12} \geqslant d''_{12}$. □

4. Hom-computable and representation-directed serial coalgebras

The study of the directing modules of an artin algebra comes from different motivations. On the one hand, they are treated as a generalization of the modules lying in a postprojective or a preinjective component (or more generally, in an acyclic component) of the Auslander–Reiten quiver of this algebra. Hence, they possess common properties with these modules as, for instance, they are determined up to isomorphism by their composition factors. On the other hand, they have interesting properties of their own as, for example, that any algebra having a sincere and directing module is a tilted algebra (that is, the endomorphism algebra of a hereditary algebra). It is also well known that a representation-directed algebra (all its modules are directing) is finite representation-type, see [2] for details. This section deals with representation-directed coalgebras as defined in [29]. In particular, we describe the representation-directed serial coalgebras following the ideas of the previous sections, i.e., by means of their valued Gabriel quiver and using the localization in categories of comodules. In order to do this, we shall make use of the so-called computable comodules and Hom-computable coalgebras.

Assume that C is a basic coalgebra with fixed decompositions (1) and (2). Following [29], a right C-comodule M is defined to be *computable* if, for each $i \in I_C$, the sum $\ell_i(M) = \sum_{n=1}^{\infty} \ell_i(M[n])$, called the composition S_i-length of M, is finite, where $M[n] = \operatorname{soc}^n M/\operatorname{soc}^{n-1} M$ and $\ell_i(M[n])$ is the number of times that the simple comodule S_i appears as a summand in a semisimple decomposition of $M[n]$. We denote by comp^C (respectively by ${}^C\operatorname{comp}$) the full subcategory of $\mathcal{M}^C$ (respectively of ${}^C\mathcal{M}$) whose objects are computable comodules. The coalgebra C is said to be *Hom-computable* if every indecomposable injective right C-comodule is computable or, equivalently (cf. [29]), if $\operatorname{Hom}_C(E_i, E_j)$ has finite K-dimension for any two indecomposable injective right C-comodules E_i and E_j. Therefore, by the duality $D : \operatorname{inj}^C \to {}^C\operatorname{inj}$ stated in [6], the notion of Hom-computability is left–right symmetric. We now describe Hom-computable serial coalgebras. For that purpose we give a version for coalgebras of the Periodicity Theorem proved by Eisenbud and Griffith in [9].

Lemma 4.1. *Let C be a coalgebra and N a uniserial right C-comodule. If M is a subcomodule of N then M is uniserial and, moreover,* $\operatorname{soc}^t M = \operatorname{soc}^t N$ *for any positive integer t such that*

$\operatorname{soc}^t M \neq \operatorname{soc}^{t-1} M$. *As a consequence, M is uniserial if and only if every subcomodule of M is uniserial.*

Proof. Obviously, if $M \leqslant N$ then $\operatorname{soc} M = \operatorname{soc} N = S$ is a simple comodule. Let us now assume that $\operatorname{soc}^k M = \operatorname{soc}^k N$ for each $k \leqslant t-1$, and also $\operatorname{soc}^{t-1} M \neq \operatorname{soc}^t M$. Then

$$0 \neq \frac{\operatorname{soc}^t M}{\operatorname{soc}^{t-1} M} \leqslant \frac{\operatorname{soc}^t N}{\operatorname{soc}^{t-1} N} \cong S_t,$$

where S_t is a simple comodule. That is, $\operatorname{soc}^t M / \operatorname{soc}^{t-1} M \cong S_t$. Thus M is uniserial and, by its definition, $\operatorname{soc}^t M = \operatorname{soc}^t N$. □

Proposition 4.2 *(Periodicity Theorem). Let C be an indecomposable serial coalgebra and E_0 an indecomposable injective right C-comodule. Let the sequence of composition factors of E_0 be $S_0 = E_0[1] = \operatorname{soc} E$, $S_1 = E_0[2]$, $S_2 = E_0[3], \ldots$. Suppose that $S_1 \cong S_k$ for some $k > 1$, and let $h \neq 1$ be the smallest such integer. Then the valued Gabriel quiver of C is $\widetilde{\mathbb{A}}_h$, and $S_m \cong S_n$ if and only if $m \equiv n \pmod h$. If there is no such an h, then $S_m \cong S_n$ implies $m = n$.*

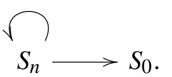
Proof. Let us assume that $S_0 \ncong S_h$ for any $h > 1$. Suppose also that $S_n \cong S_m$ for some $m \neq n$ and, furthermore, this is the first repetition, i.e., $S_i \ncong S_j$ for $i, j < m$. Let us consider the injective right C-comodule $E = E_0 \oplus E_n = Ce$, where $e = e_0 + e_n$. Then, by Proposition 3.1 and its proof, eCe is serial, $T_e(E_0) = \overline{E}_0$ is uniserial and its Loewy series is

$$S_0 \subseteq T_e\big(\operatorname{soc}^n E_0\big) \subseteq T_e\big(\operatorname{soc}^m E_0\big) \subseteq \cdots \subseteq \overline{E}_0,$$

where $\operatorname{soc}^2 \overline{E}_0 / S_0 \cong S_n$ and $\operatorname{soc}^3 \overline{E}_0 / \operatorname{soc}^2 \overline{E}_0 \cong S_n$. Let now $M = \overline{E}_0 / \operatorname{soc}^2 \overline{E}_0 \leqslant \overline{E}_n / S_n$. Then

$$S_n \cong \operatorname{soc}^3 \overline{E}_0 / \operatorname{soc}^2 \overline{E}_0 = \operatorname{soc} M \subseteq \operatorname{soc}(\overline{E}_n / S_n).$$

Thus there is a loop in the vertex S_n of the valued Gabriel quiver of eCe, Q_e. In addition, $\operatorname{soc}^2 \overline{E}_0 / S_0 \cong S_n$, so Q_e contains the subquiver



By Theorem 2.5, eCe is not serial. Hence m must equal n.

Suppose now that $S_h \cong S_0$ for some $h \neq 1$ and, moreover, it is the smallest such integer. First, by [21, Theorem 1.9], there is a path in (Q_C, d_C) of length h starting and ending at S_0, i.e., there is a cycle in (Q_C, d_C). By Theorem 2.5, $(Q_C, d_C) = \widetilde{\mathbb{A}}_h$. By a reasoning similar to the one done above, we may prove that $S_i \ncong S_j$ for any $i, j < h$ with $i \neq j$. Therefore it remains to show that $S_{l+h} \cong S_l$ for any $l > 1$. We denote by M the right comodule $E_0 / \operatorname{soc}^h E_0 \leqslant E_0$. Then, for any $l > 0$

$$S_{l+h} \cong \frac{\operatorname{soc}^{l+h+1} E_0}{\operatorname{soc}^{l+h} E_0} \cong \frac{\frac{\operatorname{soc}^{l+h+1} E_0}{\operatorname{soc}^h E_0}}{\frac{\operatorname{soc}^{l+h} E_0}{\operatorname{soc}^h E_0}} \overset{\blacktriangle}{\cong} \frac{\operatorname{soc}^{l+1} M}{\operatorname{soc}^l M} \overset{\blacktriangledown}{\cong} \frac{\operatorname{soc}^{l+1} E_0}{\operatorname{soc}^l E_0} \cong S_l,$$

where in ▲ we use [21, Lemma 1.4] and in ▼ we use Lemma 4.1. □

Proposition 4.3. *Let C be an indecomposable serial coalgebra. C is Hom-computable if and only if one of the following conditions holds*:

(a) *The right valued Gabriel quiver of C are either ${}_\infty\mathbb{A}_\infty$, or $\mathbb{A}_\infty$, or ${}_\infty\mathbb{A}$, or $\mathbb{A}_n$ for some $n \geqslant 1$.*
(b) *The right valued Gabriel quiver of C is $\widetilde{\mathbb{A}}_n$ for some $n \geqslant 1$, and C is finite-dimensional.*

Proof. Let us assume that C verifies either the condition (a) or the condition (b). First, if C is finite-dimensional then $\mathrm{Hom}_C(E_i, E_j)$ is finite-dimensional for each pair of indecomposable injective comodules. Now, if the valued Gabriel quiver of C is either $\mathbb{A}_\infty$, or ${}_\infty\mathbb{A}_\infty$, or ${}_\infty\mathbb{A}$, or $\mathbb{A}_n$ for some $n \geqslant 1$; then, by the Periodicity Theorem, for each indecomposable injective E and each $i \in I_C$, $\ell_i(E)$ is one or zero. Then C is Hom-computable.

Conversely, it is enough to prove that if $(Q_C, d_C) = \widetilde{\mathbb{A}}_n$ for some $n \geqslant 1$, and C is infinite-dimensional, then C is not Hom-computable. Now, since C is socle-finite, this is a consequence of [28, Corollary 2.10]. □

Following [30], we say that a finitely copresented indecomposable comodule M is said to be directing if there is no chain

$$M \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots\cdots \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M$$

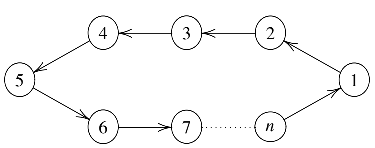
where M_i is finitely copresented and indecomposable for any $i = 1, \ldots, n$ and f_j is a non-zero non-isomorphism for any $j = 1, \ldots, n+1$. A coalgebra is said to be right (left) representation-directed if each finitely copresented indecomposable right (left) comodule is directing. Let us now classify serial representation-directed coalgebras in terms of their valued Gabriel quiver:

Proposition 4.4. *Let C be an indecomposable serial coalgebra. The following statements are equivalent*:

(a) *C is right representation-directed.*
(a′) *C is left representation-directed.*
(b) *The right valued Gabriel quiver of C are either ${}_\infty\mathbb{A}_\infty$, or $\mathbb{A}_\infty$, or ${}_\infty\mathbb{A}$, or $\mathbb{A}_n$ for some $n \geqslant 1$.*
(b′) *The left valued Gabriel quiver of C are either ${}_\infty\mathbb{A}_\infty$, or $\mathbb{A}_\infty$, or ${}_\infty\mathbb{A}$, or $\mathbb{A}_n$ for some $n \geqslant 1$.*

Proof. Since, by Proposition 2.4, the left valued Gabriel quiver of C is the opposite to the right one, it is enough to prove that C is right representation-directed if and only if (Q_C, d_C) is one of the above valued quivers.

Let us assume that $(Q_C, d_C) = \widetilde{\mathbb{A}}_n$ for some $n \geqslant 1$. We use the following labels in the vertices (we omit the labels of the arrows):



.

Then we may consider the chain

$$S_1 \xrightarrow{inc} \operatorname{soc}^2 E_1 \xrightarrow{p} S_n \xrightarrow{inc} \cdots\cdots \xrightarrow{p} S_2 \xrightarrow{inc} \operatorname{soc}^2 E_2 \xrightarrow{p} S_1$$

where p is the projection of $\operatorname{soc}^2 E_i$ onto $\operatorname{soc}^2 E_i/S_i \cong S_{i-1}$ (or S_n if $i = 1$), and *inc* is the inclusion of S_i in $\operatorname{soc}^2 E_i$. We recall that, for each $i = 1, \ldots, n$, $S_i \neq \operatorname{soc}^2 E_i$ since S_i is not an isolated point without loops in Q_C. Thus the morphisms *inc* and p are not isomorphisms. Finally, $\operatorname{soc}^2 E_i$ is indecomposable and, since C is right serial, finitely cogenerated. Thus C is not right representation-directed.

Let us now suppose that Q_C is one of the remaining quivers. By Proposition 4.3, C is Hom-computable and then, by [29, Proposition 2.13(c) and Lemma 6.4], C is right representation-directed if and only if each socle-finite localized coalgebra of C is right representation-directed. For each finite subset $\mathfrak{S} \subseteq \{S_i\}_{i\in C}$, by Proposition 3.4, the localized coalgebra D associated to $\mathfrak{S}$ is socle-finite, serial and Hom-computable. Therefore its valued Gabriel quiver is either $\mathbb{A}_n$ for some $n \geqslant 1$, or $\widetilde{\mathbb{A}}_m$ for some $m \geqslant 1$. Nevertheless, by the Periodicity Theorem, the second case is not possible and then (Q_D, d_D) is $\mathbb{A}_n$ for some $n \geqslant 1$. Then D is finite-dimensional and right representation-directed. □

5. Finite-dimensional comodules over serial coalgebras

This section is devoted to give a complete list of all indecomposable finite-dimensional right comodules over a serial coalgebra and a description of the Auslander–Reiten quiver of the category $\mathcal{M}_f^C$. We recall from [5] that any finite-dimensional indecomposable comodule M over a serial coalgebra C is uniserial, and then, there exists an integer $t \geqslant 1$ such that $\operatorname{soc}^t M = M$. Thus M is a right D-comodule, where D is the subcoalgebra of C, $\operatorname{soc}^t C = \bigoplus_{i\in I_C} \operatorname{soc}^t E_i$ (and then it is serial, cf. [5]). We refer the reader to [6,7,17,22] and [26] for definitions and terminology concerning almost split sequences and the Auslander–Reiten quiver of a coalgebra, see also [2] for the classical theory for finite-dimensional algebras.

Theorem 5.1. *Let C be a serial coalgebra. The following statements hold*:

(a) *Each finite-dimensional indecomposable right C-comodule is isomorphic to* $\operatorname{soc}^k E$ *for some positive integer k and some indecomposable injective right C-comodule.*
(b) *The category of finite-dimensional right C-comodules has almost split sequences. Furthermore, for each indecomposable non-injective right C-comodule* $\operatorname{soc}^k E$, *the almost split sequence starting on this comodule is*

$$0 \longrightarrow \operatorname{soc}^k E \xrightarrow{\binom{i}{p}} \operatorname{soc}^{k+1} E \oplus \frac{\operatorname{soc}^k E}{\operatorname{soc} E} \xrightarrow{(q\ -j)} \frac{\operatorname{soc}^{k+1} E}{\operatorname{soc} E} \longrightarrow 0,$$

where i and j are the standard inclusions and p and q are the standard projections.

Proof. The proof is dual to the one for finite-dimensional algebras given in [2, Theorem 4.1]. Alternately, the results of [7] can be applied to see that the standard almost split sequences over sufficiently large serial finite-dimensional (subco)algebras are the almost split sequences over the whole coalgebra, by using direct limits. □

By applying Theorem 5.1, we can easily calculate the Auslander–Reiten quiver of a serial coalgebra. For example, we do it for hereditary serial path coalgebras. The reader may observe that the quivers occurring here also appear in the theory of coverings of bounded quivers, see [3] or [25].

Type ${}_\infty\mathbb{A}_\infty$. Let Q be the quiver

$$\cdots \xrightarrow{\alpha_{-2}} (-2) \xrightarrow{\alpha_{-1}} (-1) \xrightarrow{\alpha_0} (0) \xrightarrow{\alpha_1} (1) \xrightarrow{\alpha_2} (2) \xrightarrow{\alpha_3} \cdots$$

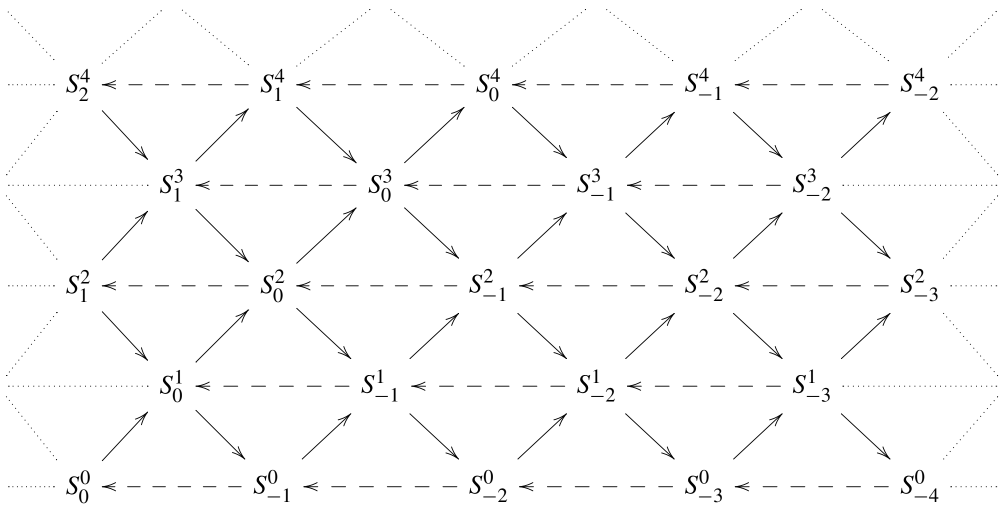
and let $C = KQ$ be the path coalgebra of Q. By Theorem 2.5, C is serial. Let E_i be the indecomposable injective right C-comodule associated to the vertex i, that is, $E_i = Ke_i \oplus (\bigoplus_{t \geqslant 0} K\alpha_i \cdots \alpha_{i-t})$, where e_i is the stationary path at i. Let

$$S_i^k = \operatorname{soc}^k E_i = Ke_i \oplus \left(\bigoplus_{t=0}^{k-1} K\alpha_i \cdots \alpha_{i-t} \right).$$

Now, since $\operatorname{soc}^k E_i / \operatorname{soc} E_i \cong \operatorname{soc}^{k-1} E_{i-1}$ for any k and i, the almost split sequences are the following:

$$0 \longrightarrow S_i^k \longrightarrow S_i^{k+1} \oplus S_{i-1}^{k-1} \longrightarrow S_{i-1}^k \longrightarrow 0,$$

for each $k \geqslant 0$ and each $i \in \mathbb{Z}$. Therefore, the Auslander–Reiten quiver of C is the following.



where each dashed arrow $Y \leftarrow - - X$ means that $Y = \tau(X)$, where $\tau = \mathrm{DTr}$ is the Auslander–Reiten translation, see [6] for definitions and details.

Type ${}_\infty\mathbb{A}$. Let Q be the quiver

$$\cdots \xrightarrow{\alpha_4} (4) \xrightarrow{\alpha_3} (3) \xrightarrow{\alpha_2} (2) \xrightarrow{\alpha_1} (1) \xrightarrow{\alpha_0} (0)$$

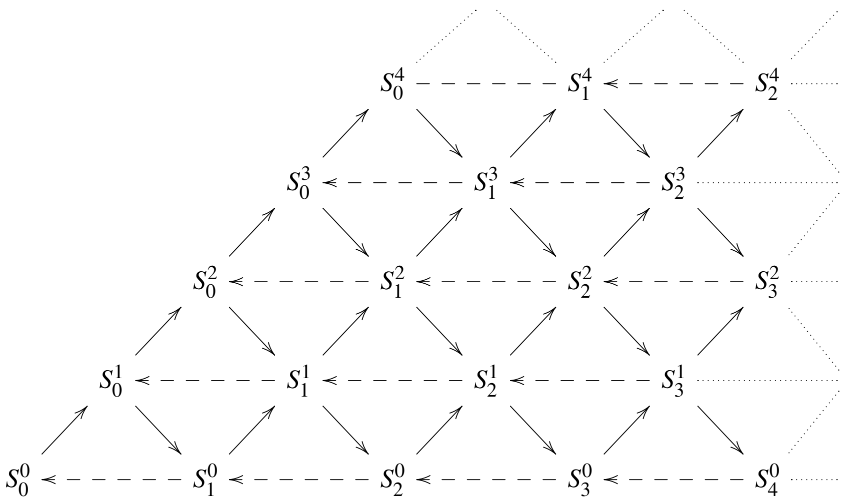
and let $C = KQ$ be the path coalgebra of Q. Again, by Theorem 2.5, C is serial. Let E_i be the indecomposable injective right C-comodule associated to the vertex i, that is, $E_i = Ke_i \oplus (\bigoplus_{t\geqslant 0} K\alpha_i \cdots \alpha_{i-t})$, where e_i is the stationary path at i. Let

$$S_i^k = \operatorname{soc}^k E_i = Ke_i \oplus \left(\bigoplus_{t=0}^{k-1} K\alpha_i \cdots \alpha_{i-t}\right).$$

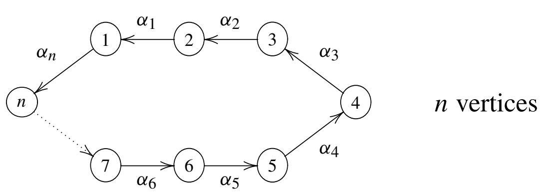
Since $\operatorname{soc}^k E_i / \operatorname{soc} E_i \cong \operatorname{soc}^{k-1} E_{i+1}$ for any k and i, the almost split sequences are the following:

$$0 \longrightarrow S_i^k \longrightarrow S_i^{k+1} \oplus S_{i+1}^{k-1} \longrightarrow S_{i+1}^k \longrightarrow 0,$$

for each $i, k \geqslant 0$. Therefore, the Auslander–Reiten quiver of C is the following.



Type $\widetilde{\mathbb{A}}_n$, $n \geqslant 1$. Let Q be the quiver



and let $C = KQ$ be the path coalgebra of Q. Clearly, C is serial. Let E_i be the indecomposable injective right C-comodule associated to the vertex i, that is, $E_i = Ke_i \oplus (\bigoplus_{t\geqslant 0} K\alpha_{[i]} \cdots \alpha_{[i-t]})$ where $[p] \equiv p \pmod n$ for any $p > 0$. Let

$$S_i^k = \operatorname{soc}^k E_i = Ke_i \oplus \left(\bigoplus_{t=0}^{k-1} K\alpha_{[i]} \cdots \alpha_{[i-t]}\right).$$

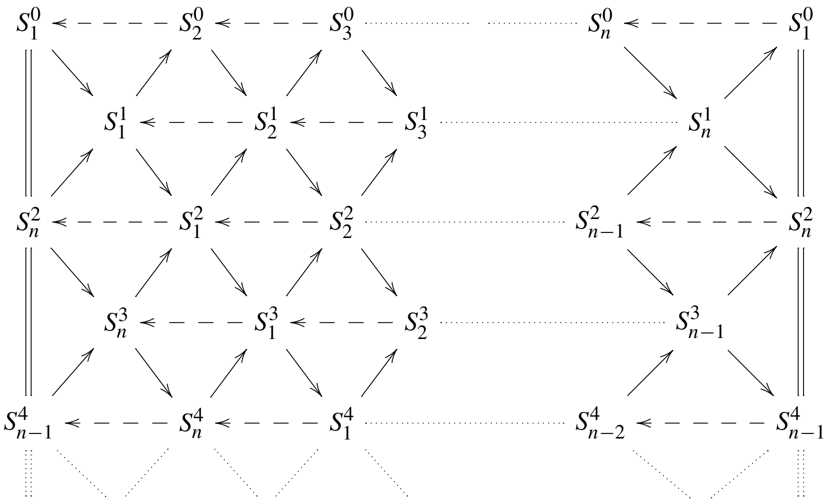
Now, since $\operatorname{soc}^k E_n / \operatorname{soc} E_n \cong \operatorname{soc}^{k-1} E_1$ and $\operatorname{soc}^k E_i / \operatorname{soc} E_i \cong \operatorname{soc}^{k-1} E_{i+1}$ for each $i = 1, \ldots, n-1$, for any $k \geqslant 0$, the almost split sequences are the following:

$$0 \longrightarrow S_i^k \longrightarrow S_i^{k+1} \oplus S_{i+1}^{k-1} \longrightarrow S_{i+1}^k \longrightarrow 0,$$

for each $i = 1, \ldots, n-1$ and $k \geqslant 0$; and

$$0 \longrightarrow S_n^k \longrightarrow S_n^{k+1} \oplus S_1^{k-1} \longrightarrow S_1^k \longrightarrow 0,$$

for any $k \geqslant 0$. Therefore, the Auslander–Reiten quiver of C is the following.



The Auslander–Reiten quiver of the remaining serial path coalgebras are described in [22].

Remark 5.2. Observe that, for each indecomposable finite-dimensional comodule $M = \operatorname{soc}^n E$, $\tau(M) = \operatorname{soc}^{k+1} E / \operatorname{soc} E$, and then length M = length $\tau(M)$. That is, the comodules lying in the same τ-orbit has the same length.

6. A theorem of Eisenbud and Griffith for coalgebras

We finish the paper by a version of the theorem of Eisenbud and Griffith [9, Corollary 3.2] for coalgebras. We recall that this theorem asserts that every proper quotient of a hereditary noetherian prime ring is serial. Obviously, first we need a translation of the concepts from ring terminology to the notions used in coalgebra theory. About hereditariness, the concept is well known in coalgebras (cf. [20]) and no explanation is needed. The "coalgebraic" version of noetherianess is the so-called co-noetherianess (cf. [11]). We recall that a comodule M is said to be co-noetherian if every quotient of M is embedded in a finite direct sum of copies of C. Nevertheless, we shall use a weaker concept: strictly quasi-finiteness [11], namely, M is strictly quasi-finite if every quotient of M is quasi-finite. This is due to fact that we may reduce the problem to socle-finite coalgebras and then, under this condition, both classes of comodules coincide

[11, Proposition 1.6]. Finally, following [16], a coalgebra is called prime if for any subcoalgebras $A, B \subseteq C$ such that $A \wedge B = C$, then $A = C$ or $B = C$. For the convenience of the reader we present the following example:

Example 6.1. Let C be a hereditary colocal coalgebra such that $C/S \cong C \oplus C$, where S is the unique simple comodule (or subcoalgebra). We prove that C is not co-noetherian.

Let us consider the subcomodule N_2 of C which yields the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & N_2 & \longrightarrow & S & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle (0,i)} & & \\ 0 & \longrightarrow & S & \longrightarrow & C & \longrightarrow & C \oplus C & \longrightarrow & 0. \end{array}$$

Then, $N_2 \subseteq \operatorname{soc}^2 C$ and $N_2/S \cong S$. Now,

$$C/N_2 \cong (C/S)/(N_2/S) \cong (C \oplus C)/S \cong (C \oplus C \oplus C) = C^3.$$

Analogously, let N_3 be the subcomodule of C which yields the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & S & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle (0,0,i)} & & \\ 0 & \longrightarrow & N_2 & \longrightarrow & C & \longrightarrow & C \oplus C \oplus C & \longrightarrow & 0. \end{array}$$

Again, $N_3/N_2 \cong S$ and

$$C/N_3 \cong (C/N_2)/(N_3/N_2) \cong (C \oplus C \oplus C)/S \cong (C \oplus C \oplus C \oplus C) = C^4.$$

If we continue in this way, we obtain an increasing family of subcomodules $\{N_t\}_{t \geqslant 1}$, where $N_1 = S$, such that $C/N_t \cong C^{t+1}$ for any $t \geqslant 1$. Let us consider the uniserial subcomodule $N = \bigcup_{t \geqslant 1} N_t$ of C whose composition series (or Loewy series) is given by

$$0 \subset S \subset N_2 \subset N_3 \subset \cdots \subset N.$$

The comodule C/N has infinite-dimensional socle. To see this, for each $i \geqslant 0$, consider the short exact sequence

$$0 \longrightarrow N/N_i \longrightarrow C/N_i \longrightarrow C/N \longrightarrow 0$$

which yields the exact sequence

$$0 \longrightarrow \operatorname{soc}(N/N_i) \longrightarrow \operatorname{soc}(C/N_i) \longrightarrow \operatorname{soc}(C/N),$$

where $\operatorname{soc}(N/N_i) \cong S$ and $\operatorname{soc}(C/N_i) \cong S^{i+1}$. Thus $\dim_K \operatorname{soc}(C/N) \geqslant i \cdot \dim_K S$ for any $i \geqslant 1$. Consequently, C is not co-noetherian.

Theorem 6.2. *Let C be a basic socle-finite coalgebra over an arbitrary field. If C is prime, hereditary and co-noetherian then C is serial.*

Proof. If C is colocal, then the valued Gabriel quiver of C is either a single point or a vertex with a loop labeled by a pair (d', d''). Now, if $d' \geqslant 2$ or $d'' \geqslant 2$, proceeding as in Example 6.1, C is not co-noetherian. Thus $d' = d'' = 1$ and, by Theorem 2.5, C is serial. Assume then that C is not colocal. Let us first develop some properties about the valued Gabriel quiver of a localized coalgebra of C. These are inspired by the ones obtained in [15] for path coalgebras. Let us suppose that S_x, S_y and S_z are three simple C-comodules (where S_x could equals S_z) such that there is path in (Q_C, d_C)

$$S_x \xrightarrow{(d_1,d_2)} S_y \xrightarrow{(c_1,c_2)} S_z.$$

Let $e \in C^*$ be an idempotent and eCe the localized coalgebra associated to e whose quotient functor we denote by T_e. Assume that S_x and S_z are torsion-free and S_y is torsion. Since C is hereditary, $E_z/S_z \cong \bigoplus_{j\in J} E_j^{r_j} \oplus \bigoplus_{t\in T} E_t^{r_t}$, where S_j is torsion for all $j \in J$ and S_t is torsion-free for all $t \in T$, and r_α is a positive integer for any $\alpha \in J \cup T$. Now, since there is an arrow from S_y to S_z, $y \in J$ and $E_y^{r_y} \subseteq E_z/S_z$, where $r_y = c_1$. Then $T_e(E_y^{c_1}) \subseteq T_e(E_z/S_z)$. Finally, since $S_x^{d_1} \subseteq T_e(E_y) \cong T_e(E_y/S_y)$ then $S_x^{d_1c_1} \subseteq T_e(E_z/S_z) = \overline{E}_z/S_z$. That is, there exists an arrow

$$S_x \xrightarrow{(h_1,h_2)} S_z$$

in (Q_{eCe}, d_{eCe}) such that $h_1 \geqslant d_1c_1$. By Proposition 2.4, it is easy to see that $h_2 \geqslant d_2c_2$. Note that the hereditariness is a left–right symmetric property.

By an easy induction one may prove that if there is a path

$$S_x \xrightarrow{(a_0,b_0)} S_1 \xrightarrow{(a_1,b_1)} \cdots \longrightarrow S_{n-1} \xrightarrow{(a_{n-1},b_{n-1})} S_n \xrightarrow{(a_n,b_n)} S_z \tag{3}$$

such that S_i is torsion, for all $i = 1, \dots, n$. Then there is an arrow

$$S_x \xrightarrow{(h_1,h_2)} S_z$$

in (Q_{eCe}, d_{eCe}) such that $h_1 \geqslant a_0a_1 \cdots a_n$ and $h_2 \geqslant b_0b_1 \cdots b_n$. Furthermore, following this procedure, one may prove that if $\mathfrak{P} = \{p^l\}_{l\in\Lambda}$ is non-empty, where $\mathfrak{P}$ is the set of all possible paths p^l in (Q_C, d_C) as described in (3), i.e., starting at S_x, ending at S_y and whose intermediate vertices are torsion, then there is an arrow

$$S_x \xrightarrow{(h_1,h_2)} S_z$$

in (Q_{eCe}, d_{eCe}) such that $h_1 = \sum_l a_0^l a_1^l \cdots a_{n_l}^l$ and $h_2 = \sum_l b_0^l b_1^l \cdots b_{n_l}^l$. Here we have denoted by $a_0^l, \dots, a_{n_l}^l$ and by $b_0^l, \dots, b_{n_l}^l$ the first and the second component, respectively, of the labels of the arrows whose composition build the path p^l. We refer the reader to [21] for more details about injective comodules and the localization functors.

Now we consider a primitive orthogonal idempotent $e_x \in C^*$ and e_xCe_x the localized coalgebra of C associated to e_x. By [11, Proposition 1.8] and [10, p. 376, Corollary 5], e_xCe_x is co-noetherian and hereditary, respectively. Therefore, following the colocal case, the valued

Gabriel quiver of e_xCe_x must be a single point or a vertex with a loop labeled by $(1,1)$. As a consequence, by the above considerations, each vertex of the valued Gabriel quiver of C is inside of at most one cycle and, if exists, the arrows of that cycle are labeled by $(1,1)$.

Finally, we prove that for each pair of vertices of Q_C there is a cycle passing through these two vertices. This yields the statement of the theorem since, together with the above conditions, the only possible quiver is $(Q_C, d_C) = \widetilde{\mathbb{A}}_n$ for some $n \geqslant 1$ and then C is serial.

Fix two different simple comodules S_x and S_y. Let e_x and e_y be the primitive orthogonal idempotents in C^* associated to S_x and S_y, respectively. We set $e = e_x + e_y$. By [16, Proposition 4.1], eCe is prime. First, let us suppose that there is no path in Q_C from S_x to S_y nor vice versa. Then eCe has two connected components and, by [28, Corollary 2.4(b)], eCe is not indecomposable. Thus eCe is not prime (cf. [16, Lemma 1.4]). Now, suppose that there is a path from S_x to S_y but there is no path from S_y to S_x. Then the valued Gabriel quiver of eCe is a subquiver of the following quiver:

$$\overset{\circlearrowleft}{S_x} \xrightarrow{(a,b)} \overset{\circlearrowleft}{S_y}\ .$$

By [21, Lemma 3.7], $e_xCe_y = T_x(E_y) \neq 0$ and $e_yCe_x = T_y(E_x) = 0$. Therefore, there is a vector space direct sum decomposition $eCe = e_xCe_x \oplus e_xCe_y \oplus e_yCe_y$. A straightforward calculation shows that the linear map

$$\Psi : D = \begin{pmatrix} e_yCe_y & e_xCe_y \\ 0 & e_xCe_x \end{pmatrix} \longrightarrow eCe,$$

between eCe and the bipartite coalgebra D (in the sense of [18]), given by

$$\Psi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = a + b + c$$

is an isomorphism of coalgebras, see [4] for definitions and details about the structures of the spaces e_xCe_x, e_yCe_y and e_xCe_y. We recall that the coalgebra structure of the bipartite coalgebra D is given by the formulae:

- $\Delta(a+b+c) = \Delta_y(a) + \rho_y(b) + \rho_x(b) + \Delta_x(c)$, where ρ_y and ρ_x are the e_yCe_y–e_xCe_x-bicomodule structure maps of e_xCe_y; and Δ_x and Δ_y are the comultiplication of the coalgebras e_yCe_y and e_xCe_x, respectively.
- $\epsilon(a+b+c) = \epsilon_y(a) + \epsilon_x(b)$, where ϵ_y and ϵ_x are the counit of the coalgebras e_yCe_y and e_xCe_x, respectively.

Then $eCe = e_yCe_y \wedge e_xCe_x$ and therefore eCe is not prime. $\square$

Now we prove the Eisenbud–Griffith Theorem for coalgebras.

Corollary 6.3. *If C is a subcoalgebra of a prime, hereditary and strictly quasi-finite* (*left and right*) *coalgebra over an arbitrary field, then C is serial.*

Proof. By [5, Proposition 1.5], we may assume that C is prime, hereditary and strictly quasi-finite itself. Let $e \in C^*$ be an idempotent such that the localized coalgebra eCe is socle-finite.

By [10, Corollary 5, p. 376], [16, Proposition 4.1] and [11, Proposition 1.8], eCe is hereditary, prime and right strictly quasi-finite, respectively. Moreover, by [11, Proposition 1.5], eCe is co-noetherian. Therefore, by Theorem 6.2, eCe is serial. Thus the result follows from Proposition 3.4. □

Remark 6.4. Observe that, whilst the hypothesis of Eisenbud–Griffith Theorem for rings requires a proper homomorphic image, in Corollary 6.3 it is not needed to assume C is a proper subcoalgebra. This due to the fact that every comodule over any coalgebra is locally finite.

References

[1] E. Abe, Hopf Algebras, Cambridge University Press, 1977.
[2] I. Assem, D. Simson, A. Skowroński, Elements of Representation Theory of the Associative Algebras, vol. 1: Techniques of Representation Theory, London Math. Soc. Stud. Texts, vol. 65, Cambridge University Press, London, 2006.
[3] K. Bongartz, P. Gabriel, Covering spaces in representation theory, Invent. Math. 65 (1982) 331–378.
[4] J. Cuadra, J. Gómez-Torrecillas, Idempotents and Morita–Takeuchi theory, Comm. Algebra 30 (2002) 2405–2426.
[5] J. Cuadra, J. Gómez-Torrecillas, Serial coalgebras, J. Pure Appl. Algebra 189 (2004) 89–107.
[6] W. Chin, M. Kleiner, D. Quinn, Almost split sequences for comodules, J. Algebra 249 (2002) 1–19.
[7] W. Chin, M. Kleiner, D. Quinn, Local theory of almost split sequences for comodules, Ann. Univ. Ferrara Sez. VII (N.S.) 51 (2005) 183–196.
[8] W. Chin, S. Montgomery, Basic coalgebras, in: AMS/IP Stud. Adv. Math., vol. 4, 1997, pp. 41–47.
[9] D. Eisenbud, P. Griffith, Serial rings, J. Algebra 17 (1971) 389–400.
[10] P. Gabriel, Des categories abeliennes, Bull. Soc. Math. France 90 (1962) 323–448.
[11] J. Gómez-Torrecillas, C. Năstăsescu, B. Torrecillas, Localization in coalgebras. Applications to finiteness conditions, J. Algebra Appl. 6 (2007) 233–243.
[12] J.A. Green, Locally finite representations, J. Algebra 41 (1976) 137–171.
[13] P. Jara, L.M. Merino, G. Navarro, On path coalgebras of quivers with relations, Colloq. Math. 102 (2005) 49–65.
[14] P. Jara, L.M. Merino, G. Navarro, Localization in tame and wild coalgebras, J. Pure Appl. Algebra 211 (2007) 342–359.
[15] P. Jara, L. Merino, G. Navarro, J.F. Ruíz, Localization in coalgebras, stable localizations and path coalgebras, Comm. Algebra 34 (2006) 2843–2856.
[16] P. Jara, L. Merino, G. Navarro, J.F. Ruíz, Prime path coalgebras, preprint, arXiv: 0802.0692v1 [math.RA].
[17] J. Kosakowska, D. Simson, Hereditary coalgebras and representations of species, J. Algebra 293 (2005) 457–505.
[18] J. Kosakowska, D. Simson, Bipartite coalgebras and a reduction functor for coradical square complete coalgebras, Colloq. Math. 112 (2008) 89–129.
[19] S. Montgomery, Hopf Algebras and Their Actions on Rings, CBMS Reg. Conf. Ser. Math., vol. 82, Amer. Math. Soc., 1993.
[20] C. Nastasescu, B. Torrecillas, Y.H. Zhang, Hereditary coalgebras, Comm. Algebra 24 (1996) 1521–1528.
[21] G. Navarro, Some remarks on localization in coalgebras, Comm. Algebra (2008), in press, preprint, arXiv: math.RA/0608425.
[22] S. Nowak, D. Simson, Locally Dynkin quivers and hereditary coalgebras whose left comodules are direct sums of finite dimensional comodules, Comm. Algebra 30 (2002) 455–476.
[23] N. Popescu, Abelian Categories with Applications to Rings and Modules, London Math. Soc. Monogr., vol. 3, Academic Press, London/New York, 1973.
[24] D.E. Radford, On the structure of pointed coalgebras, J. Algebra 77 (1982) 1–14.
[25] C. Riedtmann, Algebren, Darstellunngsköcher, Überlagerungen und zurück, Comment. Math. Helv. 55 (1980) 199–224.
[26] D. Simson, Coalgebras, comodules, pseudocompact algebras and tame comodule type, Colloq. Math. 90 (2001) 101–150.
[27] D. Simson, Path coalgebras of quivers with relations and a tame-wild dichotomy problem for coalgebras, in: Lect. Notes Pure Appl. Math., vol. 236, 2005, pp. 465–492.
[28] D. Simson, Irreducible morphisms, the Gabriel-valued quiver and colocalizations for coalgebras, Int. J. Math. Sci. 72 (2006) 1–16.

[29] D. Simson, Hom-computable coalgebras, a composition factors matrix and the Euler bilinear form of an Euler coalgebra, J. Algebra 315 (2007) 42–75.
[30] D. Simson, Localising embeddings of comodule categories with applications to tame and Euler coalgebras, J. Algebra 312 (2007) 455–494.
[31] M.E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[32] M. Takeuchi, Morita theorems for categories of comodules, J. Fac. Sci. Univ. Tokyo 24 (1977) 629–644.
[33] D. Woodcock, Some categorical remarks on the representation theory of coalgebras, Comm. Algebra 25 (1997) 2775–2794.