Diagonally convex directed polyominoes and even trees: a bijection and related issues

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Abstract

We present a simple bijection between diagonally convex directed (DCD) polyominoes with $n$ diagonals and plane trees with $2n$ edges in which every vertex has even degree (even trees), which specializes to a bijection between parallelogram polyominoes and full binary trees. Next we consider a natural definition of symmetry for DCD-polyominoes, even trees, ternary trees, and non-crossing trees, and show that the number of symmetric objects of a given size is the same in all four cases.

Résumé

On présente une bijection simple entre polyominos dirigés diagonalement convexes (DCD) avec $n$ diagonales et arbres planaires avec $2n$ arêtes dont les sommets sont de degré pair (arbres pairs); par restriction on obtient une bijection entre polyominos parallélogrammes et arbres binaires. On considère aussi une définition naturelle de symétrie parmi les polyominos DCD, arbres pairs, arbres ternaires et arbres sans croisement, et on prouve que le nombre d’objets symétriques d’une taille donnée est le même dans les quatre cas. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The Catalan numbers (M1459)\(^2\), defined by \(1/(n+1)\binom{2n}{n}\) for \(n=0,1,2,\ldots\), are probably the most frequently occurring combinatorial numbers after the binomial coefficients. For example, in [17] there are given 66 combinatorial structures that are counted by these numbers.

The next sequence in the hierarchy of generalized Catalan numbers is \(1,1,3,12,55,273,1428,\ldots\), defined by \(1/(2n+1)\binom{3n}{n}\), (M2926). In addition to the four instances that we will consider in more detail, the sequence counts (see Problems 5.45–47 in [17] for a discussion of several of these families):

(i) paths in the \((x,y)\) plane from \((0,0)\) to \((2n,n)\) with steps \((1,0)\) and \((0,1)\), that never pass above the line \(y=\frac{1}{2}x\) [11];
(ii) paths in the \((x,y)\) plane from \((0,0)\) to \((3n,0)\) with steps \((1,1)\) and \((1,-2)\), that never pass below the \(x\)-axis;
(iii) certain two-line arrays (see [3]);
(iv) tree-involutions (see [7]);
(v) ways of decomposing the circular permutation \((1,2,\ldots,n+1)\) into a product of \(n-1\) transpositions (see [7,8]);
(vi) ways of associating \(n\) applications of a given nonassociative ternary operation (see [4,11]);
(vii) ways of dissecting a convex \(2n+2\)-gon into \(n\) quadrilaterals by drawing \(n-1\) diagonals, no two of which intersect in its interior (see [11]).

Next we mention the four families of objects that we will consider.

Ternary trees. These are ordered trees in which each node has outdegree 0 or 3 (see, for instance, [15, p. 127]). The size of a ternary tree is defined to be the number of its internal nodes. In Fig. 1a we show a ternary tree of size 4. In all drawings of trees throughout the paper, the root is always the topmost vertex.

Non-crossing trees. These are trees drawn on the plane having as vertices the vertices of a convex polygon and whose edges do not cross (see [6,10,13]). We consider them as being rooted, i.e., one vertex is distinguished. The size of a non-crossing tree is

\(^2\)These numbers are identifiers of the sequences in The Encyclopedia of Integer Sequences [16].
defined to be the number of its edges. In Fig. 1b we show a non-crossing tree of size 5.

Even trees. These are ordered trees in which each node has even outdegree (see [18]). The size of an even tree is defined to be half of the number of edges. In Fig. 1c we show an even tree of size 6.

In each of the above three examples, making use of the simple way in which set operations on combinatorial objects translate into operations on their counting generating functions (see, for example [15]), one easily sees that the generating function $g = g(z)$ satisfies the equation

$$g = 1 + zg^3,$$

where $z$ marks number of internal nodes, number of edges, and number of edge pairs, respectively. From this equation, applying the Lagrange inversion formula, a straightforward computation gives

$$g(z) = \sum_{n=0}^{\infty} \frac{1}{2n+1} \left( \binom{3n}{n} \right) z^n$$

so that in each case the number of objects of size $n$ is equal to $1/(2n+1)(\binom{3n}{n})$. We also obtain

$$g(z)^2 = \sum_{n=0}^{\infty} \frac{1}{n+1} \left( \binom{3n+1}{n} \right) z^n,$$

which will be needed later.

Diagonally convex directed polyominoes (DCD-polyominoes). A polyomino is a finite connected union of elementary cells (i.e. squares $[i,i+1] \times [j,j+1] \subseteq \mathbb{R} \times \mathbb{R}$ with $i$ and $j$ integers), whose interior is also connected. Polyominoes are defined up to translation. A polyomino $P$ is said to be directed if every cell of $P$ can be reached from a distinguished cell, called the root, by a path which is contained in $P$ and uses only north and east steps. We will consider that the cell $[0,1] \times [0,1]$ is the root. A directed polyomino is said to be diagonally convex if all cells with centers on a line of slope $-1$ form a continuous chain of diagonal neighbors. These continuous chains of cells are the diagonals of the DCD-polyomino. The size of a DCD-polyomino is defined to be the number of its diagonals (we remark that the most usual way of counting polyominoes is according to area or perimeter).

DCD-polyominoes have been investigated in [1,2,5,9,14]. For a diagonal $D$ of a DCD-polyomino we define its shadow as the union of the cells that are either upper or right neighbors of the cells of $D$ (they extend diagonally south-east from the upper neighbor of the top cell of $D$ to the right neighbor of the bottom cell of $D$). Let $D_1, D_2, \ldots, D_n$ be the diagonals of a DCD-polyomino of size $n$, numbered from the root upwards in the NE direction. Clearly, $D_1$ consists only of one cell (the root). Due to the fact that the polyomino is directed, each $D_i$ $(2 \leq i \leq n)$ lies in the shadow of $D_{i-1}$. A DCD-polyomino of size $n$ can be coded by two sequences $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$, each consisting of $n$ non-negative integers. The number $a_i$ $(b_i)$ is called
the \(i\)th upper loss (lower loss) and is defined, for \(i \in \{2, 3, \ldots, n\}\), as the number of cells in the shadow of \(D_{i-1}\) and situated above (below) the cells of \(D_i\), while \(a_1 = b_1 = 0\).

The encoding of DCD-polyominoes by these numbers has been introduced in [9].

To illustrate the above defined concepts, we consider the DCD-polyomino \(P\) of Fig. 2 (its contour is drawn with a heavy line). \(P\) has size 7, the cells of the seven diagonals being numbered accordingly. The shaded cells form the shadow of the fourth diagonal.

The upper losses \(a_i\), the lower losses \(b_i\), and the diagonal lengths \(d_i\) of \(P\) are given in Table 1.

The rest of the paper is organized as follows. In Section 2 we establish a simple bijection between DCD-polyominoes of size \(n\) and even trees of size \(n\). This bijection has several interesting features that we discuss in detail, in particular the fact that it specializes to a bijection between parallelogram polyominoes and full binary trees. After carefully checking the literature, it appears that this bijection is not the composition of known bijections between parallelogram polyominoes and Dyck words, and known bijections between Dyck words and full binary trees.

In Section 3 we consider a natural definition of symmetry for the four families of objects under study, and show that the number of symmetric objects of size \(n\) is the same in all four cases. We remark that symmetry classes of polyominoes have been considered elsewhere [12].

2. A bijection between DCD-polyominoes and even trees

Bijections between DCD-polyominoes and other combinatorial structures have been defined before. More precisely, rather complex bijections between DCD-polyominoes and ternary trees are given in [5, 14] and a bijection between DCD-polyominoes with \(n\) diagonals and paths in the \((x, y)\) plane from \((0, 0)\) to \((2n, n)\) with steps \((1, 0)\) and \((0, 1)\),
and never passing above the line \( y = \frac{1}{2}x \) is given in [9]. This latter, by composition, yields another bijection between DCD-polyominoes and ternary trees.

In this section we present a new simple bijection between DCD-polyominoes of size \( n \) and even trees of size \( n \). This bijection, as will be seen later, has the desired feature of commuting with naturally defined symmetries on DCD-polyominoes and on even trees. In addition (i) under this bijection, the statistics “one-half of the perimeter of a DCD-polyomino” and “number of leaves of an even tree” correspond to each other, and (ii) the restriction of this bijection to parallelogram polyominoes is a bijection between the latter and full binary trees.

Consider a DCD-polyomino of size \( n \), with upper losses \( a_1, a_2, \ldots, a_n \), lower losses \( b_1, b_2, \ldots, b_n \), and diagonal lengths

\[
d_i = i - \sum_{j=1}^{i} (a_j + b_j) > 0 \quad (i = 1, 2, \ldots, n) \tag{3}
\]

(see [9], Proposition 1). It is immediate that

\[
d_{i-1} = d_i + a_i + b_i - 1 \quad (i = 1, 2, \ldots, n; \quad d_0 = 0). \tag{4}
\]

The even tree associated to this DCD-polyomino is constructed in the following manner. We start with a root of degree \( 2d_n \), i.e. we hang \( 2d_n \) leaves on the root. We label the leftmost leaf by \( A_1 \), the rightmost one by \( B_1 \), and we hang on them \( 2a_n \) and \( 2b_n \) leaves, respectively. We have now \( 2d_n + 2a_n + 2b_n - 2 = 2d_{n-1} \) unlabeled leaves. Note that \( 2d_{n-1} \geq 2 \) if \( n > 1 \). We traverse the tree in left (right) preorder \(^3\) and we label the first unlabeled leaf by \( A_2 \) \((B_2)\). On \( A_2 \) and \( B_2 \) we hang \( 2a_{n-1} \) \( 2b_{n-1} \) leaves, respectively. Now we have \( 2d_{n-1} + 2a_{n-1} + 2b_{n-1} - 2 = 2d_{n-2} \) unlabeled leaves. Again \( 2d_{n-2} \geq 2 \) if \( n > 2 \). We continue in this manner and after the \( n \)th labeling (with \( A_n \) and \( B_n \)) we will have no unlabeled leaves \((2d_1 + 2a_1 + 2b_1 - 2 = 0)\). The obtained even tree is defined to be the image of the given DCD-polyomino. Making use of (3), with \( i = n \), one can see easily that the even tree has \( 2d_n + 2 \sum_{j=1}^{n} (a_j + b_j) = 2n \) edges, i.e. its size is \( n \). In Fig. 3 we show the successive steps for obtaining the even tree corresponding to the DCD-polyomino of Fig. 2.

To obtain the inverse mapping, it is easy to retrace the steps of the above construction. Namely, omitting the root of a given even tree, we label its nodes alternatively in left and right preorder, by \( A_1, B_1, A_2, B_2, \ldots, A_n, B_n \). For each \( i \in \{1, 2, \ldots, n\} \) we define

\[
a_i = \frac{1}{2} \text{outdegree of } A_{n+1-i}, \quad b_i = \frac{1}{2} \text{outdegree of } B_{n+1-i}.
\]

Now, the inverse image of the given even tree is the DCD-polyomino defined by the sequences of upper losses \( a_1, a_2, \ldots, a_n \) and lower losses \( b_1, b_2, \ldots, b_n \).

Remark. There is an alternative procedure producing the same bijection which reads the losses from left to right instead of from right to left. Given a DCD-polyomino of size \( n \) with losses \( a_i \) and \( b_i \) as before, construct inductively a sequence of even

\(^3\) The usual preorder traversal is called here left preorder traversal and then, right preorder traversal has the obvious meaning.
trees $T_1, T_2, \ldots, T_n$ as follows. $T_1$ is the even tree made up of a root with exactly two children. Suppose we have constructed $T_{i-1}$ with root $R$. Add two new children $A_i$ and $B_i$ to $R$ being, respectively, the leftmost and the rightmost children. Then remove the leftmost $2a_i$ (the rightmost $2b_i$) subtrees of $T_{i-1}$ together with the edges connecting them to $R$, and hang them on $A_i$ (on $B_i$) (see Fig. 4). The resulting tree is $T_i$. When $i = n$ we are finished.

It can be checked that this procedure gives the same even tree from a given DCD-polyomino as the previous method.
Next we show that the semiperimeter of a DCD-polyomino is equal to the number of leaves of the corresponding even tree. It is known (Proposition 1 in [9]) and easy to prove that the semiperimeter of a DCD-polyomino is equal to the number of its zero losses. However, from the definition of our bijection it follows that each zero loss of a DCD-polyomino contributes a leaf to the corresponding even tree.

Another statistics correspondence follows at once from the definition of the bijection. Namely, the length of the last diagonal of a DCD-polyomino (called “pointure” in [14]) is equal to one-half of the root degree of the corresponding even tree.

Finally, a parallelogram polyomino is a polyomino bounded by two lattice paths which start together and end together but never meet in between. Equivalently, in a way more convenient for our purpose, a parallelogram polyomino is a DCD-polyomino such that (i) all its losses are equal to 0 or 1 and (ii) the length of the last diagonal is 1. Now, given a parallelogram polyomino, from the definition of the bijection it follows that the corresponding even tree is a full binary tree (since we keep hanging either 0 or 2 leaves on the existing nodes). Hence, we obtain a bijection between parallelogram polyominoes of semiperimeter \(n+1\) and full binary trees with \(n\) internal nodes.

3. Symmetry

In each of the four families of combinatorial objects that we investigate, we can consider a natural definition of symmetry.

Ternary trees. For a ternary tree \(\tau\), we define its reflection \(\tau^r\) by interchanging recursively left and right children at every node. Then we say that \(\tau\) is symmetric if \(\tau = \tau^r\) (see Fig. 5a).

Non-crossing trees. For a non-crossing tree \(T\), we define its reflection \(T^r\) with respect to a bisector through the root (here we assume \(T\) is constructed on the vertices of a regular polygon). Then we say that \(T\) is symmetric if \(T^r = T\) (see Fig. 5b).

Even trees. Reflection and symmetry are defined as for ternary trees (see Fig. 5c for a symmetric even tree).

DCD-polyominoes. For a DCD-polyomino \(P\), we define its reflection \(P^r\) with respect to the bisector of the first quadrant. Then we say that \(P\) is symmetric if \(P^r = P\) (see Fig. 5d).

Theorem 1. The number of symmetric objects of size \(n\) is the same in all cases. Moreover, it is given by

\[
s_n = \begin{cases} 
\frac{1}{n+1} \binom{3n/2}{n/2} & \text{if } n \text{ is even,} \\
\frac{2}{n+1} \binom{(3n-1)/2}{(n-1)/2} & \text{if } n \text{ is odd.}
\end{cases}
\]  

(5)

By a lattice path we mean a path with steps \((1,0)\) and \((0,1)\).
This can be shown by using generating functions. If $s_n$ is the number of symmetric objects of size $n$, in each case one shows that

$$S(z) = \sum_{n=0}^{\infty} s_n z^n = g(z^2) + z g(z^2)^2,$$

where

$$g(z) = 1 + z g(z^3).$$

For instance, in the case of ternary trees, let $\tau_1, \tau_2, \tau_3$ be the left, middle and right ternary trees attached to the root of a ternary tree $\tau$. If $\tau$ is symmetric, then $\tau_2$ is symmetric, $\tau_1$ is any ternary tree, and $\tau_3$ is its reflection. Hence

$$S(z) = 1 + z S(z) g(z^2),$$

from where

$$S(z) = \frac{1}{1 - z g(z^2)} = g(z^2) + z g(z^2)^2.$$

Now, making use of relations (1) and (2), we obtain easily the expressions in (5). The sequence defined by (5) is obtained by alternating the terms of the sequences M2926 and M1782, while the latter is the convolution of the former with itself.
However, instead of the individual enumeration of the symmetric objects in each of the four cases, it seems more interesting to define symmetry-preserving bijections between the four families of the considered combinatorial objects.

A bijection $P \rightarrow P'$ between DCD-polyominoes and even trees has been defined in Section 2. This bijection preserves symmetry since $(P')^r = (P')^r$. Indeed, the upper (lower) losses of $P$ are the lower (upper) losses of $P'$ and then, by the definition of the bijection, the even tree $(P')^r$ is the reflection of the even tree $P'$.

**Remark.** Here we would like to mention that for DCD-polyominoes the first part of formula (5) follows also bijectively. Namely, to a DCD-polyomino of size $n$ with upper losses $(a_1, a_2, \ldots, a_n)$ and lower losses $(b_1, b_2, \ldots, b_n)$ we associate the DCD-polyomino of size $2n$ having $(a_1, b_1, a_2, b_2, \ldots, a_n, b_n)$ as upper and lower losses. It is easy to see that these two identical sequences define a symmetric DCD-polyomino and that the mapping can be reversed.

**A bijection between ternary trees and even trees.** To the empty ternary tree we make correspond the empty even tree. Given a ternary tree $\tau$ with $\tau_1, \tau_2, \tau_3$ as before, define recursively an even tree $\tau'$ as indicated in Fig. 6.

It follows easily by induction that through this bijection (i) the size is preserved, (ii) the numbers of left and right leaves of a ternary tree sum up to the number of leaves of the corresponding even tree, and (iii) the length of the central path of the ternary tree is equal to half of the degree of the root of the corresponding even tree. In addition, symmetry is preserved since, clearly, $(\tau')^r = (\tau')^r$.

**A bijection between even trees and non-crossing trees.** We consider a non-crossing tree as an ordered sequence of butterflies attached to the root, where a butterfly is an ordered pair of non-crossing trees (the wings) with the roots identified. To the empty even tree we make correspond the empty non-crossing tree. Let $\epsilon$ be an even tree, and let $\epsilon_1, \epsilon_2, \ldots, \epsilon_{2d}$ be the $2d$ subtrees attached to its root. Then we define recursively a non-crossing tree $\epsilon'$ as indicated in Fig. 7.

It follows easily by induction that: (i) the size is preserved, (ii) the number of leaves of the even tree is equal to the number of empty wings of the corresponding non-crossing tree, and (iii) half of the degree of the root of an even tree is equal to the degree of the root of the corresponding non-crossing tree. In addition, symmetry is preserved since, clearly, $(\epsilon')^r = (\epsilon')^r$.

The reader can check that the objects in Fig. 5 correspond to each other under the above bijections.
Remark. From (5) it follows that we have just found four more manifestations of the sequence $1/(2n + 1)\left(\frac{3n}{n}\right)$: (i) symmetric ternary trees with $2n$ internal nodes; (ii) symmetric non-crossing trees with $2n$ edges; (iii) symmetric even trees with $4n$ edges; (iv) symmetric DCD-polyominoes with $2n$ diagonals.

References