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The variational iteration method: A reliable analytic tool for solving linear and nonlinear wave equations

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Abstract

In this work, He's variational iteration method (VIM) is used for analytic treatment of the linear and the nonlinear wave equations in bounded and unbounded domains. Wave-like equations are also investigated. The method is capable of reducing the size of calculation and easily overcomes the difficulty of the perturbation technique or Adomian polynomials. The study highlights the power of the VIM method.

Keywords: Variational iteration method; Wave equations

1. Introduction

The variational iteration method (VIM) developed in 1999 by He in [1–10] will be used to study the linear wave equation, nonlinear wave equation, and wave-like equation in bounded and unbounded domains. The method has been proved by many authors [11–22] to be reliable and efficient for a wide variety of scientific applications, linear and nonlinear as well. It was shown by many authors that this method is more powerful than existing techniques such as the Adomian method [23,24], perturbation method, etc. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purposes. The method is effectively used in [1–16] and the references therein. The perturbation method suffers from the computational workload, especially when the degree of nonlinearity increases. Moreover, the Adomian method suffers from the complicated algorithms used to calculate the Adomian polynomials that are necessary for nonlinear problems. The VIM has no specific requirements, such as linearization, small parameters, etc. for nonlinear operators.

A substantial amount of research work has been directed to the study of the linear wave equations, the nonlinear wave equations, the wave-like equations [17] in bounded domains, and the wave equation in an unbounded domain given by

$$u_{tt} = u_{xx} + f(u), \quad 0 < x < L, \tag{1}$$

$$u_{tt} = u_{xx} + F(u) + g(x,t), \quad 0 < x < L, \tag{2}$$

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$$u_{tt} = \frac{x^2}{2} u_{xx}, \quad 0 < x < L, \tag{3}$$

and

$$u_{tt} = u_{xx}, \quad \infty < x < \infty, \tag{4}$$

respectively. The functions $f(u)$, $F(u)$ and $g(x,t)$ are linear, nonlinear and source functions respectively. The wave equation plays an important role in various physical problems. Study of the wave equation is needed in diverse areas of engineering and scientific applications.

A vast amount of research work has been invested in the study of wave equations (1)–(4). The standard methods are the separation of variables method, D'Alembert method and many others. It is the goal of this work to effectively employ the variational iteration method (VIM) to establish exact solutions. As stated before, the method is reliable and efficient for handling linear and nonlinear problems, homogeneous or inhomogeneous, and in a bounded domain or unbounded domain, in a straightforward manner. Unlike the Adomian decomposition method, where computational algorithms are normally used to deal with the nonlinear terms, the VIM is used directly with no requirement or restrictive assumptions for the nonlinear terms.

Another important advantage is that the VIM method is capable of greatly reducing the size of calculations while still maintaining high accuracy of the numerical solution. In what follows we will highlight briefly the main points of the method, where details can be found in [1–16] and the references therein.

The aim of this work is twofold. First we aim to investigate the four physical wave models to obtain exact solutions without any restrictive assumptions that may change the physical behavior of the solution. Second we aim to confirm the power of He's variational iteration method in handling scientific and engineering problems.

2. He's variational iteration method

Consider the differential equation

$$Lu + Nu = g(x,t), \tag{5}$$

where L and N are linear and nonlinear operators respectively, and $g(x,t)$ is the source inhomogeneous term. In [1–10], He proposed the variational iteration method where a correction functional for Eq. (5) can be written as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi) \right) \mathrm{d}\xi, \tag{6}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta \tilde{u}_n = 0$. It is required first to determine the Lagrangian multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t)$, $n \geq 0$, of the solution $u(x,t)$ will be readily obtained upon using the Lagrangian multiplier obtained and by using any selective function u_0. The initial values $u(x,0)$ and $u_t(x,0)$ are usually used for the selective zeroth approximation u_0. Having λ determined, then several approximations $u_j(x,t)$, $j \geq 0$, can be determined. Consequently, the solution is given by

$$u = \lim_{n \to \infty} u_n. \tag{7}$$

In what follows, we will apply the VIM method to four physical models to illustrate the strength of the method and to establish exact solutions for these models.

3. The homogeneous wave equation

We first consider the homogeneous wave equation

$$\begin{aligned} & u_{tt} = u_{xx} - 3u, \quad 0 < x < \pi, t > 0, \\ & \text{B.C } u(0,t) = \sin(2t), \qquad u(\pi,t) = -\sin 2t, \\ & \text{I.C } u(x,0) = 0, \qquad u_t(x,0) = 2\cos x. \end{aligned} \tag{8}$$

It is to be noted that an additional term $-3u$ is added to the traditional wave equation. This term arises when each element of the string is subject to an additional force which is proportional to its displacement.

The correction functional for (8) reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2(\tilde{u}_n)(x,\xi)}{\partial x^2} + 3\tilde{u}_n(x,\xi)\right)\mathrm{d}\xi. \tag{9}$$

This yields the stationary conditions

$$\begin{aligned} &1-\lambda' = 0,\\ &\lambda(\xi = t) = 0,\\ &\lambda'' = 0. \end{aligned} \tag{10}$$

This in turn gives

$$\lambda = \xi - t. \tag{11}$$

Substituting this value of the Lagrangian multiplier into the functional (9) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2(\tilde{u}_n)(x,\xi)}{\partial x^2} + 3\tilde{u}_n(x,\xi)\right)\mathrm{d}\xi, \quad n \geq 0. \tag{12}$$

Considering the given initial values, we can select $u_0(x,t) = 2t\cos x$. Using this selection in (12) we obtain the following successive approximations:

$$\begin{aligned} u_0(x,t) &= 2t\cos x,\\ u_1(x,t) &= 2t\cos x - \frac{(2t)^3}{3!}\cos x,\\ u_2(x,t) &= 2t\cos x - \frac{(2t)^3}{3!}\cos x + \frac{(2t)^5}{5!}\cos x,\\ u_3(x,t) &= 2t\cos x - \frac{(2t)^3}{3!}\cos x + \frac{(2t)^5}{5!}\cos x - \frac{(2t)^7}{7!}\cos x,\\ &\vdots,\\ u_n(x,t) &= \cos x\left((2t) - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \frac{(2t)^7}{7!} + \cdots\right). \end{aligned} \tag{13}$$

Recall that

$$u = \lim_{n\to\infty} u_n, \tag{14}$$

which gives the exact solution

$$u(x,t) = \cos x \sin 2t. \tag{15}$$

4. The inhomogeneous wave equation

We next consider the inhomogeneous nonlinear wave equation

$$\begin{aligned} &u_{tt} = u_{xx} + u + u^2 - xt - xt^2, \quad 0 < x < \pi, t > 0,\\ &\text{B.C } u(0,t) = 0, \qquad u(\pi,t) = \pi t,\\ &\text{I.C } u(x,0) = 0, \qquad u_t(x,0) = x. \end{aligned} \tag{16}$$

The correction functional for (16) reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2(\tilde{u}_n)(x,\xi)}{\partial x^2} + \tilde{u}_n(x,\xi) + \tilde{u}_n^2(x,\xi) - x\xi - x^2\xi^2\right)\mathrm{d}\xi. \tag{17}$$

This yields the stationary conditions

$$\begin{aligned} &1-\lambda' = 0,\\ &\lambda(\xi = t) = 0,\\ &\lambda'' = 0. \end{aligned} \tag{18}$$

This in turn gives

$$\lambda = \xi - t. \tag{19}$$

Substituting this value of the Lagrangian multiplier into the functional (17) gives the iteration formula

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t)\\ &+ \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 (\tilde{u}_n)(x,\xi)}{\partial x^2} + \tilde{u}_n(x,\xi) + \tilde{u}_n^2(x,\xi) - x\xi - x^2\xi^2\right)\mathrm{d}\xi, \quad n \geq 0. \end{aligned} \tag{20}$$

Considering the given initial values, we can select $u_0(x,t) = xt$. Using this selection in (20) we obtain the following successive approximations:

$$\begin{aligned} &u_0(x,t) = xt,\\ &u_1(x,t) = xt,\\ &u_2(x,t) = xt,\\ &u_3(x,t) = xt,\\ &\vdots,\\ &u_n(x,t) = xt. \end{aligned} \tag{21}$$

This gives the exact solution

$$u(x,t) = xt. \tag{22}$$

5. The wave-like equation

We now consider the wave-like equation [17]

$$\begin{aligned} &u_{tt} = \frac{x^2}{2} u_{xx}, \quad 0 < x < 1, t > 0,\\ &\text{B.C } u(0,t) = 0, \qquad u(1,t) = \sinh t,\\ &\text{I.C } u(x,0) = 0, \qquad u_t(x,0) = x^2. \end{aligned} \tag{23}$$

The correction functional for (23) is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{x^2}{2}\frac{\partial^2 (\tilde{u}_n)(x,\xi)}{\partial x^2}\right)\mathrm{d}\xi. \tag{24}$$

This yields the stationary conditions

$$\begin{aligned} &1-\lambda' = 0,\\ &\lambda(\xi = t) = 0,\\ &\lambda'' = 0, \end{aligned} \tag{25}$$

so that

$$\lambda = \xi - t. \tag{26}$$

Substituting this value of the Lagrangian multiplier into the functional (24) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{x^2}{2}\frac{\partial^2(\tilde{u}_n)(x,\xi)}{\partial x^2}\right)\mathrm{d}\xi, \quad n \geq 0. \tag{27}$$

Considering the given initial values, we can select $u_0(x,t) = x^2 t$. Using this selection in (27) yields the following successive approximations:

$$\begin{aligned}
u_0(x,t) &= x^2 t,\\
u_1(x,t) &= x^2 t + x^2\frac{t^3}{3!},\\
u_2(x,t) &= x^2 t + x^2\frac{t^3}{3!} + x^2\frac{t^5}{5!},\\
u_3(x,t) &= x^2 t + x^2\frac{t^3}{3!} + x^2\frac{t^5}{5!} + x^2\frac{t^7}{7!},\\
&\vdots,\\
u_n(x,t) &= x^2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots\right),
\end{aligned} \tag{28}$$

and using the fact that

$$u = \lim_{n\to\infty} u_n, \tag{29}$$

that leads to the exact solution

$$u(x,t) = x^2 \sinh t. \tag{30}$$

6. The wave equation in unbounded domain

We finally study the wave equation in an unbounded domain [17]

$$\begin{aligned}
&u_{tt} = u_{xx}, \quad -\infty < x < \infty, t > 0,\\
&\text{I.C } u(x,0) = \sin x, \qquad u_t(x,0) = 0.
\end{aligned} \tag{31}$$

It is to be noted that this equation is usually solved by the D'Alembert method. Because of the unbounded domain, boundary conditions are not given. The correction functional for (31) is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2(\tilde{u}_n)(x,\xi)}{\partial x^2}\right)\mathrm{d}\xi. \tag{32}$$

This yields the stationary conditions

$$\begin{aligned}
&1 - \lambda' = 0,\\
&\lambda(\xi = t) = 0,\\
&\lambda'' = 0,
\end{aligned} \tag{33}$$

so that

$$\lambda = \xi - t. \tag{34}$$

Substituting this value of the Lagrangian multiplier into the functional (32) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2(\tilde{u}_n)(x,\xi)}{\partial x^2}\right)\mathrm{d}\xi, \quad n \geq 0. \tag{35}$$

Considering the given initial values, we can select $u_0(x,t) = \sin x$. Using this selection in (35) we obtain the following successive approximations:

$$\begin{aligned}
u_0(x,t) &= \sin x, \\
u_1(x,t) &= \sin x - \frac{t^2}{2!}\sin x, \\
u_2(x,t) &= \sin x - \frac{t^2}{2!}\sin x + \frac{t^4}{4!}\sin x, \\
u_3(x,t) &= \sin x - \frac{t^2}{2!}\sin x + \frac{t^4}{4!}\sin x - \frac{t^6}{6!}\sin x, \\
&\vdots, \\
u_n(x,t) &= \sin x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right),
\end{aligned} \tag{36}$$

and using the fact that

$$u = \lim_{n\to\infty} u_n, \tag{37}$$

that leads to the exact solution

$$u(x,t) = \sin x \cos t. \tag{38}$$

7. Discussion

There are two main goals that we aimed for this work. The first is employing the powerful variational iteration to investigate four physical wave models. The second is showing the power of this method and its significant features. The two goals are achieved.

It is obvious that the method gives rapidly convergent successive approximations without any restrictive assumptions or transformation that may change the physical behavior of the problem. He's variational iteration method gives several successive approximations through using the iteration of the correction functional. Moreover, the VIM reduces the size of calculations by not requiring the tedious Adomian polynomials, and hence the iteration is direct and straightforward. The VIM uses the initial values for selecting the zeroth approximation, and boundary conditions, when given for bounded domains, can be used for justification only.

For nonlinear equations that arise frequently in expressing nonlinear phenomena, He's variational iteration method facilitates the computational work and gives the solution rapidly as compared with the Adomian method. For concrete problems where an exact solution does not exist, a few approximations can be used for numerical purposes. The VIM method is reliable and promising.

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