Discrete Mathematics 25 (1979) 89–90.

NOTE
ON COGRAPHIC REGULAR MATROIDS

James G. OXLEY

*Mathematical Institute, 24–29 St. Giles, Oxford OX1 3LB, England.**

Received 16 August 1977

The purpose of this note is to prove the following result.

Theorem 1. *A regular matroid is cographic if and only if it has no parallel minor isomorphic to $M(K_5)$ and no series minor isomorphic to $M(K_{3,3})$.*

The matroid terminology used here will in general follow Welsh [4]. In particular a matroid will be called *cographic* if it is isomorphic to the cocycle (bond) matroid of a graph. If M is a matroid on a set S and $T \subseteq S$, we shall denote the *restriction* and *contraction* of M to $S \setminus T$ by $M \setminus T$ and M/T respectively. The matroid N is a *parallel minor* of M if $N \cong M/U \setminus T$ where each element of T is parallel to an element of M/U. If N^* is a parallel minor of M^*, then N is a *series minor* of M.

The following result is due to Tutte [3, 9.42].

Theorem 2. *A regular matroid is cographic if and only if it has no minor isomorphic to $M(K_5)$ or $M(K_{3,3})$.*

Bixby [1, Theorem 5.1] showed that one can replace "minor" by "series minor" in the statement of this theorem. Bixby [1, Theorem 6.1] also characterized which regular matroids are cographic by a list of five forbidden *parallel* minors which includes $M(K_5)$ and $M(K_{3,3})$. Theorem 1 is motivated by Bixby's results though it is considerably easier to prove.

If M is a matroid on the set S and $e \in S$, then we call M a *non-trivial* single element extension of $M \setminus e$ if e is not a component of M and e is not parallel to any element of $S \setminus e$.

Lemma 3. *Neither $M^*(K_{3,3})$ nor $M(K_5)$ has a non-trivial single element extension which is also regular.*

Proof. Since every vertex of $K_{3,3}$ has degree 3, $M^*(K_{3,3})$ has no non-trivial single element extension which is also cographic. Thus if N is a regular non-trivial single

* Present address: Mathematics Department, Australian National University, Canberra, Australia.

element extension of $M^*(K_{3,3})$, then N has rank 4 and, by Theorem 2, N contains $M(K_5)$ as a minor. That is, $M^*(K_{3,3})$ is a restriction of $M(K_5)$; a contradiction.

It is known (J.H. Mason, private communication, 1977) that, for all positive integers n, $M(K_n)$ has no regular non-trivial single element extension. For completeness we indicate a short proof of this for $n = 5$. Using the fact that $M(K_5)$ is isomorphic to the Euclidean matroid of the rank 4 Desargues configuration (see, for example, [2, p.141]), it is easy to show that every non-trivial single element extension of $M(K_5)$ has a seven-point plane as a minor. However, the only binary seven-point plane is the Fano matroid and this is not regular (see, for example, [4, Theorem 10.4.1]).

Proof of Theorem 1. Let M be a regular matroid which is not cographic. Then, by Theorem 2, M has a minor isomorphic to $M(K_5)$ or $M(K_{3,3})$. Suppose the former and choose $U \subseteq S$ such that $M/U \backslash T \cong M(K_5)$ and no non-empty subset T' of T satisfies $M/U \cup T') \backslash (T \backslash T') \cong M(K_5)$. Then by Lemma 3 and the choice of U, every element of T is parallel to some element of M/U. That is, $M(K_5)$ is a parallel minor of M.

If M has $M(K_{3,3})$ as a minor, then choose a subset T of S such that $M \backslash T/U \cong M(K_{3,3})$ and no non-empty subset U' of U satisfies $M \backslash (T \cup U')/(U \backslash U') \cong M(K_{3,3})$. Then $M^*(K_{3,3}) \cong M^*/T \backslash U$. Again by Lemma 3 and the choice of T, each element of U is parallel to an element of M^*/T. That is, $M(K_{3,3})$ is a series minor of M.

The converse follows by Theorem 2.

Acknowledgement

The author thanks CSIRO (Australia) for their generous financial support.

References

[1] R.E. Bixby, Kuratowski's and Wagner's theorems for matroids, J. Combinatorial Theory 22(B) (1977) 31–53.

[2] J.H. Mason, Matroids as the study of geometrical configurations, in: M. Aigner, ed., Higher Combinatorics (D. Reidel, Dordrecht, 1977) 133–176.

[3] W.T. Tutte, Lectures on matroids, J. Res. Nat. Bur. Standards Sect. B 69B (1965) 1–47.

[4] D.J.A. Welsh, Matroid Theory, London Math. Soc. Monographs No. 8, (Academic Press, New York, 1976).