Towards an Optimum Test for Non-additivity in Tukey’s Model

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In a two-way classification model with one observation per cell, the existence of an optimum test for non-additivity is investigated under Tukey’s model. An expression for the local conditional (given the complete sufficient statistic under the null hypothesis) power of any invariant similar test is provided. It turns out that no optimum test exists in the class of invariant similar tests. It is further shown that there exist two exact $F$ tests, different from Tukey’s test, which are locally more powerful than Tukey’s test under suitable restrictions on the main effects. Local unbiasedness of Tukey’s test as well as the proposed tests is also established. © 1991 Academic Press, Inc.

1. INTRODUCTION

For a two-way layout with one observation per cell, it is well known that the usual analysis of variance is not possible if interaction terms are present, unless some assumptions are made on the interaction. The first attempt to provide a test for the significance of the interaction is due to Tukey [17], who, from heuristic considerations, derived an $F$ test, though without assuming any specific functional form for the interaction. However, as pointed out later by Ward and Dick [18], Scheffe [14], and Graybill [4], Tukey’s test is really meaningful when the interaction is of a particular form. To be specific, consider the model

$$y_{ij} = \mu + \tau_i + \beta_j + \lambda \tau_i \beta_j + e_{ij}$$

$$i = 1, 2, ..., v; \quad j = 1, 2, ..., b,$$

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where \( y_{ij} \)'s are the observations, \( \mu \) is an overall mean, \( \tau_i \)'s and \( \beta_j \)'s are the main effects (treatment effects and block effects, respectively), \( \lambda \) is the interaction parameter, and \( e_{ij} \)'s are i.i.d. \( N(0, \sigma^2) \) random variables. It is assumed that (1.1) is a fixed effects model, and the problem is to test \( H_0: \lambda = 0 \) vs \( H_1: \lambda \neq 0 \). In this context, Tukey's test can be described as follows. Let \( \hat{\tau}_i \) and \( \hat{\beta}_j \) denote the least squares estimates of \( \tau_i \) and \( \beta_j \), respectively, under \( H_0 \). If \( SS_{\text{Residual}} \) denotes the residual sum of squares under \( H_0 \), then Tukey's test rejects \( H_0 \) for large values of

\[
W_0 = \frac{\{(b-1)(v-1)-1\} R_0^2}{SS_{\text{Residual}} - R_0^2},
\]

where

\[
R_0^2 = \frac{(\sum_{i,j} \hat{\tau}_i \hat{\beta}_j y_{ij})^2}{(\sum_i \hat{\tau}_i^2)(\sum_j \hat{\beta}_j^2)}. \tag{1.3}
\]

It is also known that, under \( H_0 \), \( W_0 \) in (1.2) is distributed as central \( F \) with degrees of freedom \((1, (b-1)(v-1)-1)\) (cf. Scheffe [14, p. 133]). A justification for model (1.1) along with a generalization is given in Scheffe [14, Sect. 4.8].

An extension of (1.1) is given in Milliken and Graybill [12]; see also Rao [13, p. 251]. The literature on this problem appears to deal almost exclusively with the estimation of \( \lambda \) in (1.1) and the construction of an exact \( F \) test for various models similar to (1.1). We refer to Sinha, Saharay, and Mukhopadhyay [16] for results on estimation, and Hegemann and Johnson [5] for results on tests. However, so far nothing is known by way of optimality properties of Tukey's test or the existence of any optimum test for the above testing problem, although some power computations for Tukey's test are reported in Ghosh and Sharma [2] and Hegemann and Johnson [5].

Our object in this paper is to make an attempt in this direction, using the tool of invariance. For the model (1.1), a group which leaves the testing problem invariant is described in the next section. Based on the expansion of the probability ratio of the nonnull to null distributions of a maximal invariant, an expression for the local conditional (given the complete sufficient statistic under the null hypothesis) power of any invariant similar test is provided. Tukey's test turns out to be invariant, but not optimal, even locally, in the class of invariant similar tests. In fact, a locally best test does not exist even after reducing the problem through invariance and similarity. It is shown however that when \( \| \tau \|/\sigma \) and \( \| \beta \|/\sigma \) are small, two exact \( F \) tests different from, and locally better than, Tukey's test exist. Such a result may not be surprising in view of the observation of Hegemann and
Johnson [5] that Tukey's test has considerably small power when \( \| \tau \|/\sigma \) and \( \| \beta \|/\sigma \) are small. In the next section, the test statistics corresponding to the proposed tests are denoted by \( W_1 \) and \( W_2 \). It is observed that the choice between the two tests based on \( W_1 \) and \( W_2 \) depends on the relative magnitudes of \( \| \tau \|/\sigma \) and \( \| \beta \|/\sigma \). The \( F \) test based on \( W_2 \) is recommended if it is known that \( \| \tau \|/\sigma \) is small compared to \( \| \beta \|/\sigma \). Otherwise the \( F \) test based on \( W_1 \) is recommended. We would like to point out that if the above assumption on the magnitudes of \( \| \tau \|/\sigma \) and \( \| \beta \|/\sigma \) is violated, we would be risking a substantial loss in power if the tests proposed in this paper are used, instead of Tukey's test. In the sequel we have also established the local unbiasedness of Tukey's test as well as the proposed tests.

Our results demonstrate the difficulties associated with the problem even for the simple model (1.1). It may be noted that for ANOVA models, the invariance approach has been fruitfully applied to derive optimum tests in several situations (see Seifert [15] and Mathew and Sinha [10, 11]). In conclusion we may mention that several other models as well as appropriate tests for non-additivity have been proposed in the literature; see Gollob [3], Johnson and Graybill [6], Mandel [8, 9], and Williams [20]. While it is not difficult to study the power properties of Tukey's test and the proposed tests under these models (the nonnull distribution of each of these test statistics essentially follows from Ghosh and Sharma [2, Sect. 2]), the problem of derivation of an optimum test remains open. We hope to address these problems in the future.

2. DERIVATION OF THE TESTS

Let \( y_{ij} = (y_{ij1}, y_{ij2}, ..., y_{ijn})' \), \( j = 1, 2, ..., b \), \( y = (y_{11}, ..., y_{vn})' \), \( \bar{y}_v = (y_{v1}, y_{v2}, ..., y_{vb})' \), \( i = 1, 2, ..., v \), \( F_1 = 1_b \otimes I_v \), \( F_2 = I_b \otimes 1_v \), \( \tau = (\tau_1, \tau_2, ..., \tau_v)' \), and \( \beta = (\beta_1, \beta_2, ..., \beta_b)' \). Here \( 1_m \) denotes the \( m \)-component vector of ones. Then (1.1) can be written as

\[
y = \mu 1_v + F_1 \tau + F_2 \beta + \lambda (\beta \otimes \tau) + e,
\]

where \( e \) is the vector of \( e_i \)'s. It is assumed that \( \sum_{i=1}^v \tau_i = 0 = \sum_{j=1}^b \beta_j \), and the problem is to test \( H_0 : \lambda = 0 \) vs \( H_1 : \lambda \neq 0 \). Further, it is reasonable to assume that there is a number \( \gamma \) \((0 < \gamma < \infty)\) satisfying \( \| \tau \|/\sigma < \gamma \) and \( \| \beta \|/\sigma < \gamma \). This assumption is made in the rest of the paper.

Let \( \bar{y}_{ij} = (1/v) y_{ij1} \), \( i = 1, 2, ..., b \), \( \bar{y}_v = (1/b) y_{v1} \), \( i = 1, 2, ..., v \), \( \bar{y}_v = (1/v) y_{1v} \) and \( SS_{\text{Residual}} = \sum_{i,j} (y_{ij} - \bar{y}_{ij} - \bar{y}_i - \bar{y}_j + \bar{y}_v)^2 \). Then clearly a complete sufficient statistic under \( H_0 \) is given by

\[
T_0 = (\bar{y}_v, \bar{y}_1, ..., \bar{y}_{b-1}, \bar{y}_1, ..., \bar{y}_{v-1}, SS_{\text{Residual}}).
\]
Let $\mathcal{P}_m$ denote the group of $m \times m$ orthogonal matrices $P_m$ satisfying $P_m 1_m = \pm 1_m$. Then the testing problem under consideration is left invariant by the group $G = \{ g ; g = (a, c, P_b, P_v), c > 0, a \text{ arbitrary real}, P_b \in \mathcal{P}_b \text{ and } P_v \in \mathcal{P}_v \}$ acting on $y$ as

$$ gy = c \left( (P_b \otimes P_v) y + a 1_{bv} \right). \quad (2.3) $$

The invariance of Tukey's test follows from (1.2) and (1.3) by noting that $\beta = (1/v) \{(I_b - (1/b) 1_b 1_b') \otimes y\} y$, $\dot{\beta} = (1/b) \{(I_v - (1/v) 1_v 1_v') \otimes y\} y$ and $\sum_{i,j} \dot{\beta}_j y_{ij} = (\beta \otimes \dot{\beta}) y$. Writing $\lambda^* = \lambda \sigma$, $\tau^* = (1/\sigma) \tau$, and $\beta^* = (1/\sigma) \beta$, it is also clear that a maximal invariant in the parameter space is given by $\lambda^*$, $\| \tau^* \|$, and $\| \beta^* \|$.

In order to derive an optimum invariant test, we first derive the density ratio, $R$, of the nonnull to null distributions of a maximal invariant using Wijsman's representation theorem [19]. Let $dP_m$ denote the left invariant Haar measure on the compact group $\mathcal{P}_m$ (for a characterization of $\mathcal{P}_m$ and a description of $dP_m$, see Lemma 5.1 in the Appendix). Then $da(dc/c) dP_b dP_v$ is a left invariant measure on $G$. Applying Wijsman's [19] theorem, we obtain

$$ R = \int_{\mathcal{P}_m} \int_{\mathcal{P}_m} \int_{0}^{\infty} \int_{\infty}^{\infty} f(y/H_i) J^{-1} da(dc/c) dP_b dP_v \int_{\mathcal{P}_m} \int_{\mathcal{P}_m} \int_{0}^{\infty} \int_{\infty}^{\infty} f(y/H_0) J^{-1} da(dc/c) dP_b dP_v, \quad (2.4) $$

where $J$ is the Jacobian of the transformation $y \to gy$, $gy$ is given by (2.3), and $f(y/H_i)$ denotes the normal density of $y$ under $H_i$, $i = 0, 1$. $R$ is simplified in the Appendix (see Eq. (5.13)). Let $B = (y_{i,1} - y_{i,2}, ..., y_{i,b} - y_{i,2})'$, $T = (y_{i,1} - y_{i,2}, ..., y_{i,b} - y_{i,2})'$, $B^* = B/\sqrt{SS_{Total}}$, $T^* = T/\sqrt{SS_{Total}}$, $w = (1_b P_b 1_b) \tau^* P_v T + (1_v P_v 1_v) \beta^* P_b B$ and $SS_{Total} = \sum_{i,j} (y_{ij} - y_{..})^2$. Then from Eqs. (5.11) and (5.13) we have

$$ R = \exp \left\{ -\frac{\lambda^* y^*}{2} \right\} dP_v dP_b, \quad (2.5) $$

where $n = bv$. It may be noted that the denominator of $R$, say $h(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|)$, depends on $\| \beta^* \|, \| \tau^* \|$ and the complete sufficient statistic $T_0$ given in (2.2), and is obviously independent of $\lambda^*$ (see
Eqs. (5.19) and (5.20) for its actual computation, which justifies the notation used. In view of our assumption about \( \tau^* \), \( \beta^* \), and the fact that \( \sqrt{b} \| B^* \| < 1 \), \( \sqrt{\tau} \| T^* \| < 1 \), the uniform convergence of the denominator of (2.5) is obvious from its derivation given in the Appendix. The same conclusion also follows for the numerator with the added assumption that \( \lambda^* \) is bounded. It is clear from (2.5) that there is no uniformly most powerful test in the class of invariant similar tests.

To derive a locally best invariant (LBI) similar test, we expand \( R \) in (2.5) around \( \lambda^* = 0 \). It is shown in the Appendix (see the arguments below equation (5.13)) that the integrals in both the numerator and denominator of (2.5) vanish when \( r \) is odd, and furthermore, the coefficient of any odd power of \( \lambda^* \) in the numerator of (2.5) also vanishes. Hence, \( R \) can be expanded (around \( \lambda^* = 0 \)) as

\[
R = 1 + \lambda^* [ - \frac{1}{2} \| \tau^* \|^2 \| \beta^* \|^2 + \{ h(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|) \}^{-1} \psi_2 ]
\]

+ remainder term, \hspace{2cm} (2.6)

where

\[
\psi_2 = \sum_{r=2}^{\infty} \Gamma \left( \frac{n+r-1}{2} \right) \frac{2^{n+r-1}}{r!} \left( r \right) \times \int_{\gamma} \int_{\gamma} \frac{w^{r-2} \{ y'(P_b \otimes P_v')(\beta^* \otimes \tau^*) \}^2}{(SS_{Total})^{r/2}} dP_v dP_b. \hspace{2cm} (2.7)
\]

Clearly, the remainder term in (2.6) is \( o(\lambda^{r/2}) \) uniformly in \( y \) and trivially so in \( \tau^* \) and \( \beta^* \) (recall the assumption \( \| \tau^* \| < y \) and \( \| \beta^* \| < y \)). Since the terms in the infinite series in (2.7) involve \( \beta^* \) and \( \tau^* \), it follows that, in general, a LBI similar test also does not exist. In order to explore the possibility of obtaining meaningful LBI similar tests even under some restrictions on \( \beta^* \) and \( \tau^* \), and get a real feeling for what \( \psi_2 \) actually represents, we have evaluated \( \psi_2 \) exactly. This is given in Eq. (5.35) in the Appendix, which reads

\[
\psi_2 = \pi_0^* \| \beta^* \|^4 \| \tau^* \|^4 \left\{ \kappa(6, 1) + \sum_{r=8}^{\infty} \sum_{u=1}^{(r-2)/2-1} \left( \frac{r-2}{2} - u \right) \kappa(6, 1) + \sum_{r=6}^{\infty} \kappa \left( r, \frac{r-2}{2} \right) \frac{r-2}{2} \| \tau^* \| \| T^* \| \| B^* \| \right\} 
\]

\[
+ \frac{1}{2} \left( \pi_1^* - b \| T^* \|^{4} \right) \| \beta^* \|^2 \| \tau^* \|^4 
\]

\[
\times \left\{ \kappa(4, 1) + \sum_{r=6}^{\infty} \kappa \left( r, \frac{r-2}{2} \right) \frac{r-2}{2} \| \tau^* \| \| T^* \| \| T^* \| \right\}
\]

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\]

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\]

\[
\times \left\{ \kappa(4, 1) + \sum_{r=6}^{\infty} \kappa \left( r, \frac{r-2}{2} \right) \frac{r-2}{2} \| \tau^* \| \| T^* \| \| T^* \| \right\}
\]
OPTIMUM TESTS IN TUKEY'S MODEL

\[ + \| \beta^* \|^2 \| B^* \|^2 \left\{ \kappa(6, 1) \right. \]
\[ + \sum_{r=8}^{\infty} \left( r-2 \right)/2 \left\{ \kappa(r, 0) \right. \]
\[ + \frac{1}{2} \left( \pi^*_2 - \psi \| B^* \|^4 \| \beta^* \|^4 \| \tau^* \|^2 \right) \]
\[ \times \left[ \left\{ \kappa(4, 0) \right. \] + \sum_{r=6}^{\infty} \frac{r-2}{2} \| \beta^* \| r-4 \| B^* \| r-4 \left. \right\} \]
\[ + \| \tau^* \|^2 \| T^* \|^2 \left\{ \kappa(6, 1) \right. \]
\[ + \sum_{r=8}^{\infty} \left. \frac{r-2}{2} - u \right\} \]
\[ \times h_{r, u}(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|) \right\} \]
\[ + h_1(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|) \right. \]  \hspace{1cm} (2.8)

where \( \kappa(r, u), h_{r, u}(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|) \) and \( h_1(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|) \) are respectively given by (5.28), (5.31), and (5.36), and, furthermore,

\[ \pi^*_0 = \frac{1}{(SS_{Total})^2} \left\{ \sum_{j=1}^{b} \sum_{i=1}^{v} (\bar{y}_{..} - \bar{y}_{i..})(\bar{y}_{..} - \bar{y}_{i..})y_{ij} \right\}^2 \]  \hspace{1cm} (2.9)

\[ \pi^*_1 = \frac{1}{(SS_{Total})^2} \sum_{j=1}^{b} \left\{ \sum_{i=1}^{v} (\bar{y}_{..} - \bar{y}_{i..})^2 y_{ij} \right\} \]  \hspace{1cm} (2.10)

and

\[ \pi^*_2 = \frac{1}{(SS_{Total})^2} \sum_{j=1}^{b} \left\{ \sum_{i=1}^{v} (\bar{y}_{..} - \bar{y}_{i..})^2 y_{ij} \right\} \]  \hspace{1cm} (2.11)

It is highly interesting to observe how the test statistic \( \pi^*_0 \), which essentially corresponds to Tukey's test, and the two other statistics \( \pi^*_1 \) and \( \pi^*_2 \), with no apparent interpretation, emerge as vital components of \( \psi_2 \). The above representation of \( \psi_2 \) and hence of \( R \) indicates that, up to \( o(\lambda^2) \), the statistics \( B^*, T^* \), \( \pi^*_0, \pi^*_1, \) and \( \pi^*_2 \) are sufficient for \( \lambda^*, \lambda^* \), and \( \beta^* \) under \( H_1 \). Since under \( H_0 \), \( T^* \), and \( B^* \) are sufficient for \( \tau^* \) and \( \beta^* \), the representation also brings out clearly the role of \( \pi^*_0, \pi^*_1, \) and \( \pi^*_2 \) as possible candidates for locally optimum test statistics.

We are now in a position to give an expression for the local behavior of the power function of an invariant similar size \( \alpha \) test for testing \( H_0: \lambda = 0 \) vs \( H_1: \lambda \neq 0 \). It should be noted that since \( T_0 \) is complete sufficient under \( H_0 \) a similar test of size \( \alpha \) will also have its conditional size, given \( T_0 \), equal to \( \alpha \) (see Lehmann [7, p.140]). Using (2.6) and (2.8), we get the following result.
THEOREM 2.1. Consider the problem of testing $H_0: \lambda = 0$ vs $H_1: \lambda \neq 0$ in the model (1.1), where $\tau$ and $\beta$ satisfy $\|\tau\|/\sigma < \gamma$ and $\|\beta\|/\sigma < \gamma$ for some $0 < \gamma < \infty$. Let $T_0, \lambda^*, \tau^*, \beta^*, T^*, B^*, h(\|\tau^*\|, \|\beta^*\|, \|T^*\|, \|B^*\|), \pi^*_0, \pi^*_1$, and $\pi^*_2$ be as described above. If $\phi(y)$ is the test function of an invariant similar size $\alpha$ test, then the conditional local power of $\phi(y)$, conditionally given $T_0$, can be evaluated as

$$
\alpha \left(1 - \frac{1}{2} \lambda^* \|\tau^*\|^2 \|\beta^*\|^2\right) + \lambda^* \left\{ h(\|\tau^*\|, \|\beta^*\|, \|T^*\|, \|B^*\|) \right\}^{-1} \times \left[ E_{H_0}(\phi(y) \pi^*_0 \mid T_0) \|\beta^*\|^4 \|\tau^*\|^4 \left\{ \kappa(6, 1) \right\} \right.
$$

$$
+ \sum_{r=8}^{\infty} \sum_{u=1}^{(r-2)/2-1} \left(\frac{r-2}{2} - u\right) u h_{r,u}(\|\tau^*\|, \|\beta^*\|, \|T^*\|, \|B^*\|) \right\} \left[ \frac{1}{2} \left\{ E_{H_0}(\phi(y) \pi^*_1 \mid T_0) - b \|T^*\| \|\tau^*\| \right\} \|\beta^*\|^2 \|\tau^*\|^4
$$

$$
\times \left\{ \kappa(4, 1) + \sum_{r=6}^{\infty} \kappa \left(\frac{r-2}{2}, \frac{r-2}{2}, \|\tau^*\|, \|T^*\|, \|B^*\|\right) \right\} \right. \left[ \lambda^* \|\tau^*\|^{r-4} \|\tau^*\| \right) \right.$$

$$
+ \|\beta^*\|^2 \|B^*\|^2 \left\{ \kappa(6, 1) \right\}
$$

$$
+ \sum_{r=8}^{\infty} \sum_{u=1}^{(r-2)/2-1} u h_{r,u}(\|\tau^*\|, \|\beta^*\|, \|T^*\|, \|B^*\|) \right\} \left[ \frac{1}{2} \left\{ E_{H_0}(\phi(y) \pi^*_2 \mid T_0) - c \|B^*\| \right\} \|\beta^*\|^4 \|\tau^*\|^2
$$

$$
\times \left\{ \kappa(4, 0) + \sum_{r=6}^{\infty} \kappa(r, 0) \frac{r-2}{2} \|\beta^*\|^{r-4} \|B^*\|^{r-4} \right\} \right. \left[ \lambda^* \|\tau^*\|^2 \|T^*\|^2 \left\{ \kappa(6, 1) + \sum_{r=8}^{\infty} \sum_{u=1}^{(r-2)/2-1} \left(\frac{r-2}{2} - u\right) \right. \right.
$$

$$
\times h_{r,u}(\|\tau^*\|, \|\beta^*\|, \|T^*\|, \|B^*\|) \right\}
$$

$$
+ h_1(\|\tau^*\|, \|\beta^*\|, \|T^*\|, \|B^*\|) \alpha \right] + o(\lambda^2),
$$

where $\kappa(r, u), h_{r,u}(\|\tau^*\|, \|\beta^*\|, \|T^*\|, \|B^*\|)$ and $h_1(\|\tau^*\|, \|\beta^*\|, \|T^*\|, \|B^*\|)$ are respectively given by (5.28), (5.31), and (5.36).
Let us now carefully examine the conditional local power of $\phi(y)$ given in Theorem 2.1. It is clear that when both $\|\tau^*\|$ and $\|\beta^*\|$ are small, the leading terms in the conditional local power are $E_{H_0}(\phi(y) \pi_{1}^* | T_0)$ and $E_{H_0}(\phi(y) \pi_{2}^* | T_0)$. However, when $\|\tau^*\|$ and $\|\beta^*\|$ are arbitrary (except for the boundedness assumption), no single term dominates the conditional local power. Thus $E_{H_0}(\phi(y) \pi_{1}^* | T_0)$ is never the leading term in the conditional local power. Our numerical computations show that the tests based on $\pi_1^*$ and $\pi_2^*$ have significantly more local power compared to Tukey's test when $\|\tau^*\|$ and $\|\beta^*\|$ are small (see Table I). However, Tukey's test outperforms the tests based on $\pi_1^*$ and $\pi_2^*$ when $\|\tau^*\|$ and $\|\beta^*\|$ are large (see Table II). In general, if both $\|\tau^*\|$ and $\|\beta^*\|$ are small and if the magnitude of $\|\tau^*\|$ relative to $\|\beta^*\|$ is known, then the test can be described using a linear combination of $\pi_1^*$ and $\pi_2^*$. Under the further assumption that $\|\beta^*\|$ is small relative to $\|\tau^*\|$, it follows from Theorem 2.1 that the LBI similar test rejects $H_0$ for large values of $\pi_1^*$

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TABLE II
Simulated Power of the Tests Based on $W_0$ (Tukey's Test), and $W_1$ and $W_2$ (the Proposed Tests) for $b = v = 6$ and for Large Values of $\tau^2/\sigma^2$ and $\beta^2/\sigma^2$ Based on 5000 Simulations

<table>
<thead>
<tr>
<th>$\sigma^2\lambda^2$</th>
<th>$\tau^2/\sigma^2$</th>
<th>$\beta^2/\sigma^2$</th>
<th>Power of $W_0$</th>
<th>Power of $W_1$</th>
<th>Power of $W_2$</th>
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<td>4</td>
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</table>

conditionally given $T_0$ or, equivalently, for large values of $\bar{\pi}_1$ conditionally given $T_0$, where

$$\bar{\pi}_1 = \sum_{j=1}^{b} \left\{ \sum_{i=1}^{v} (\bar{y}_i, - \bar{y}_i)(y_{ij} - \bar{y}_j, - \bar{y}_j, + \bar{y}_j) \right\}^2.$$  \hspace{1cm} (2.12)

In this case the test simplifies to an exact unconditional $F$ test, as we shall show. Let

$$z_j = \sum_{i=1}^{v} (\bar{y}_i, - \bar{y}_j)(y_{ij} - \bar{y}_j, - \bar{y}_j, + \bar{y}_j), \hspace{1cm} j = 1, 2, \ldots, b.$$  \hspace{1cm} (2.13)

Then from (2.12)

$$\bar{\pi}_1 = \sum_{j=1}^{b} z_j^2.$$  \hspace{1cm} (2.14)

Let $\mathbf{z} = (z_1, z_2, \ldots, z_b)'$ and $SS_T = \sum_{i=1}^{v} (\bar{y}_i, - \bar{y}_i)^2$. It is readily verified that, under $H_0$, $(y_{ij} - \bar{y}_j, - \bar{y}_j, + \bar{y}_j)$ is independent of $\bar{y}_i$, and thus, given $\bar{y}_i$. $(i = 1, 2, \ldots, v)$, $\mathbf{z}$ is multivariate normal with mean 0 and covariance matrix $\sigma^2 SS_T (I_b - (1/b) \mathbf{1}_b \mathbf{1}_b')$. Hence, under $H_0$, given $\bar{y}_i$. $(i = 1, 2, \ldots, v)$, $\mathbf{z}/\sigma \sqrt{SS_T}$ is multivariate normal with mean zero and covariance matrix $(I_b - (1/b) \mathbf{1}_b \mathbf{1}_b')$, which is also its unconditional distribution. Let $Z_1$ be a $b \times (b - 1)$ matrix satisfying $Z_1'Z_1 = I_{b-1}$ and $Z_1'Z_1' = (I_b - (1/b) \mathbf{1}_b \mathbf{1}_b')$. Then $Z_1' \mathbf{1}_b = 0$ and $Z_1' \mathbf{z}/\sigma \sqrt{SS_T} \sim N(0, I_{b-1})$. Using $\mathbf{1}_b' \mathbf{z} = 0$, it now follows that

$$\frac{z'Z_1'Z_1'z}{\sigma^2 SS_T} = \frac{z'z}{\sigma^2 SS_T} = \frac{\bar{\pi}_1}{\sigma^2 SS_T}.$$
and, thus, under $H_0$, $\pi_1/\sigma^2 SS_T$ has a chi square distribution with $b - 1$ degrees of freedom. Also, under $H_0$, $(1/\sigma^2) SS_{Residual}$ has a chi square distribution with $(b - 1)(v - 1)$ degrees of freedom. Noting that the difference $SS_{Residual} - \pi_1/SS_T$ is nonnegative and applying a result in Rao [13, p. 187] it follows that, under $H_0$, $(1/\sigma^2) (SS_{Residual} - \pi_1/SS_T)$ has a chi square distribution with $(b - 1)(v - 1) - (b - 1) = (b - 1)(v - 2)$ degrees of freedom and is distributed independently of $\pi_1/SS_T$. Hence, under $H_0$,

$$W_1 = (v - 2) \frac{\pi_1}{SS_T} \left( \frac{SS_{Residual} - \pi_1}{SS_T} \right),$$

has an $F$ distribution with degrees of freedom $((b - 1), (b - 1)(v - 2))$.

Similarly in the set up of Theorem 2.1, under the dual assumption that $\|\tau^*\|$ is small relative to $\|\beta^*\|$, the $LI$ similar test rejects $H_0$ for large values of

$$W_2 = (b - 2) \frac{\pi_2}{SS_B} \left( \frac{SS_{Residual} - \pi_2}{SS_B} \right),$$

where $\pi_2 = \sum_{i=1}^v \{ \sum_{j=1}^b (\bar{y}_i - \bar{y}_.,) (y_{ij} - \bar{y}_i - \bar{y}_., + \bar{y}_.,) \}$ and $SS_B = \sum_{j=1}^b (\bar{y}_i - \bar{y}_.,)^2$. Under $H_0$, $W_2$ is distributed as central $F$ with degrees of freedom $((v - 1), (b - 2)(v - 1))$.

**Remark 2.1.** Since the expression for the conditional local power given in Theorem 2.1 is valid for any invariant similar test, it is also valid for Tukey’s test as well as for the $F$ tests based on $E_1$ and $\pi_*$. 

**Remark 2.2.** Our computations (not reported here) show that the coefficients of higher powers of $\lambda^*$ in the expansion of the density ratio $R$ involve quantities other than $\|B^*\|$, $\|T^*\|$, $\pi_{0*}$, $\pi_{1*}$, and $\pi_{2*}$, thus justifying our observation that there does not exist a UMPI test.

**Remark 2.3.** Using the expression for the conditional local power given in Theorem 2.1, it is possible to compute the unconditional power of Tukey’s test as well as that of the $F$ tests based on $\pi_1$ and $\pi_2$ when $\|\tau^*\|$ and $\|\beta^*\|$ are small. The unconditional power can also be evaluated using the nonnull densities of $W_0$, $W_1$, and $W_2$ given in Section 4. However, these computations are not reported here.

### 3. Simulation Results

To examine the amount of improvement in local power by the use of the test statistics $W_1$ and $W_2$ over Tukey’s test, we have simulated the power of all the three tests for $b = v = 6$, $\lambda^2 \sigma^2 = \frac{1}{8}$, $\lambda^2 \sigma^2 = \frac{1}{4}$, and $\lambda^2 \sigma^2 = \frac{3}{4}$, $\tau^* \sigma^2 = \frac{1}{3}$, $1$, $\frac{5}{3}$, $2$, $\frac{10}{3}$,
and 4, and $\beta^t/\sigma^2 = \frac{1}{3}, 1, \frac{5}{3}, 2, \frac{10}{3},$ and 4. For small values of $\tau^t/\sigma^2$ and $\beta^t/\sigma^2$, the power for the three tests appear in Table I. As expected, the performance of the tests based $W_1$ and $W_2$, as appropriate, is much superior to that of Tukey’s test. For large values of $\tau^t/\sigma^2$ and $\beta^t/\sigma^2$, values of the power for the three tests appear in Table II. Surprisingly, Tukey’s test outperforms the other two tests in this case. Based on these findings, for the testing problem $H_0: \lambda = 0$ vs $H_1: \lambda \neq 0$ under the model (1.1), we thus recommend the use of Tukey’s test when $\tau^t/\sigma^2$ and $\beta^t/\sigma^2$ are large and the use of the two tests based on $W_1$ and $W_2$ when $\tau^t/\sigma^2$ and $\beta^t/\sigma^2$ are small. In the latter case, the choice between the two tests based on $W_1$ and $W_2$ should be based on the relative magnitudes of $\tau^t/\sigma^2$ and $\beta^t/\sigma^2$. The $F$ test based on $W_2$ is recommended if it is known that $\tau^t/\sigma^2$ is small compared to $\beta^t/\sigma^2$. Otherwise, the $F$ test based on $W_1$ is recommended.

4. Local Unbiasedness

In this section we shall show that Tukey’s test as well as the two tests suggested in Section 2 are locally unbiased for any $\tau$ and $\beta$. Let $W_0^*, W_1^*, W_2^*$ be $W_0$, $W_1 = (1/(v-2)) W_1$ and $W_2 = (1/(b-2)) W_2$, where $W_0$, $W_1$, and $W_2$ are the test statistics considered in Section 2 (recall that $W_0$ given in (1.2) corresponds to Tukey’s test). We shall first obtain the nonnull densities of $W_0^*$, $W_1^*$, and $W_2^*$. For $w > 0$ and $f_1, f_2$ positive integers, let

$$H_{i,j}(w; f_1, f_2) = \int_0^\infty \frac{x^{f_1/2 + i - 1}}{(i! j!) B(i + f_1/2, j + f_2/2)(1 + x)^{i + j + (f_1 + f_2)/2}} \, dx.$$ (4.1)

In (4.1), $B(\cdot, \cdot)$ denotes the beta function. As proved in Ghosh and Sharma [2, Sect. 2] or Hegemann and Johnson [5, (2.2)], the cdf of $W_0^*$ is given by

$$P(W_0^* \leq w_0) = \exp \left\{ -\frac{\lambda^*}{2} \| \tau^* \|^2 \| \beta^* \|^2 \right\}$$

$$\times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \frac{\lambda^*}{2} \| \tau^* \|^2 \| \beta^* \|^2 \right)^{i+j} E[(U_1 U_2)^i (1 - U_1 U_2)^j]$$

$$\times H_{i,j}(w_0; 1, (b - 1)(v - 1) - 1),$$ (4.2)

where $U_1$ and $U_2$ are independent random variables with noncentral Beta distributions $B'(1, v - 2, b \| \tau^* \|^2/2)$ and $B'(1, b - 2, v \| \beta^* \|^2/2)$, respectively, and $w_0 > 0$. Similarly, it can be shown that the cdf’s of $W_1^*$ and $W_2^*$ are respectively given by
\[ P(W_1^* \leq w_1) = \exp \left\{ -\frac{\lambda^*}{2} \| \tau^* \|^2 \| \beta^* \|^2 \right\} \]
\[ \times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \frac{\lambda^*}{2} \| \tau^* \|^2 \| \beta^* \|^2 \right)^{i+j} E[U'_1(1 - U_1)'] \]
\[ \times H_{i,j}(w_1; b - 1, (b - 1)(v - 2)) \quad (4.3) \]

and
\[ P(W_2^* \leq w_2) = \exp \left\{ -\frac{\lambda^*}{2} \| \tau^* \|^2 \| \beta^* \|^2 \right\} \]
\[ \times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \frac{\lambda^*}{2} \| \tau^* \|^2 \| \beta^* \|^2 \right)^{i+j} E[U'_2(1 - U_2)'] \]
\[ \times H_{i,j}(w_2; v - 1, (b - 2)(v - 1)), \quad (4.4) \]

where \( w_1 \) and \( w_2 \) are both positive.

**Local unbiasedness of Tukey's test.** To prove the local unbiasedness of Tukey's test, we shall show that the first derivative (with respect to \( \lambda^* \)) of the power function of the test, evaluated at \( \lambda^* = 0 \), is nonnegative for every \( \| \tau^* \| \) and \( \| \beta^* \| \). From (4.2), after evaluating the derivative, it can be seen that the local unbiasedness of Tukey's test follows if we can show that the inequality
\[ E(U_1 U_2) H_{1,0}(w_0) + E(1 - U_1 U_2) H_{0,1}(w_0) \leq H_{0,0}(w_0) \quad (4.5) \]
holds for every \( \| \tau^* \| \), \( \| \beta^* \| \), and for all \( w_0 > 0 \), where we write \( H_{i,j}(w_0) \) instead of \( H_{i,j}(w_0; 1, (b - 1)(v - 1) - 1) \). To establish (4.5), we use the well-known fact about the representation of noncentral beta variables, namely, that there exist independent random variables \( L_1 \) and \( L_2 \) following Poisson distributions with parameters \( b \| \tau^* \|^2/2 \) and \( v \| \beta^* \|^2/2 \), respectively, such that \( U_1 | L_1 = l_1 \sim B(1 + 2l_1, v - 1) \) and \( U_2 | L_2 = l_2 \sim B(1 + 2l_2, b - 1) \). This yields
\[ E(U_1 | L_1 = l_1) = \frac{1 + 2l_1}{v - 1 + 2l_1}, \quad E(U_2 | L_2 = l_2) = \frac{1 + 2l_2}{b - 1 + 2l_2}. \quad (4.6) \]

Write
\[ p(l_1, l_2) = E(U_1 | L_1 = l_1) E(U_2 | L_2 = l_2) = \frac{(1 + 2l_1)(1 + 2l_2)}{(v - 1 + 2l_1)(b - 1 + 2l_2)}. \quad (4.7) \]

From (4.5), (4.6), and (4.7), it follows that, for establishing (4.5), it is enough to show that
\[ p(l_1, l_2) H_{1,0}(w_0) + (1 - p(l_1, l_2)) H_{0,1}(w_0) \leq H_{0,0}(w_0) \quad (4.8) \]
for all $l_1 \geq 0$ and $l_2 \geq 0$. Let $f_{1,0}(x), f_{0,1}(x)$, and $f_{0,0}(x)$ denote the integrands of $H_{1,0}(w_0)$, $H_{0,1}(w_0)$, and $H_{0,0}(w_0)$, respectively, given by

$$f_{1,0}(x) = \frac{1}{B \left( \frac{3}{2}, \frac{(b-1)(v-1) - 1}{2} \right)} \frac{x^{1/2}}{(1+x)^{(b-1)(v-1)+2}^2}$$

$$f_{0,1}(x) = \frac{1}{B \left( \frac{1}{2}, \frac{(b-1)(v-1) + 1}{2} \right)} \frac{x^{-1/2}}{(1+x)^{(b-1)(v-1)+2}^2}$$

$$f_{0,0}(x) = \frac{1}{B \left( \frac{1}{2}, \frac{(b-1)(v-1) - 1}{2} \right)} \frac{x^{-1/2}}{(1+x)^{(b-1)(v-1)-1}^2}.$$  \hfill (4.9)  

$$\psi(w_0) = \frac{\int_0^w \left[ p(l_1, l_2) f_{1,0}(x) + (1-p(l_1, l_2)) f_{0,1}(x) \right] dx}{\int_0^w f_{0,0}(x) dx}. \hfill (4.12)$$

Then (4.8) is equivalent to

$$\int_0^w \left[ p(l_1, l_2) f_{1,0}(x) + (1-p(l_1, l_2)) f_{0,1}(x) \right] dx \leq \int_0^w f_{0,0}(x) dx, \hfill (4.13)$$

which in turn is equivalent to

$$\psi(w_0) \geq 1.$$

Since $\psi(0) = 1$, (4.14) will follow if we can show that $\psi(w_0)$ is increasing in $w_0$. Towards this, write

$$f_{w_0}^*(x) = \frac{f_{0,0}(x)}{\int_0^w f_{0,0}(x) dx} I_{\{x \geq w_0\}} \hfill (4.15)$$

where $I_{\{x \geq w_0\}}$ denotes the indicator function of the set $\{x: x \geq w_0\}$. Then we have

$$\psi(w_0) = E_{f_{w_0}^*} \left[ \frac{p(l_1, l_2) f_{1,0}(x) + (1-p(l_1, l_2)) f_{0,1}(x)}{f_{0,0}(x)} \right]. \hfill (4.16)$$

Clearly, the family $\{f_{w_0}^*(x): w_0 \geq 0\}$ admits monotone likelihood ratio in $x$. Hence, in view of (4.16), it is enough to show that $(p(l_1, l_2) f_{1,0}(x)$ +
(1 - \(p(l_1, l_2)\))f_{0.1}(x)/f_{0.0}(x) is increasing in \(x\) (see [7, p. 85]). Using the expressions for \(f_{1.0}(x), f_{0.1}(x),\) and \(f_{0.0}(x)\) given in (4.9)–(4.11), we obtain

\[
\frac{p(l_1, l_2)f_{1.0}(x) + (1 - p(l_1, l_2))f_{0.1}(x)}{f_{0.0}(x)} = (b - 1)(v - 1) \left\{ \frac{p(l_1, l_2)}{(b - 1)(v - 1) - 1} \right\} \frac{1}{1 + x}. 
\]

Direct differentiation shows that this expression is increasing in \(x\) if

\[
p(l_1, l_2) \geq \frac{1 - p(l_1, l_2)}{(b - 1)(v - 1) - 1}.
\]

It is readily verified that this inequality is true for all \(l_1 \geq 0\) and \(l_2 \geq 0\) using the expression (4.7) for \(p(l_1, l_2)\), which establishes the local unbiasedness of Tukey's test.

The local unbiasedness of the tests based on \(W_1\) and \(W_2\) can be established by adopting the above arguments with obvious modifications in the expressions.

**APPENDIX**

**LEMMA 5.1.** Let \(\mathcal{P}_m\) be the group of \(m \times m\) orthogonal matrices \(P_m\) satisfying \(P_m1_m = \pm 1_m\). Let \(Z\) be a \(m \times (m - 1)\) matrix such that \(Z'Z = 1_{m-1}\) and \(Z'1_m = 0\).

(i) If \(P_m \in \mathcal{P}_m\), then \(Z'P_mZ\) is a \((m - 1) \times (m - 1)\) orthogonal matrix. Conversely, any \((m - 1) \times (m - 1)\) orthogonal matrix can be expressed as \(Z'P_mZ\) for some \(P_m \in \mathcal{P}_m\).

(ii) \(P_m \in \mathcal{P}_m\) satisfies \(P_m1_m = 1_m\) (resp. \(P_m1_m = -1_m\)) if and only if \(P_m = ZQZ' + (1/m)1_m1'_m\) (resp. \(P_m = ZQZ' - (1/m)1_m1'_m\)) for some \((m - 1) \times (m - 1)\) orthogonal matrix \(Q\). Hence \(\mathcal{P}_m = \{ZQZ' + (\delta/m)1_m1'_m : \delta = +1, \text{ or } -1, \text{ and } Q \text{ is any } (m - 1) \times (m - 1) \text{ orthogonal matrix}\}\).

(iii) The left invariant probability measure \(dP_m\) on \(\mathcal{P}_m\) can be expressed as \(dQ\,d\delta\), where \(dQ\) is the uniform distribution on the group of \((m - 1) \times (m - 1)\) orthogonal matrices \(Q\), and \(d\delta\) is the discrete uniform distribution assigning mass \(\frac{1}{2}\) to each of the points \(\delta = 1\) and \(\delta = -1\).

**Proof.** (i) Using the properties of \(Z\) and the fact that \(ZZ' = 1_m - (1/m)1_m1'_m\), it follows that if \(P_m \in \mathcal{P}_m\), then \(Z'P_mZ\) is orthogonal. Conversely, if \(Q\) is a \((m - 1) \times (m - 1)\) orthogonal matrix, then we can write \(Q = Z'(ZQZ' + (1/m)1_m1'_m)\). Writing \(P_m = ZQZ' + (1/m)1_m1'_m\), it is easily verified that \(P_m \in \mathcal{P}_m\).
(ii) If the orthogonal matrix \( P_m \) satisfies \( P_m I_m = I_m \), we shall show that \( P_m = ZQZ' + (1/m) \) \( I_m I_m' \) for some \((m-1) \times (m-1)\) orthogonal matrix \( Q \). Since \( P_m I_m = I_m \), \((P_m - (1/m) I_m I_m') I_m = 0\). Hence \( P_m - (1/m) I_m I_m' = MZ' \), for some matrix \( M \). Since \( P_m' I_m = I_m \) also holds, we get \( P_m' - (1/m) I_m I_m' = ZQZ' \), for some matrix \( Q \). The orthogonality of \( Q \) follows from part (i).

(iii) This is immediate from part (ii).

Derivation of (2.5)

In view of invariance, we assume \( \mu = 0 \) and \( \sigma^2 = 1 \), and write \( \lambda^* = \lambda \sigma \), \( \beta^* = (1/\sigma) \beta \) and \( \tau^* = (1/\sigma) \tau \). For \( g^* \) given by (2.3), the Jacobian of \( y \rightarrow g^* \) is easily seen to be \( c^{-n} \). Then the numerator, say \( N \), of (2.4) is

\[
N = \int \exp \left\{ -\frac{1}{2} \left( \eta' \eta + a^2 c^2 n + 2ac \eta' \mathbf{1}_n \right) \right\} \, da \, c^{n-1} \, dc \, dP_b \, dP_v ,
\]  

(5.1)

where

\[
\eta = c(P_b \otimes P_v) \, y - F_1 \tau^* - F_2 \beta^* - \lambda^* (\beta^* \otimes \tau^*)
\]

(5.2)

and the single integral sign in (5.1) is used to denote the four integrals in the numerator of (2.4). Note that

\[
N = \int \exp \left\{ -\frac{1}{2} \left[ \eta' \eta - \frac{\eta' \mathbf{1}_n}{n} \right] \right\} \, da \, c^{n-1} \, dc \, dP_b \, dP_v
\]

(5.3)

where the single integral in (5.3) now denotes the three integrals corresponding to \( c, P_b, \) and \( P_v \). Writing \( SS_{\text{Total}} = \sum_{i,j} (y_{ij} - \bar{y}_i)^2 = \sum_{i,j} y_{ij}^2 - n\bar{y}_i^2 \), and using \( (\eta' \mathbf{1}_n)^2 = \{ cy'(P_b \otimes P_v)(1_b \otimes 1_v) \}^2 = c^2 (\sum_{i,j} y_{ij})^2 = n^2 c^2 \bar{y}_i^2 \), we obtain

\[
\eta' \eta - \frac{(\eta' \mathbf{1}_n)^2}{n} = [F_1 \tau^* + F_2 \beta^* + \lambda^* (\beta^* \otimes \tau^*)]' [F_1 \tau^* + F_2 \beta^* + \lambda^* (\beta^* \otimes \tau^*)]
\]

\[
- 2cy'(P_b \otimes P_v) [F_1 \tau^* + F_2 \beta^* + \lambda^* (\beta^* \otimes \tau^*)] + c^2 SS_{\text{Total}} .
\]

(5.4)

Using (5.4), (5.3) simplifies to

\[
N = K_1 \int \exp \left[ -\frac{1}{2} \{ c^2 SS_{\text{Total}}
\]

\[
- 2cy'(P_b \otimes P_v) (F_1 \tau^* + F_2 \beta^* + \lambda^* (\beta^* \otimes \tau^*)) \} \right] c^{n-2} \, dc \, dP_b \, dP_v .
\]

(5.5)
where

\[ K_1 = \sqrt{\frac{2\pi}{n}} \exp \left\{ -\frac{1}{2} \left[ F_1 \tau^* + F_2 \beta^* + \lambda^*(\beta^* \otimes \tau^*) \right]' \right. \]

\[ \times \left. \left[ F_1 \tau^* + F_2 \beta^* + \lambda^*(\beta^* \otimes \tau^*) \right] \right\} \]

\[ = \sqrt{\frac{2\pi}{n}} \exp \left\{ -\frac{1}{2} \left( F_1 \tau^* + F_2 \beta^* \right)' \left( F_1 \tau^* + F_2 \beta^* \right) \right\} \]

\[ \times \exp \left\{ -\frac{\lambda^*}{2} (\tau^* \tau^*) (\beta^* \beta^*) \right\}. \] (5.6)

Expanding the factor involving \( \tau^* \) and \( \beta^* \) in (5.5) and simplifying, we obtain

\[ N = K_1 \sum_{r=0}^{\infty} \int_{\mathcal{S}_0^l} \int_{\mathcal{S}_0^r} \left\{ y'(P'_b \otimes P'_v)(F_1 \tau^* + F_2 \beta^* + \lambda^*(\beta^* \otimes \tau^*)) \right\}' dP_v dP_b, \]

\[ \times \int_0^{\infty} \exp \left\{ -\frac{c^2}{2} \frac{SS_{Total}}{r!} \right\} \frac{c^{n+r-2}}{r!} dc \]

\[ = K_1 (SS_{Total})^{-\frac{(n-1)}{2}} \sum_{r=0}^{\infty} \Gamma \left( \frac{n+r-1}{2} \right) \frac{2^{(n+r-1)/2}}{r!} \]

\[ \times \int_{\mathcal{S}_0^l} \int_{\mathcal{S}_0^r} \frac{\left\{ y'(P'_b \otimes P'_v)(F_1 \tau^* + F_2 \beta^* + \lambda^*(\beta^* \otimes \tau^*)) \right\}'} {\sqrt{SS_{Total}}} \right\} \frac{dP_v dP_b}. \] (5.7)

The denominator \( D \) of \( R \) in (2.4) is (5.7) evaluated at \( \lambda^* = 0 \). We shall show that \( D \) is a function of \( T_0 \), the complete sufficient statistic under \( H_0 \), given in (2.2). Towards this, let \( B \) and \( T \) be the vectors consisting of \((\bar{y}_j - \bar{y})_s\) and \((\bar{y}_i - \bar{y})_s\), respectively. Then

\[ y'(P'_b \otimes P'_v) F_1 \tau^* = y'(P'_b \otimes P'_v)(1_b \otimes I_v) \tau^* \]

\[ = (1_b^T P_b 1_b) \tau^* P_v T \] (5.8)

and

\[ y'(P'_b \otimes P'_v) F_2 \beta^* = (1_v^T P_v 1_v) \beta^* P_b B. \] (5.9)

Using (5.8) and (5.9) and putting \( \lambda^* = 0 \) in (5.7), it follows that \( D \) is a function of \( B \) and \( T \) only, thus establishing that \( D \) is a function of \( T_0 \) given in (2.2). Using (5.6) and (5.7), we can write

\[ D = \sqrt{\frac{2\pi}{n}} \exp \left\{ -\frac{1}{2} \left( F_1 \tau^* + F_2 \beta^* \right)' \left( F_1 \tau^* + F_2 \beta^* \right) \right\} \]

\[ \times \left( SS_{Total} \right)^{-\frac{(n-1)}{2}} h(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|). \] (5.10)
where
$$h(\|\tau^*\|, \|\beta^*\|, \|T^*\|, \|B^*\|)$$

$$= \sum_{r=0}^{\infty} \int \frac{F(n+r-1/2)^{1/2}}{r!} \left\{ \frac{w}{\sqrt{SS_{\text{Total}}}} \right\}^r dP_v dP_b$$  \hspace{1cm} (5.11)

and
$$w = y'(P_b \otimes P_v)(F_1^* + F_2^*)$$
$$= (1, P_{b1}, \tau^* P_v T + (1, P_{v1}, \beta^* P_b B$$  \hspace{1cm} (5.12)

(using (5.8) and (5.9)). (For explicit computation of the right hand side of (5.11) and justification of the notation used, see Eqs. (5.19) and (5.20) below). From (5.6), (5.7), (5.10), (5.11), and (5.12), we obtain

$$R = \frac{N}{D}$$
$$= \{h(\|\tau^*\|, \|\beta^*\|, \|T^*\|, \|B^*\|)\}^{-1} \exp \left\{ -\frac{\lambda^2}{2} \|\tau^*\|^2 \|\beta^*\|^2 \right\}$$
$$\times \sum_{r=0}^{\infty} \int \frac{F(n+r-1/2)^{1/2}}{r!} \left\{ \frac{w + \lambda^* y'(P_b \otimes P_v) (\beta^* \otimes \tau^*)}{\sqrt{SS_{\text{Total}}}} \right\}^r dP_v dP_b.$$  \hspace{1cm} (5.13)

This establishes (2.5).

We now show that the integral in (5.13) vanishes for \(r\) odd. A similar argument applies to the integral in (5.11). Applying Lemma 5.1(ii), we can write

$$P_b = Z_1 Q_1 Z_1 + \frac{\delta_1}{b} 1_b 1_b, \quad P_v = Z_2 Q_2 Z_2 + \frac{\delta_2}{v} 1_v 1_v,$$  \hspace{1cm} (5.14)

where \(\delta_1\) and \(\delta_2\) are variables, each taking values +1 and -1, \(Z_1\) and \(Z_2\) are respectively \(b \times (b-1)\) and \(v \times (v-1)\) matrices satisfying \(Z_1^T Z_1 = I_{b-1}\), \(Z_1^T 1_b = 0\), \(Z_2^T Z_2 = I_{v-1}\), and \(Z_2^T 1_v = 0\), and \(Q_1\) and \(Q_2\) are respectively \((b-1) \times (b-1)\) and \((v-1) \times (v-1)\) orthogonal matrices. Apart from a term involving \(SS_{\text{Total}}\), the integrand in (5.13), for any \(r\), is

$$\sum_{s=0}^{r} \binom{r}{s} \lambda^* \delta_1^s \delta_2^u \{y' (P_b' \otimes P_v') (\beta^* \otimes \tau^*)\}^{r-s}$$
$$= \sum_{s=0}^{r} \sum_{u=0}^{s} \binom{s}{u} \binom{r}{s} \lambda^* \delta_1^s \delta_2^u \delta_1^s \delta_2^u$$
$$\times (\tau^* Z_2 Q_2 Z_2^T T) u (\beta^* Z_1 Q_1 Z_1^T B)^{s-u}$$
$$\times \{y' (Z_1 Q_1 Z_1' \otimes Z_2 Q_2 Z_2') (\beta^* \otimes \tau^*)\}^{r-s}$$  \hspace{1cm} (5.15)

(using (5.12) and (5.14)).
From Lemma 5.1(iii), \( dP_\alpha = dQ_1 d\delta_1 \) and \( dP_\beta = dQ_2 d\delta_2 \). If \( r \) is odd, then it is easily verified that each term in (5.15) is an odd function of \( \delta_1 \) or \( \delta_2 \) or \( Q_1 \) or \( Q_2 \). Hence the integral in (5.13) vanishes for \( r \) odd. It can also be verified that if \( r \) is even, then the terms in (5.15) for odd values of \( s \) are odd functions of \( \delta_1 \) or \( \delta_2 \). Hence, for \( r \) even, in the binomial expansion of the integrand in (5.13), the integral of any term involving an odd power of \( z^* \) vanishes.

**Lemma 5.2.** Let \( X = (X_1, X_2, \ldots, X_m)' \) be an \( m \times 1 \) random vector distributed as multivariate normal, \( N(0, I_m) \). Then for any positive even integer \( s \),

1. \( E\left( \frac{X_1^{s+2}}{\|X\|^{s+2}} \right) = \frac{\Gamma\left(\frac{s+3}{2}\right)\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{s+2+m}{2}\right)\Gamma\left(\frac{1}{2}\right)} \)

2. \( E\left( \frac{X_1^2 X_2}{\|X\|^{s+2}} \right) = \frac{1}{2} \frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{s+2+m}{2}\right)\Gamma\left(\frac{1}{2}\right)} \).

**Proof.** Note that \( U = x_1^2 / \|X\|^2 \) is distributed as \( \beta(\frac{1}{2}, (m-1)/2) \). (i) follows since the expression to be evaluated is \( E(U^{(s+2)/2}) \). To prove (ii), we note that

\[
E(U^{s/2}) = E\left[ \frac{X_1^2 X'X}{(X'X)^{(s+2)/2}} \right] = E\left[ \frac{X_1^{s+2}}{(X'X)^{(s+2)/2}} + \frac{X_1^2 \sum_{j=2}^{m} X_j^2}{(X'X)^{(s+2)/2}} \right]
\]

\[
= E(U^{(s+2)/2}) + (m-1) E\left( \frac{X_1^2 X_2^2}{\|X\|^{s+2}} \right).
\]

(ii) can now be established after evaluating \( E(U^{s/2}) \).

**Lemma 5.3.** Let \( \theta, d_1, d_2 \) be \( m \times 1 \) nonnull vectors, and \( O(m) \) the group of \( m \times m \) orthogonal matrices. If \( Q \) is any \( m \times m \) orthogonal matrix and \( dQ \) denotes the uniform distribution on \( O(m) \), then for any positive even integer \( s \),

1. \( \int_{O(m)} (\theta'Qd_1)^s \, dQ = \frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{s+m}{2}\right)\Gamma\left(\frac{1}{2}\right)} \|\theta\|^s (d_1' d_1)^{s/2} \).
(ii) \[
\int_{O(m)} (\theta'Qd_1)^s (\theta'Qd_2)^2 dQ = \frac{\Gamma\left(\frac{s + 1}{2}\right) \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{s + m + 2}{2}\right) \Gamma\left(\frac{1}{2}\right)} \|\theta\|^{s + 2} (d_1^t d_1)^{\frac{s + 2}{2} - 1} \\
\times \left[ \frac{s}{2} (d_1^t d_2)^2 + \frac{1}{2} (d_1^t d_1)(d_2^t d_2) \right].
\]

(iii) \[
\int_{O(m)} (\theta'Qd_1)^s Q'\theta'Q dQ = \frac{\Gamma\left(\frac{s + 1}{2}\right) \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{s + m + 2}{2}\right) \Gamma\left(\frac{1}{2}\right)} \|\theta\|^{s + 2} (d_1^t d_1)^{\frac{s + 2}{2} - 1} \\
\times \left[ \frac{s}{2} d_1^t d_1 + \frac{1}{2} (d_1^t d_1) I_m \right].
\]

**Proof.** Part (i) of the lemma follows from part (ii) by taking \(d_1 = d_2\) and replacing \(s\) by \(s - 2\) in part (ii). Part (iii) follows from part (ii) by writing \((\theta'Qd_2)^2 = d_2^t Q'\theta'Qd_2\) and dropping \(d_2\) from both sides of (ii). Thus we shall prove only part (ii). We first use an argument given in Eaton [1, pp. 274–275]. Let \(\varepsilon_1\) be the \(m\)-component vector with first entry one and zeros elsewhere. Then \(\theta\) and \(\|\theta\| \varepsilon_1\) have the same length. Hence there exists \(Q_1 \in O(m)\) such that \(\theta = \|\theta\| Q_1 \varepsilon_1\). Using the invariance of \(dQ\), we see that the integral to be evaluated is

\[
\|\theta\|^{s + 2} \int_{O(m)} (\varepsilon_1^t Qd_1)^s (\varepsilon_1^t Qd_2)^2 dQ.
\]

Let \(X, X_1, \ldots, X_{m-1}\) be \(m\) independent \(m\)-component random vectors, each distributed as \(N(0, I_m)\). Let \(S\) be the \(m \times m\) random orthogonal matrix obtained by applying the Gram–Schmidt orthogonalization procedure to the rows of \((X, X_1, \ldots, X_{m-1})'\). The first row of \(S\) can be taken as \(X'/\|X\|\). With such a choice of \(S\), applying Proposition 7.2 in Eaton [1, p. 234], we obtain

\[
\int_{O(m)} (\varepsilon_1^t Qd_1)^s (\varepsilon_1^t Qd_2)^2 dQ = E[(\varepsilon_1^t Sd_1)^s (\varepsilon_1^t Sd_2)^2] \\
= E\left[\frac{(X'd_1)^s (X'd_2)^2}{\|X\|^{s + 2}}\right].
\]

**Case (i)** \(d_1\) is not a scalar multiple of \(d_2\). Let \(Q_0\) be an \(m \times m\) orthogonal matrix such that the first two elements of \(\bar{Y} = Q_0 X\) are \(Y_1 = (d_1^t d_1)^{-1/2} d_1^t X\) and \(Y_2 = (d_2^t d_2 - (d_1^t d_2)^2/d_1^t d_1)^{-1/2} (d_2^t d_2/d_1^t d_1) d_1^t X\). Then \(\bar{Y} \sim N(0, I_m)\) and
The right-hand side of (5.18) can be evaluated using Lemma 5.2, resulting in the expression given in Lemma 5.3(ii).

Case (ii) \( \mathbf{d}_1 \) is a scalar multiple of \( \mathbf{d}_2 \). It is enough to complete the proof assuming that \( \mathbf{d}_1 = \mathbf{d}_2 \). Then from (5.17), the quantity to be evaluated is \( E[(X'\mathbf{d}_1)^2 \|X\|^s + 1] \). Let \( Q_0 \) be an orthogonal matrix such that the first element of \( \mathbf{Y} = Q_0 \mathbf{X} \) is \( Y_1 = (\mathbf{d}_1' \mathbf{d}_1)^{-1/2} \mathbf{d}_1' \mathbf{X} \). The rest of the argument is similar to Case (i).

This completes the proof of Lemma 5.3.

We shall now explicitly evaluate \( h(\|\tau^*\|, \|\beta^*\|, \|\mathbf{T}^*\|, \|\mathbf{B}^*\|) \) given in (5.11) and \( \psi_2 \) given in (2.7). From (5.11), using the expression for \( w \) in (5.12), we have

\[
h(\|\tau^*\|, \|\beta^*\|, \|\mathbf{T}^*\|, \|\mathbf{B}^*\|) = \sum_{r=0}^{\infty} \Gamma \left( \frac{n+r-1}{2} \right) \frac{2^{(n+r-1)/2}}{r!} \times \int_{\mathcal{P}_b} \int_{\mathcal{P}_v} \left\{ (1_b^\prime P_b 1_b) \tau^* P_v \mathbf{T}^* + (1_v^\prime P_v 1_v) \beta^* P_b \mathbf{B}^* \right\}^r dP_v dP_b. \quad (5.19)
\]

Recall that the integral in (5.19) vanishes for \( r \) odd. Also, for \( r \) even, if we consider the binomial expansion of the integrand in (5.19), the integral of any term involving an odd power of \( \tau^* P_v \mathbf{T}^* \) or \( \beta^* P_b \mathbf{B}^* \) also vanishes. Hence writing \( r = 2l \) (where \( l \) is a positive integer), and noting that \( 1_b^\prime P_b 1_b = \pm b \) and \( 1_v^\prime P_v 1_v = \pm v \), we have

\[
\int_{\mathcal{P}_b} \int_{\mathcal{P}_v} \left\{ (1_b^\prime P_b 1_b) \tau^* P_v \mathbf{T}^* + (1_v^\prime P_v 1_v) \beta^* P_b \mathbf{B}^* \right\}^r dP_v dP_b = \sum_{u=0}^{l} \frac{2^{2l}}{2^{2u}} b^{2u} v^{2l-2u} \int_{\mathcal{P}_b} \int_{\mathcal{P}_v} (\tau^* P_v \mathbf{T}^* \beta^* P_b \mathbf{B}^*)^{2l-2u} dP_v dP_b
\]

\[
= \sum_{u=0}^{l} \frac{2^{2l}}{2^{2u}} b^{2u} v^{2l-2u} \int_{O(b-1)} \int_{O(v-1)} (\tau^* Z_{b-1} Z_{v-1}^\prime \mathbf{T}^* \beta^* Z_{b-1} Z_{v-1}^\prime) dQ_1 dQ_2
\]

(using (5.14))
The last expression in (5.20) follows by applying Lemma 5.3(i). We have also used the fact that $\|Z_1a_1\| = \|a_1\|$ and $\|Z_2a_2\| = \|a_2\|$, whenever $a_1$ and $a_2$ are respectively $b \times 1$ and $v \times 1$ vectors satisfying $a_1'1_b = 0$ and $a_2'1_v = 0$. (5.19) and (5.20) yield an explicit representation of $h(\|\tau^*\|, \|\beta^*\|, \|T^*\|, \|B^*\|)$ (as an infinite series).

In order to simplify $\psi_2$ in (2.7), we shall first evaluate the integral in (2.7) whenever $r$ is an even number ($r \geq 2$). Using the expression (5.12) for $w$, we note that

$$
\int_{\mathbb{R}_b} \int_{\mathbb{R}_b} \{w^{r-2}(P_b' \otimes P_b')(\beta^* \otimes \tau^*)\}^2 dP_b dP_b' = \sum_{u=0}^{(r-2)/2} \binom{r-2}{2u} b^{2u} r^{r-2-2u}
$$

$$\times y' \left( \left( \int_{\mathbb{R}_b} (\beta^* P_b B)^{r-2-2u} P_b' \beta^* \tau^* P_b dP_b \right)^2 \right) \otimes \left( \int_{\mathbb{R}_v} (\tau^* P_v T)^{2u} P_v' \tau^* \tau^* P_v dP_v \right) y. \quad (5.21)
$$

Using (5.14), Lemma 5.2(iii), and the fact that $Z_1 Z_1' = I_b - (1/b) 1_b 1_b'$, we obtain

$$
\int_{\mathbb{R}_b} (\beta^* P_b B)^{r-2-2u} P_b' \beta^* \tau^* P_b dP_b
$$

$$= \int_{Q(b-1)} (\beta^* Z_1 Q_1' Z_1 B)^{r-2-2u} Z_1 Q_1 Z_1' \beta^* \beta^* Z_1 Q_1 Z_1' dQ_1
$$

$$= \frac{\Gamma\left(\frac{r-2u-1}{2}\right) \Gamma\left(\frac{b-1}{2}\right)}{\Gamma\left(\frac{r-2u+b-1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \|\beta^*\|^{r-2u} \|B\|^{r-2u-4}
$$

$$\times \left[ \left( \frac{r}{2} - u \right) BB' + \frac{1}{2} \|B\|^2 \left( I_b - \frac{1}{b} 1_b 1_b' \right) \right]. \quad (5.22)
$$
Similarly, we also obtain
\[
\int_{\gamma_0} (\tau^* P_v T)^{2u} P_v \tau^* P_v dP_v
\]
\[
= \int_{\gamma_0(v-1)} (\tau^* Z_2 Z_2 T)^{2u} Z_2 Q_2 Z_2 \tau^* Z_2 Q_2 Z_2 dQ_2
\]
\[
= \frac{\Gamma\left(\frac{2u+1}{2}\right) \Gamma\left(\frac{v-1}{2}\right)}{\Gamma\left(\frac{2u+v+1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \left\| \tau^* \right\|^{2u+2} \left\| T \right\|^{2(u-1)}
\]
\[
\times \left[ u TT' + \frac{1}{2} \left\| T \right\|^2 \left( I_v - \frac{1}{v} 1_v 1_v' \right) \right].
\]
(5.23)

In view of (5.22) and (5.23), for simplifying (5.21) we need to evaluate
\[
y' \left[ \left\{ \left( \frac{r-2}{2} - u \right) BB' + \frac{1}{2} \left\| B \right\|^2 \left( I_b - \frac{1}{b} 1_b 1_b' \right) \right\} \right.
\]
\[
\otimes \left\{ u TT' + \frac{1}{2} \left\| T \right\|^2 \left( I_v - \frac{1}{v} 1_v 1_v' \right) \right\} \left. \right] y.
\]

Let
\[
\pi_0 = \left\{ y'(B \otimes T) \right\}^2 = \left\{ \sum_{j=1}^b \sum_{i=1}^v (\bar{y}_{-j} - \bar{y}_{..})(\bar{y}_{i} - \bar{y}_{..}) y_{ij} \right\}^2
\]
(5.24)
\[
\pi_1 = y'(I_b \otimes TT') y = \sum_{j=1}^b \left\{ \sum_{i=1}^v (\bar{y}_{i} - \bar{y}_{..}) y_{ij} \right\}^2
\]
(5.25)
and
\[
\pi_2 = y'(BB' \otimes I_v) y = \sum_{i=1}^v \left\{ \sum_{j=1}^b (\bar{y}_{-j} - \bar{y}_{..}) y_{ij} \right\}^2.
\]
(5.26)

Straightforward algebra gives
\[
y'(BB' \otimes TT') y = \pi_0, \quad y'(\left( I_b - \left( \frac{1}{b} \right) 1_b 1_b' \right) \otimes TT') y = \pi_1 - b \left\| T \right\|^4, \quad \text{and} \quad y'(BB' \otimes (I_v - \left( \frac{1}{v} \right) 1_v 1_v')) y = \pi_2 - v \left\| B \right\|^4.
\]
Furthermore, \(y'(I_b - \left( \frac{1}{b} \right) 1_b 1_b') \otimes (I_v - \left( \frac{1}{v} \right) 1_v 1_v') y = SS_{Residual}.\) Using these observations we obtain
\[
y' \left[ \left\{ \left( \frac{r-2}{2} - u \right) BB' + \frac{1}{2} \left\| B \right\|^2 \left( I_b - \frac{1}{b} 1_b 1_b' \right) \right\} \right.
\]
\[
\otimes \left\{ u TT' + \frac{1}{2} \left\| T \right\|^2 \left( I_v - \frac{1}{v} 1_v 1_v' \right) \right\} \left. \right] y
\]
\[
\begin{align*}
&= \left( \frac{r-2}{2} - u \right) \omega_0 + \frac{1}{2} u \| B \| ^2 \{ \pi_1 - b \| T \| ^4 \} \\
&\quad + \left( \frac{r-2}{2} - u \right) \| T \| ^2 \{ \pi_2 - v \| B \| ^4 \} \\
&\quad + \frac{1}{4} \| B \| ^2 \| T \| ^2 \text{SS}_{\text{Residual}}. \tag{5.27}
\end{align*}
\]

It may be noted that \( \pi_1 - b \| T \| ^4 \) and \( \pi_2 - v \| B \| ^4 \) are both positive. Let \( \kappa(r, u) \) \((0 \leq u \leq (r-2)/2, r \geq 2)\) be defined as

\[
\kappa(r, u) = \Gamma \left( \frac{n + r - 1}{2} \right) \Gamma \left( \frac{r-2}{2} \right) \Gamma \left( \frac{2u + r - 2u - 2}{2} \right) \Gamma \left( \frac{2u + 1}{2} \right) \Gamma \left( \frac{1}{2} \right) \frac{\Gamma \left( \frac{r-2u - 1}{2} \right)}{\Gamma \left( \frac{r-2u + b - 1}{2} \right)} \frac{\Gamma \left( \frac{b-1}{2} \right)}{\Gamma \left( \frac{b + 1}{2} \right)} \frac{\Gamma \left( \frac{v-1}{2} \right)}{\Gamma \left( \frac{2u + v + 1}{2} \right)} \frac{\Gamma \left( \frac{1}{2} \right)}{r!} 2^{(n+r-1)/2} \left( \frac{r}{2} \right) b^{2u} (-r-2) \tag{5.28}
\]

Also let

\[
\begin{align*}
\pi_0^* &= \frac{\pi_0}{(SS_{\text{Total}})^3}, \\
\pi_1^* &= \frac{\pi_1}{(SS_{\text{Total}})^2}, \\
\pi_2^* &= \frac{\pi_2}{(SS_{\text{Total}})^2}, \\
\text{SS}_{\text{Residual}}^* &= \frac{\text{SS}_{\text{Residual}}}{SS_{\text{Total}}}. \tag{5.29}
\end{align*}
\]

From (5.21)-(5.23), (5.27)-(5.29), we thus obtain

\[
\begin{align*}
\psi_2 &= \sum_{r=2}^{\infty} \Gamma \left( \frac{n + r - 1}{2} \right) \frac{2^{(n+r-1)/2}}{r!} \left( \frac{r}{2} \right) (SS_{\text{Total}})^{-r/2} \\
&\times \int_{\mathcal{D}_b} \int_{\mathcal{D}_v} w'^{-2} \{ y'(P'_b \otimes P'_v)(\beta^* \otimes \tau^*) \}^2 dP'_v dP'_b \\
&= \sum_{r=2}^{\infty} \sum_{u=0}^{(r-2)/2} \kappa(r, u) \\
&\times (SS_{\text{Total}})^{-r/2} \| \beta^* \| ^{r-2} \| \tau^* \| ^{2u+2} \| B \| ^{r-2u-4} \| T \| ^{2u-2} \\
&\times y' \left[ \left\{ \left( \frac{r-2}{2} - u \right) BB' + \frac{1}{2} \| B \| ^2 \left( I_b - \frac{1}{b} \mathbf{1}_b \mathbf{1}_b \right) \right\} \otimes \left( uTT' + \frac{1}{2} \| T \| ^2 \left( I_v - \frac{1}{v} \mathbf{1}_v \mathbf{1}_v \right) \right) \right] y
\end{align*}
\]
\[= \pi_0^* \sum_{r = 2}^{\infty} \sum_{u = 0}^{(r-2)/2} \kappa(r, u) \times \left(\frac{r - 2 - u}{2} - u\right) u \| \beta^* \| |r - 2u|^{2u+2} |\tau^*|^{2u+2} \| B^* \| |r - 2u - 4| \| T^* \|^{2u - 2} \]
\[+ \frac{1}{2} \| B^* \|^2 (\pi_1^* - b \| T^* \|^4) \]
\[\times \sum_{r = 2}^{\infty} \sum_{u = 0}^{(r-2)/2} \kappa(r, u) u \| \beta^* \| |r - 2u|^{2u+2} |\tau^*|^{2u+2} \| B^* \| |r - 2u - 4| \| T^* \|^{2u - 2} \]
\[+ \frac{1}{2} \| T^* \|^2 (\pi_2^* - v \| B^* \|^4) \]
\[\times \sum_{r = 2}^{\infty} \sum_{u = 0}^{(r-2)/2} \kappa(r, u) \left(\frac{r - 2 - u}{2} - u\right) u \| \beta^* \| |r - 2u|^{2u+2} |\tau^*|^{2u+2} \| B^* \| |r - 2u - 4| \| T^* \|^{2u - 2} \]
\[\times \| B^* \| |r - 2u - 4| \| T^* \|^{2u - 2} \]
\[+ \frac{1}{4} S_{\text{Residual}}^* \sum_{r = 2}^{\infty} \sum_{u = 0}^{(r-2)/2} \kappa(r, u) u \| \beta^* \| |r - 2u|^{2u+2} |\tau^*|^{2u+2} \| B^* \| |r - 2u - 2| \| T^* \|^{2u - 2}. \]

(5.30)

We now arrange the coefficients of \(\pi_0^*\), \(\pi_1^* - b \| T^* \|^4\), and \(\pi_2^* - v \| B^* \|^4\) in (5.30) in ascending powers of \(|\tau^*|^2\) and \(|\beta^*|^2\) to reveal the leading terms when \(|\tau^*|^2\) and \(|\beta^*|^2\) are small. For \(r \geq 6\) and \(1 \leq u \leq (r-2)/2 - 1\), let

\[h_{r,u}(|\tau^*|^2, |\beta^*|^2, |T^*|^2, |B^*|^2) = \kappa(r, u) u \| \beta^* \| |r - 2u - 4| \| \tau^* |^{2u+2} \| B^* \| |r - 2u - 4| \| T^* \|^{2u - 2}. \]

(5.31)

Since \(((r - 2)/2 - u) u\) is zero when \(u = 0\) or \((r - 2)/2\), the coefficient of \(\pi_0^*\) in (5.30) can be readily expressed as

\[\| B^* \|^4 \| \tau^* |^4 \sum_{r = 6}^{\infty} \sum_{u = 1}^{(r-2)/2 - 1} \left(\frac{r - 2}{2} - u\right) u h_{r,u}(|\tau^*|^2, |\beta^*|^2, |T^*|^2, |B^*|^2) \]
\[= \| B^* \|^4 \| \tau^* |^4 \]
\[\times \left\{ \kappa(6, 1) + \sum_{r = 8}^{\infty} \sum_{u = 1}^{(r-2)/2 - 1} \left(\frac{r - 2}{2} - u\right) u h_{r,u}(|\tau^*|^2, |\beta^*|^2, |T^*|^2, |B^*|^2) \right\}. \]

(5.32)

Upon dropping the term corresponding to \(u = 0\) and separating the term corresponding to \(u = (r - 2)/2\), the coefficient of \(\frac{1}{2}(\pi_1^* - b \| T^* \|^4)\) in (5.30) can be expressed as
\[ \| \beta^* \|^4 \| \tau^* \|^4 \left\{ \kappa(4, 1) + \sum_{r=6}^{\infty} \kappa \left( r, \frac{r-2}{2} \right) \frac{r-2}{2} \| \tau^* \|^{r-4} \| T^* \|^{r-4} \right\} \\
+ \| \beta^* \|^4 \| \tau^* \|^4 \| B^* \|^2 \\
\times \left\{ \kappa(6, 1) + \sum_{r=8}^{\infty} \sum_{u=1}^{(r-2)/2-1} u h_{r,u}(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|) \right\}. \tag{5.33} \]

Analogously, dropping the term corresponding to \( u = (r-2)/2 \) and separating the term corresponding to \( u = 0 \), the coefficient of \( \frac{1}{2}(\pi^*_z - b \| B^* \|^4) \) in (5.30) can be expressed as
\[ \| \beta^* \|^4 \| \tau^* \|^2 \left\{ \kappa(4, 0) + \sum_{r=6}^{\infty} \kappa(r, 0) \frac{r-2}{2} \| \beta^* \|^{r-4} \| B^* \|^{r-4} \right\} \\
+ \| \beta^* \|^4 \| \tau^* \|^4 \| T^* \|^2 \\
+ \left\{ \kappa(6, 1) + \sum_{r=8}^{\infty} \sum_{u=1}^{(r-2)/2-1} \left( \frac{r-2}{2} - u \right) h_{r,u}(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|) \right\}. \tag{5.34} \]

Hence, from (5.30), (5.32)–(5.34), \( \psi_2 \) can be written as
\[ \psi_2 = \pi_0^* \| \beta^* \|^4 \| \tau^* \|^4 \left\{ \kappa(6, 1) + \sum_{r=8}^{\infty} \sum_{u=1}^{(r-2)/2-1} \left( \frac{r-2}{2} - u \right) u \right\} \\
\times h_{r,u}(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|) \right\} \]
\[ + \frac{1}{2} (\pi^*_z - b \| T^* \|^4) \| \beta^* \|^2 \| \tau^* \|^4 \\
\times \left[ \left\{ \kappa(4, 1) + \sum_{r=6}^{\infty} \kappa \left( r, \frac{r-2}{2} \right) \frac{r-2}{2} \| \tau^* \|^{r-4} \| T^* \|^{r-4} \right\} \\
+ \| \beta^* \|^2 \| B^* \|^2 \left\{ \kappa(6, 1) \\
+ \sum_{r=8}^{\infty} \sum_{u=1}^{(r-2)/2-1} u h_{r,u}(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|) \right\} \right] \]
\[ + \frac{1}{2} (\pi^*_z - v \| B^* \|^4) \| \beta^* \|^4 \| \tau^* \|^2 \\
\times \left[ \left\{ \kappa(4, 0) + \sum_{r=6}^{\infty} \kappa(r, 0) \frac{r-2}{2} \| \beta^* \|^{r-4} \| B^* \|^{r-4} \right\} \right]. \]
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\[ + \| \tau^* \|^2 \| T^* \|^2 \left\{ \kappa(6, 1) + \sum_{r=2}^{\infty} \sum_{u=0}^{(r-2)/2-1} \left( \frac{r-2}{2} - u \right) \right\} \times h_{r,u}(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|) \]

\[ + h_1(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|), \]

where \( h_1(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|) \) is a function of \( \| \tau^* \|, \| \beta^* \|, \| T^* \|, \) and \( \| B^* \|, \) given by

\[
h_1(\| \tau^* \|, \| \beta^* \|, \| T^* \|, \| B^* \|) = \frac{1}{4} \text{SS}^{\text{Residual}} \sum_{r=2}^{\infty} \sum_{u=0}^{(r-2)/2} \kappa(r, u) \| \beta^* \|^{r-2u} \| \tau^* \|^{2u+2} \]

\[ \times \| B^* \|^{r-2u-2} \| T^* \|^{2u}. \]

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