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On the ergodic principle for Markov and quadratic stochastic processes and its relations

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Abstract

In the paper we prove that a quadratic stochastic process satisfies the ergodic principle if and only if the associated Markov process satisfies one.

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1. Introduction

It is known that Markov processes are well-developed field of mathematics which have various applications in physics, biology and so on. But there are some physical models which cannot be described by such processes. One of such models is a model related to population genetics. Namely, this model is described by quadratic stochastic processes (see [7] for review). To define it, we denote

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$$\ell^1 = \left\{ x = (x_n) : \|x\|_1 = \sum_{n=1}^{\infty} |x_n| < \infty;\; x_n \in \mathbb{R} \right\},$$
$$S = \{x \in \ell^1 : x_n \geqslant 0;\; \|x\|_1 = 1\}.$$

Hence this process is defined as follows (see [1,9]): consider a family of functions $\left\{ P_{ij,k}^{[s,t]} : i, j, k \in \mathbb{N},\; s,\; t \in \mathbb{R}^+,\; t - s \geqslant 1 \right\}$. This family is said to be *quadratic stochastic process* (*q.s.p.*) if for fixed $s, t \in \mathbb{R}_+$, it satisfies the following conditions:

(i) $P_{ij,k}^{[s,t]} = P_{ji,k}^{[s,t]}$ for any $i, j, k \in \mathbb{N}$.

(ii) $P_{ij,k}^{[s,t]} \geqslant 0$ and $\sum_{k=1}^{\infty} P_{ij,k}^{[s,t]} = 1$ for any $i, j, k \in \mathbb{N}$.

(iii) An analogue of Kolmogorov–Chapman equation; here there are two variants: for the initial point $x^{(0)} \in S$, $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots)$ and $s < r < t$ such that $t - r \geqslant 1, r - s \geqslant 1$.

(iii$_A$)

$$P_{ij,k}^{[s,t]} = \sum_{m,l=1}^{\infty} P_{ij,m}^{[s,r]} P_{ml,k}^{[r,t]} x_l^{(r)},$$

where $x_k^{(r)}$ is defined as follows:

$$x_k^{(r)} = \sum_{i,j=1}^{\infty} P_{ij,k}^{[0,r]} x_i^{(0)} x_j^{(0)}.$$

(iii$_B$)

$$P_{ij,k}^{[s,t]} = \sum_{m,l,g,h=1}^{\infty} P_{im,l}^{[s,r]} P_{jg,h}^{[s,r]} P_{lh,k}^{[r,t]} x_m^{(s)} x_g^{(s)}.$$

We say that the q.s.p. $\left\{ P_{ij,k}^{s,t} \right\}$ is of *type* (A) or (B) if it satisfies the fundamental equations (iii$_A$) or (iii$_B$), respectively. In this definition the functions $P_{ij,k}^{[s,t]}$ denote the probability that under the interaction of the elements i and j at time s the element k comes into effect at time t. Since for physical, chemical and biological phenomena, a certain time is necessary for the realization of an interaction, we shall take the greatest such time to be equal to 1 (see the Boltzmann model [4] or the biological model [7]). Thus the probability $P_{ij,k}^{[s,t]}$ is defined for $t - s \geqslant 1$. It should be noted that the quadratic stochastic processes are related to quadratic transformations (see [5,7]) in the same way as Markov processes are related to linear transformations.

The equations (iii$_A$) and (iii$_B$) can be interpreted as different laws of behavior of the "offspring". Some examples of q.s.p. were given in [1,3]. We note that quadratic processes of type (A) were considered in [1,9]. One of the central problems in this theory is the study of limit behaviors of q.s.p. An ergodic principle is such a concept relating to limit behaviors (see [6]). In [10] some conditions were given for q.s.p. to satisfy this principle. It is known [1,2] that a certain Markov process can be defined by means of q.s.p., therefore, it is interesting to know the following question: if this Markov process satisfies the ergodic principle, then would q.s.p. satisfy that principle? The answer to this question helps us to find more conditions for fulfilling the ergodic principle for q.s.p., since the theory of Markov processes is a well-developed filed. In this paper we are going to solve the formulated question for discrete time q.s.p.

We note that a part of the results were announced in [8,10].

2. Ergodic principle for quadratic stochastic processes

In this section we will answer to the above formulated question for discrete time q.s.p. Before doing it, we prove some results concerning Markov processes.

In the sequel we will consider discrete time q.s.p., i.e. for $\left\{P^{s,t}_{ij,k}\right\}$ the numbers s,t belong to $\mathbb{N}$.

Recall that a matrix (Q_{ij}) is called *stochastic* if

$$Q_{ij} \geqslant 0; \quad \sum_{j=1}^{\infty} Q_{ij} = 1.$$

First recall that a family of stochastic matrices $\left\{\left(Q^{m,n}_{ij}\right)_{i,j\in\mathbb{N}} : m,n \in \mathbb{N}, n-m \geqslant 1\right\}$ is called *discrete time Markov process* if the following condition holds: for every $m < n < l$

$$Q^{m,l}_{ij} = \sum_{k=1}^{\infty} Q^{m,n}_{ik} Q^{n,l}_{kj}. \tag{2.1}$$

This equation is known as the Kolmogorov–Chapman equation.

A Markov process $\left\{Q^{m,n}_{ij}\right\}$ is said to satisfy *the ergodic principle* if

$$\lim_{n\to\infty} \left|Q^{m,n}_{ik} - Q^{m,n}_{jk}\right| = 0$$

is valid, for every $i, j, k, m \in \mathbb{N}$. Note that this notion firstly was introduced in [6].

Each $(Q^{m,n}_{ij})$-stochastic matrix defines a linear operator $\mathbf{Q}^{m,n} : \ell^1 \to \ell^1$ as follows:

$$(\mathbf{Q}^{m,n}(x))_j = \sum_{i=1}^{\infty} Q^{m,n}_{ij} x_i, \quad x = (x_n) \in \ell^1. \tag{2.2}$$

Stochasticity of $\left(Q^{m,n}_{ij}\right)$ implies that

$$\mathbf{Q}^{m,n}(S) \subset S \quad \text{and} \quad \|\mathbf{Q}^{m,n}x\|_1 \leqslant \|x\|_1, \quad x \in \ell^1. \tag{2.3}$$

By $\{e^{(n)}\}$ we denote standard basis of ℓ^1, i.e.

$$e^{(n)} = (\underbrace{0, 0, \ldots, 1}_{n}, \ldots), \quad n \in \mathbb{N}.$$

Now we formulate the following well-known fact (see, for example [11]).

Lemma 2.1. *A sequences $\{x_n\} \subset \ell^1$ converges weakly if and only if it converges in the norm of ℓ^1.*

We have the following:

Theorem 2.2. *Let $\left\{Q^{m,n}_{ij}\right\}$ be a Markov process. The following conditions are equivalent:*

(i) $\left\{Q^{m,n}_{ij}\right\}$ *satisfies the ergodic principle.*
(ii) *For every $i, j, m \in \mathbb{N}$ the following relation holds:*

$$\lim_{n\to\infty} \|\mathbf{Q}^{m,n}e^{(i)} - \mathbf{Q}^{m,n}e^{(j)}\|_1 = 0.$$

(iii) *For every* $\varphi, \psi \in S$ *and* $m \in \mathbb{N}$ *the following relation holds*:

$$\lim_{n\to\infty} \|\mathbf{Q}^{m,n}\varphi - \mathbf{Q}^{m,n}\psi\|_1 = 0.$$

Proof. (i)⇒(ii). The ergodic principle means that the sequence $x_{ij,m}^{(n)} = \left(Q_{ik}^{m,n} - Q_{jk}^{m,n}\right)_{k\in\mathbb{N}}$ converges weakly in ℓ^1, here $i, j, m \in \mathbb{N}$ are fixed numbers. According to Lemma 2.1 we infer that $x_{ij,m}^{(n)}$ converges strongly in ℓ^1, this means

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \left|Q_{ik}^{m,n} - Q_{jk}^{m,n}\right| \to 0.$$

On the other hand, from (2.2) we find

$$\|\mathbf{Q}^{m,n}e^{(i)} - \mathbf{Q}^{m,n}e^{(j)}\|_1 = \sum_{k=1}^{\infty} \left|Q_{ik}^{m,n} - Q_{jk}^{m,n}\right|. \tag{2.4}$$

Hence, the considered implication is proved.

(ii)⇒(iii). First consider the following elements: $\xi = \sum_{i=1}^{M} \alpha_i e^{(i)}$, $\eta = \sum_{j=1}^{N} \beta_j e^{(j)}$, where $\alpha_i, \beta_j \geqslant 0$, $\sum_{i=1}^{M} \alpha_i = \sum_{j=1}^{N} \beta_j = 1$. Using (ii) we have

$$\begin{aligned}
\|\mathbf{Q}^{m,n}\xi - \mathbf{Q}^{m,n}\eta\|_1 &= \left\| \sum_{i=1}^{M} \alpha_i \mathbf{Q}^{m,n} e^{(i)} - \sum_{j=1}^{N} \beta_j \mathbf{Q}^{m,n} e^{(j)} \right\| \\
&= \left\| \sum_{i=1}^{M}\sum_{j=1}^{N} \alpha_i\beta_j \mathbf{Q}^{m,n} e^{(i)} - \sum_{i=1}^{M}\sum_{i=1}^{N} \alpha_i\beta_j \mathbf{Q}^{m,n} e^{(j)} \right\| \\
&\leqslant \sum_{i=1}^{M}\sum_{j=1}^{N} \alpha_i\beta_j \|\mathbf{Q}^{m,n}e^{(i)} - \mathbf{Q}^{m,n}e^{(j)}\|_1 \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{2.5}$$

Now let $\varphi, \psi \in S$ and $\varepsilon > 0$. Denote

$$G = \left\{ \xi = \sum_{i=1}^{M} \alpha_i e^{(i)} : \alpha_i \geqslant 0; \sum_{i=1}^{M} \alpha_i = 1,\ M \in \mathbb{N} \right\}.$$

It is clear that G is dense in S. Therefore, there exist $\xi, \eta \in G$ such that

$$\|\varphi - \xi\|_1 < \varepsilon/3, \quad \|\psi - \eta\|_1 < \varepsilon/3.$$

According to (2.5) there is $n_0 \in \mathbb{N}$ such that

$$\|\mathbf{Q}^{m,n}\xi - \mathbf{Q}^{m,n}\eta\|_1 < \varepsilon/3, \quad \forall n \geqslant n_0.$$

Hence, by means of the above relations and (2.3) we obtain

$$\begin{aligned}
\|\mathbf{Q}^{m,n}\varphi - \mathbf{Q}^{m,n}\psi\|_1 &\leqslant \|\mathbf{Q}^{m,n}(\varphi - \xi)\|_1 + \|\mathbf{Q}^{m,n}(\psi - \eta)\|_1 + \|\mathbf{Q}^{m,n}\xi - \mathbf{Q}^{m,n}\eta\|_1 \\
&\leqslant \|\varphi - \xi\|_1 + \|\psi - \eta\|_1 + \varepsilon/3 < \varepsilon
\end{aligned}$$

for all $n \geqslant n_0$. Thus the implication is proved. The implication (iii)⇒(i) is obvious. □

Let $\left\{P_{ij,k}^{[m,n]}\right\}$ be a q.s.p. Define

$$\mathbb{H}_{ij}^{m,n} = \sum_{l=1}^{\infty} P_{il,j}^{[m,n]} x_l^{(m)}, \quad i, j \in \mathbb{N}. \tag{2.6}$$

It is clear $\left(\mathbb{H}_{ij}^{m,n}\right)$ is a stochastic matrix.

Lemma 2.3. *Let* $\left\{P_{ij,k}^{[m,n]}\right\}$ *be a q.s.p. Then* $\left\{\mathbb{H}_{ij}^{m,n}\right\}$ *is a Markov process.*

Proof. Consider two distinct cases with respect to the types of q.s.p.

Case (a). Let $P_{ij,k}^{[m,n]}$ be a q.s.p of type (A). Then we have

$$\begin{aligned}\sum_{k=1}^{\infty} \mathbb{H}_{ik}^{m,n} \mathbb{H}_{kj}^{n,l} &= \sum_{k=1}^{\infty} \left(\sum_{u=1}^{\infty} P_{iu,k}^{[m,n]} x_u^{(m)}\right) \left(\sum_{v=1}^{\infty} P_{kv,j}^{[n,l]} x_v^{(n)}\right) \\ &= \sum_{u=1}^{\infty} \left(\sum_{k,v=1}^{\infty} P_{iu,k}^{[m,n]} P_{kv,j}^{[n,l]} x_v^{(n)}\right) x_u^{(m)} \\ &= \sum_{u=1}^{\infty} P_{iu,j}^{[m,l]} x_u^{(m)} = \mathbb{H}_{ij}^{m,l}.\end{aligned}$$

So $\left\{Q_{ij}^{m,n}\right\}$ is a Markov process.

Case (b). Let $P_{ij,k}^{[m,n]}$ be a q.s.p of type (B). First consider

$$\begin{aligned}\sum_{i,j=1}^{\infty} P_{ij,k}^{[m,n]} x_i^{(m)} x_j^{(m)} &= \sum_{i,j=1}^{\infty} P_{ij,k}^{[m,n]} \left(\sum_{e,f=1}^{\infty} P_{ef,i}^{[0,m]} x_e^{(0)} x_f^{(0)}\right) \left(\sum_{c,d=1}^{\infty} P_{cd,j}^{[0,m]} x_c^{(0)} x_d^{(0)}\right) \\ &= \sum_{c,e=1}^{\infty} \left(\sum_{f,d,i,j=1}^{\infty} P_{ef,i}^{[0,m]} P_{cd,j}^{[0,m]} P_{ij,k}^{[m,n]} x_f^{(0)} x_d^{(0)}\right) x_e^{(0)} x_c^{(0)} \\ &= \sum_{e,c=1}^{\infty} P_{ec,k}^{[0,n]} x_e^{(0)} x_c^{(0)} = x_k^{(n)}.\end{aligned}$$

Therefore

$$x_k^{(n)} = \sum_{i,j=1}^{\infty} P_{ij,k}^{[m,n]} x_i^{(m)} x_j^{(m)}.$$

Using this equality check Markovianity:

$$\mathbb{H}_{ij}^{m,l} = \sum_{k=1}^{\infty} P_{ik,j}^{[m,l]} x_k^{(m)} = \sum_{k=1}^{\infty} \left(\sum_{a,b,c,d=1}^{\infty} P_{ia,b}^{[m,n]} P_{kc,d}^{[m,n]} P_{bd,j}^{[n,l]} x_a^{(m)} x_c^{(m)}\right) x_k^{(m)}$$

$$= \sum_{b,d=1}^{\infty} \left(\sum_{a=1}^{\infty} P_{ia,b}^{[m,n]} x_a^{(m)} \right) \left(\sum_{k,c=1}^{\infty} P_{kc,d}^{[m,n]} x_k^{(m)} x_c^{(m)} \right) P_{bd,j}^{[n,l]}$$

$$= \sum_{b=1}^{\infty} \left(\sum_{a=1}^{\infty} P_{ia,b}^{[m,n]} x_a^{(m)} \right) \left(\sum_{d=1}^{\infty} P_{bd,j}^{[n,l]} x_d^{(n)} \right) = \sum_{b=1}^{\infty} \mathbb{H}_{ib}^{m,n} \mathbb{H}_{bj}^{n,l}. \qquad \square$$

The defined process $\{\mathbb{H}_{ij}^{m,n}\}$ is called *associated Markov process* with respect to q.s.p. By $\mathbf{H}^{m,n}$ we denote a linear operator associated with this Markov process (see (2.2)).

We say that the *ergodic principle* holds for the q.s.p. $\{P_{ij,k}^{[m,n]}\}$ if

$$\lim_{n\to\infty} \left| P_{ij,k}^{[m,n]} - P_{uv,k}^{[m,n]} \right| = 0,$$

is valid for any $i, j, u, v, k \in \mathbb{N}$ and arbitrary $m \in \mathbb{N}$.

Define

$$R_{ij}^{m,n}(x) = \sum_{l=1}^{\infty} P_{il,j}^{[m,n]} x_l, \tag{2.7}$$

here $x = (x_n) \in S$. It is clear that for each $m, n \in \mathbb{N}$ and $x \in S$ the matrix $\left(R_{ij}^{m,n}(x)\right)$ is stochastic.

Proposition 2.4. *Let* $\{P_{il,j}^{[m,n]}\}$ *be a q.s.p. Then the following conditions are equivalent*:

(i) *For every* $i, j, k \in \mathbb{N}$ *and* $x \in S$ *the following holds*:
$$\lim_{n\to\infty} \left| R_{ik}^{m,n}(x) - R_{jk}^{m,n}(x) \right| = 0.$$

(ii) *The Markov process* $\{\mathbb{H}_{ij}^{m,n}\}$ *satisfies the ergodic principle*.

Proof. The implication (i)⇒(ii) is obvious. Therefore, let us consider the implication (ii)⇒(i). Again divide the proof into two cases.

Case (a). Let $\{P_{ij,k}^{[m,n]}\}$ be a q.s.p. of type (A). Then we have

$$R_{ik}^{m,n}(x) = \sum_{l=1}^{\infty} P_{il,k}^{[m,n]} x_l = \sum_{l=1}^{\infty} \left(\sum_{u,v=1}^{\infty} P_{il,u}^{[m,m+1]} P_{uv,k}^{[m+1,n]} x_v^{(m+1)} \right) x_l$$

$$= \sum_{l=1}^{\infty} \sum_{u=1}^{\infty} P_{il,k}^{[m,m+1]} \mathbb{H}_{uk}^{m+1,n} x_l$$

$$= \sum_{u=1}^{\infty} Q_{uk}^{m+1,n} y_u(i) = (\mathbf{H}^{m+1,n} y(i))_k,$$

where $y_u(i) = \sum_{l=1}^{\infty} P_{il,u}^{[m,m+1]} x_l$. Similarly, one gets

$$R_{jk}^{m,n}(x) = \sum_{u=1}^{\infty} \mathbb{H}_{u,k}^{[m+1,n]} y_u(j) = (\mathbf{H}^{m+1,n} y(j))_k.$$

The ergodic principle for the Markov process with Theorem 2.2 implies that

$$\|\mathbf{H}^{m+1,n} y(i) - \mathbf{H}^{m+1,n} y(j)\|_1 \to 0 \quad \text{as } n \to \infty.$$

Therefore,

$$\left|R_{ik}^{m,n}(x) - R_{jk}^{m,n}(x)\right| = |(\mathbf{H}^{m+1,n}y(i))_k - (\mathbf{H}^{m+1,n}y(j))_k| \leqslant \|\mathbf{H}^{m+1,n}y(i) - \mathbf{H}^{m+1,n}y(j)\|_1 \to 0 \quad \text{as } n \to \infty.$$

Case (b). Now suppose that $\left(P_{ij,k}^{[m,n]}\right)$ is a q.s.p. of type (B). Given $x \in S$, define operator $\mathbf{R}^{m,n}(x) : \ell^1 \to \ell^1$ as follows:

$$(\mathbf{R}^{m,n}(x))(y)_k = \sum_{i=1}^{\infty} R_{ik}^{m,n}(x) y_i,$$

$y = (y_i) \in \ell^1$. Using stochasticity of $(R_{ij}^{m,n}(x))$ we infer

$$\|(\mathbf{R}^{m,n}(x))(y)\|_1 \leqslant \|y\|_1, \quad \forall y \in \ell^1. \tag{2.8}$$

Now using $(\text{iii})_B$ we find

$$\begin{aligned} R_{ik}^{m,n+1}(x) &= \sum_{l=1}^{\infty}\left(\sum_{a,b,c,d=1}^{\infty} P_{ia,b}^{[m,n]} P_{lc,d}^{[m,n]} P_{bd,k}^{[n,n+1]} x_a^{(m)} x_c^{(m)}\right) x_l \\ &= \sum_{l=1}^{\infty}\sum_{b,d=1}^{\infty} \mathbb{H}_{ib}^{m,n}\mathbb{H}_{ld}^{m,n} P_{bd,k}^{[n,n+1]} x_l \\ &= (\mathbf{R}^{n,n+1}(y))(\mathbf{H}^{m,n}(e^{(i)}))_k, \end{aligned} \tag{2.9}$$

here $y = \mathbf{H}^{m,n}(x)$. Similarly, one gets

$$R_{jk}^{m,n+1}(x) = (\mathbf{R}^{n,n+1}(y))(\mathbf{H}^{m,n}(e^{(j)}))_k. \tag{2.10}$$

Therefore, it follows from (2.8)–(2.10) that

$$\begin{aligned} \left|R_{ik}^{m,n+1}(x) - R_{jk}^{m,n+1}(x)\right| &= |(\mathbf{R}^{n,n+1}(y))(\mathbf{H}^{m,n}(e^{(i)}) - \mathbf{H}^{m,n}(e^{(j)}))_k| \\ &\leqslant \|(\mathbf{R}^{n,n+1}(y))(\mathbf{H}^{m,n}(e^{(i)}) - \mathbf{H}^{m,n}(e^{(j)}))\|_1 \\ &\leqslant \|\mathbf{H}^{m,n}(e^{(i)}) - \mathbf{H}^{m,n}(e^{(j)})\|_1 \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This completes the proof. □

Proposition 2.5. *Let* $\left\{P_{ij,k}^{[m,n]}\right\}$ *be a q.s.p. Then the following conditions are equivalent*:

(i) $\left\{P_{ij,k}^{[m,n]}\right\}$ *satisfies the ergodic principle*.
(ii) *For every* $x \in S$ *and* $i, j, k, m \in \mathbb{N}$ *the following holds*:

$$\lim_{n\to\infty}\left|R_{ik}^{m,n}(x) - R_{jk}^{m,n}(x)\right| = 0.$$

Proof. (i)⇒(ii). Define a bilinear operator $\mathbf{P}^{[m,n]} : \ell^1 \times \ell^1 \to \ell^1$ as follows:

$$(\mathbf{P}^{[m,n]}(x, y))_k = \sum_{i,j=1}^{\infty} P_{ij,k}^{[m,n]} x_i y_j,$$

where $x = (x_n)$, $y = (y_n) \in \ell^1$. According to Lemma 2.1 from the ergodic principle we find

$$\lim_{n\to\infty} \|\mathbf{P}^{[m,n]}(e^{(i)}, e^{(j)}) - \mathbf{P}^{[m,n]}(e^{(u)}, e^{(v)})\|_1 = 0$$

for every $i, j, u, v, m \in \mathbb{N}$. Similar arguments used towards the proof of Theorem 2.2 implies that

$$\lim_{n\to\infty} \|\mathbf{P}^{[m,n]}(e^{(i)}, x) - \mathbf{P}^{[m,n]}(e^{(u)}, y)\|_1 = 0,$$

for every $x, y \in S$. Hence, we get

$$\left|R_{ik}^{m,n}(x) - R_{jk}^{m,n}(x)\right| \leqslant \|\mathbf{P}^{[m,n]}(e^{(i)}, x) - \mathbf{P}^{[m,n]}(e^{(j)}, x)\|_1 \to 0 \quad \text{as } n \to \infty.$$

Now consider the implication (ii)⇒(i). From (2.7) we have

$$\left|P_{iu,k}^{[m,n]} - P_{ju,k}^{[m,n]}\right| = \left|R_{ik}^{m,n}(e^{(u)}) - R_{jk}^{m,n}(e^{(u)})\right| \to 0 \quad \text{as } n \to \infty;$$

for every $i, j, k, u \in \mathbb{N}$. Whence one gets

$$\left|P_{ij,k}^{[m,n]} - P_{uv,k}^{[m,n]}\right| \leqslant \left|P_{ij,k}^{[m,n]} - P_{uj,k}^{[m,n]}\right| + \left|P_{ju,k}^{[m,n]} - P_{vu,k}^{[m,n]}\right| \to 0 \quad \text{as } n \to \infty,$$

here we have used equation (i) of definition q.s.p. □

Now we are ready to formulate our main result.

Theorem 2.6. *Let* $\{P_{ij,k}^{[m,n]}\}$ *be a q.s.p. The following conditions are equivalent*:

(i) $\{P_{ij,k}^{[m,n]}\}$ *satisfies the ergodic principle*.
(ii) *The Markov process* $\{\mathbb{H}_{ij}^{m,n}\}$ *satisfies the ergodic principle*.

The proof immediately follows from Propositions 2.4 and 2.5.

3. An application of the main result

In this section we give certain conditions for the Markov process which ensure fulfilling the ergodic principle for q.s.p.

Now we need some auxiliary facts.

Lemma 3.1. *Let* $\{a_n\}$ *be a nonnegative sequence which satisfies the following inequality*:

$$a_n \leqslant (1 - \lambda_n)a_{n-1} + \prod_{k=1}^{n}(1 - \lambda_k), \tag{3.1}$$

where $\lambda_n \in (0, 1)$, $\forall n \in \mathbb{N}$ *and*

$$\sum_{n=1}^{\infty} \lambda_n = \infty, \tag{3.2}$$

$$\sum_{j=1}^{n} \frac{\prod_{k=1}^{n}(1 - \lambda_k)}{(1 - \lambda_j)} \to 0 \quad \text{as } n \to \infty, \tag{3.3}$$

then $\lim_{n\to\infty} a_n = 0$.

Proof. From (3.1) by iterating we obtain

$$a_n \leqslant a_1 \prod_{i=2}^{n}(1-\lambda_i) + \sum_{j=1}^{n} \frac{\prod_{k=1}^{n}(1-\lambda_k)}{(1-\lambda_j)}. \tag{3.4}$$

The condition (3.2) with $0 < \lambda_n < 1$ implies that $\prod_{k=1}^{n}(1-\lambda_k) \to 0$ as $n \to \infty$. Hence, by means of (3.3) one gets $a_n \to 0$ as $n \to \infty$. □

Corollary 3.2. *Let $\{a_n\}$ be as above. If the sequence $\{\lambda_n\}$, $(0 < \lambda_n < 1, \forall n \in \mathbb{N})$ satisfies (3.2) and the following relations:*

$$n \prod_{k=1}^{n}(1-\lambda_k) \to 0 \quad \textit{as } n \to \infty, \tag{3.5}$$

$$\sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_i} = \mathrm{O}(n) \tag{3.6}$$

then $\lim_{n\to\infty} a_n = 0$.

Proof. It is enough to show that the conditions (3.5), (3.6) imply (3.3). Indeed, since $0 < 1-\lambda_j < 1$, we can write

$$\frac{1}{1-\lambda_j} = 1 + \varepsilon_j,$$

here ε_j is some positive number. It then follows that

$$\sum_{j=1}^{n} \frac{1}{1-\lambda_j} = n + \sum_{j=1}^{n} \varepsilon_j = n + \sum_{j=1}^{n} \frac{\lambda_j}{1-\lambda_j}.$$

It follows from (3.6) that

$$\sum_{j=1}^{n} \frac{1}{1-\lambda_j} \leqslant C \cdot n \tag{3.7}$$

for all $n \in \mathbb{N}$, here C is a constant. Therefore, from (3.7), (3.4) and (3.5) we infer that

$$\begin{aligned}\sum_{j=1}^{n} \frac{\prod_{k=1}^{n}(1-\lambda_k)}{1-\lambda_j} &= \prod_{k=1}^{n}(1-\lambda_k) \sum_{j=1}^{n} \frac{1}{1-\lambda_j} \\ &\leqslant C \cdot n \prod_{k=1}^{n}(1-\lambda_k) \to 0 \quad \text{as } n \to \infty. \qquad \square\end{aligned}$$

Theorem 3.3. *Let $\left\{Q_{ij}^{m,n}\right\}$ be a Markov process. If there exists a number $k_0 \in \mathbb{N}$ and a sequence $\{\lambda_n\}$, $0 < \lambda_n < 1$, $\forall n \in \mathbb{N}$ satisfying (3.2), (3.3) such that*

$$Q_{ik_0}^{n-1,n} \geqslant \lambda_n \quad \textit{for all } i, n \in \mathbb{N}. \tag{3.8}$$

Then the Markov process satisfies the ergodic principle.

Proof. We set

$$\sup_{i\in\mathbb{N}} Q_{ik_0}^{k,n} = M_{k,n}(k_0); \quad \inf_{i\in\mathbb{N}} Q_{ik_0}^{k,n} = m_{k,n}(k_0).$$

For $i < k < n$ we have

$$Q_{ik_0}^{i,n} = \sum_{l=1}^{\infty} Q_{il}^{i,k} Q_{lk_0}^{k,n} \leqslant M_{k,n}(k_0) \sum_{l=1}^{\infty} Q_{il}^{i,k} = M_{k,n}(k_0). \tag{3.9}$$

Similarly,

$$Q_{ik_0}^{i,n} \geqslant m_{k,n}(k_0). \tag{3.10}$$

By means of (3.8) we infer

$$Q_{ik_0}^{n-1,n} - \lambda_n Q_{jk_0}^{n-1,n} \geqslant 0 \tag{3.11}$$

for all $i, j \in \mathbb{N}$, because $0 \leqslant Q_{jk_0}^{n-1,n} \leqslant 1$. It follows:

$$\begin{aligned} Q_{ik_0}^{k-1,n} &= \sum_{l=1}^{\infty} Q_{il}^{k-1,k} Q_{lk_0}^{k,n} \\ &= \sum_{l=1}^{\infty} \left[Q_{il}^{k-1,k} - \lambda_k Q_{jl}^{k-1,k}\right] Q_{lk_0}^{k,n} + \lambda_k \sum_{l=1}^{\infty} Q_{jl}^{k-1,k} Q_{lk_0}^{k,n} \\ &\geqslant m_{k,n}(k_0) \sum_{l=1}^{\infty} \left[Q_{il}^{k-1,k} - \lambda_k Q_{jl}^{k-1,k}\right] + \lambda_k Q_{jk_0}^{k-1,n} \\ &= (1-\lambda_k) m_{k,n}(k_0) + \lambda_k Q_{jk_0}^{k-1,n}, \end{aligned}$$

whence

$$Q_{jk_0}^{k-1,n} - Q_{ik_0}^{k-1,n} \leqslant (1-\lambda_k)\left(Q_{jk_0}^{k-1,n} - m_{k,n}(k_0)\right). \tag{3.12}$$

Since (3.12) holds for any $i, j \in \mathbb{N}$, from (3.9) and (3.10) we find

$$M_{k-1,n}(k_0) - m_{k-1,n}(k_0) \leqslant (1-\lambda_k)(M_{k,n}(k_0) - m_{k,n}(k_0)). \tag{3.13}$$

So iterating the last inequality we get

$$M_{l,n}(k_0) - m_{l,n}(k_0) \leqslant \prod_{k=l+1}^{n-1} (1-\lambda_k). \tag{3.14}$$

Using (2.1) we have

$$\left|Q_{ik}^{m,n} - Q_{jk}^{m,n}\right| \leqslant \sum_{l=1}^{\infty} \left|Q_{il}^{m,n-1} - Q_{jl}^{m,n-1}\right| Q_{lk}^{n-1,n},$$

for every $i, j \in \mathbb{N}$. Hence by means of (3.8) it yields that

$$\begin{aligned} \sum_{k=1,k\neq k_0}^{\infty} \left|Q_{ik}^{m,n} - Q_{jk}^{m,n}\right| &\leqslant \sum_{l,k=1}^{\infty} \left|Q_{il}^{m,n-1} - Q_{jl}^{m,n-1}\right| Q_{lk}^{n-1,n} \\ &\quad - \sum_{l=1}^{\infty} \left|Q_{il}^{m,n-1} - Q_{jl}^{m,n-1}\right| Q_{lk_0}^{n-1,n} \\ &\leqslant (1-\lambda_n) \sum_{l=1}^{\infty} \left|Q_{il}^{m,n-1} - Q_{jl}^{m,n-1}\right|. \end{aligned} \tag{3.15}$$

Now add $\left|Q_{ik_0}^{m,n} - Q_{jk_0}^{m,n}\right|$ to both sides of (3.15). Then

$$\sum_{k=1}^{\infty}\left|Q_{ik}^{m,n} - Q_{jk}^{m,n}\right| \leqslant (1-\lambda_n)\sum_{l=1}^{\infty}\left|Q_{il}^{m,n-1} - Q_{jl}^{m,n-1}\right| + \left|Q_{ik_0}^{m,n} - Q_{jk_0}^{m,n}\right|.$$

Now by means of (3.14) and (2.4) we infer

$$\|\mathbf{Q}^{m,n}(e^{(i)}) - \mathbf{Q}^{m,n}(e^{(j)})\|_1 \leqslant (1-\lambda_n)\|\mathbf{Q}^{m,n-1}(e^{(i)}) - \mathbf{Q}^{m,n-1}(e^{(j)})\|_1 + \prod_{j=m+1}^{n-1}(1-\lambda_j).$$

Denoting $a_n = \|\mathbf{Q}^{m,n}(e^{(i)}) - \mathbf{Q}^{m,n}(e^{(j)})\|_1$ and applying Lemma 2.1 to a_n one gets that

$$\left\|\mathbf{Q}^{m,n}(e^{(i)}) - \mathbf{Q}^{m,n}(e^{(j)})\right\|_1 \to 0 \quad \text{as } n \to \infty.$$

So, according to Theorem 2.2 we obtain the required assertion. □

Now we can formulate the following:

Theorem 3.4. *Let* $\left\{P_{ij,k}^{[m,n]}\right\}$ *be a q.s.p. If there exist a number* $k_0 \in \mathbb{N}$ *and a sequence* $\{\lambda_n\}$, $(0 < \lambda_n < 1)$ *satisfying the conditions* (3.2), (3.3) *such that*

$$P_{ij,k_0}^{[n-1,n]} \geqslant \lambda_n \quad \textit{for all } i, j \in \mathbb{N}, \tag{3.16}$$

then the q.s.p. satisfies the ergodic principle.

Proof. Consider the Markov process $\left\{\mathbb{H}_{ij}^{m,n}\right\}$ associated with given q.s.p. (see (2.6)). Then (3.16) implies that

$$\mathbb{H}_{ik_0}^{n-1,n} = \sum_{l=1}^{\infty} P_{il,k_0}^{[n-1,n]} x_l^{(n-1)} \geqslant \lambda_n \sum_{l=1}^{\infty} x_l^{(n-1)} = \lambda_n.$$

Consequently, the Markov process satisfies the conditions of Theorem 3.3. Therefore the ergodic principle is valid for it. Now by means of Theorem 2.6 we infer that q.s.p. satisfies the ergodic principle. □

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