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Extremal problems on triangle areas in two and three dimensions ☆

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ABSTRACT

The study of extremal problems on triangle areas was initiated in a series of papers by Erdős and Purdy in the early 1970s. In this paper we present new results on such problems, concerning the number of triangles of the same area that are spanned by finite point sets in the plane and in 3-space, and the number of distinct areas determined by the triangles.

In the plane, our main result is an $O(n^{44/19}) = O(n^{2.3158})$ upper bound on the number of unit-area triangles spanned by n points, which is the first breakthrough improving the classical bound of $O(n^{7/3})$ from 1992. We also make progress in a number of important special cases. We show that: (i) For points in convex position, there exist n-element point sets that span $\Omega(n \log n)$ triangles of unit area. (ii) The number of triangles of minimum (nonzero) area determined by n points is at most $\frac{2}{3}(n^2 - n)$; there exist n-element point sets (for arbitrarily large n) that span $(6/\pi^2 - o(1))n^2$ minimum-area triangles. (iii) The number of acute triangles of minimum area determined by n points is $O(n)$; this is asymptotically tight. (iv) For n points in convex position, the number of triangles of minimum area is $O(n)$; this is asymptotically tight. (v) If no three points are allowed to be collinear, there are n-element point sets that span $\Omega(n \log n)$

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minimum-area triangles (in contrast to (ii), where collinearities are allowed and a quadratic lower bound holds).
In 3-space we prove an $O(n^{17/7}\beta(n)) = O(n^{2.4286})$ upper bound on the number of unit-area triangles spanned by n points, where $\beta(n)$ is an extremely slowly growing function related to the inverse Ackermann function. The best previous bound, $O(n^{8/3})$, is an old result of Erdős and Purdy from 1971. We further show, for point sets in 3-space: (i) The number of minimum nonzero area triangles is at most $n^2 + O(n)$, and this is worst-case optimal, up to a constant factor. (ii) There are n-element point sets that span $\Omega(n^{4/3})$ triangles of maximum area, all incident to a common point. In any n-element point set, the maximum number of maximum-area triangles incident to a common point is $O(n^{4/3+\varepsilon})$, for any $\varepsilon > 0$. (iii) Every set of n points, not all on a line, determines at least $\Omega(n^{2/3}/\beta(n))$ triangles of distinct areas, which share a common side.

1. Introduction

Given n points in the plane, consider the following equivalence relation defined on the set of (nondegenerate) triangles spanned by the points: two triangles are *equivalent* if they have the same area. Extremal problems typically ask for the maximum cardinality of an equivalence class, and for the minimum number of distinct equivalence classes, in a variety of cases. A classical example is when we call two segments spanned by the given points equivalent if they have the same length. Bounding the maximum size of an equivalence class is the famous *repeated distances* problem [8,18, 35,36], and bounding the minimum number of distinct classes is the equally famous *distinct distances* problem [8,18,25,34,36,38]. In this paper, we make progress on several old extremal problems on triangle areas in two and in three dimensions. We also study some new and interesting variants never considered before. Our proof techniques draw from a broad range of combinatorial tools such as the Szemerédi–Trotter theorem on point-line incidences [37], the Crossing Lemma [5,27], incidences between curves and points and tangencies between curves and lines, extremal graph theory [26], quasi-planar graphs [3], Minkowski-type constructions, repeated distances on the sphere [29], the partition technique of Clarkson et al. [13], various charging schemes, etc.

In 1967, A. Oppenheim (see [21]) asked the following question: Given n points in the plane and $A > 0$, how many triangles spanned by the points can have area A? By applying an affine transformation, one may assume $A = 1$ and count the triangles of *unit* area. Erdős and Purdy [19] showed that a $\sqrt{\log n} \times (n/\sqrt{\log n})$ section of the integer lattice determines $\Omega(n^2 \log\log n)$ triangles of the same area. They also showed that the maximum number of such triangles is at most $O(n^{5/2})$. In 1992, Pach and Sharir [30] improved the exponent and obtained an $O(n^{7/3})$ upper bound using the Szemerédi–Trotter theorem [37] on the number of point-line incidences. We further improve the upper bound by estimating the number of incidences between the points and a 4-parameter family of quadratic curves. We show that n points in the plane determine at most $O(n^{44/19}) = O(n^{2.3158})$ unit-area triangles. We also consider the case of points in convex position, for which we construct n-element point sets that span $\Omega(n \log n)$ triangles of unit area.

Braß, Rote, and Swanepoel [9] showed that n points in the plane determine at most $O(n^2)$ minimum-area triangles, and they pointed out that this bound is asymptotically tight. We introduce a simple charging scheme to first bring the upper bound down to $n^2 - n$ and then further to $\frac{2}{3}(n^2 - n)$. Our charging scheme is also instrumental in showing that a $\sqrt{n} \times \sqrt{n}$ section of the integer lattice spans $(6/\pi^2 - o(1))n^2$ triangles of minimum area. In the lower bound constructions, there are many collinear triples and most of the minimum-area triangles are obtuse. We show that there are at most $O(n)$ *acute* triangles of minimum (nonzero) area, for any n-element point set. Also, we show that n points in (strictly) convex position determine at most $O(n)$ minimum-area triangles—these bounds are best possible apart from the constant factors. If no three points are allowed to be collinear, we construct n-element point sets that span $\Omega(n\log n)$ triangles of minimum area.

Next we address analogous questions for triangles in 3-space. The number of triangles with some extremal property might go up (significantly) when one moves up one dimension. For instance, Braß, Rote, and Swanepoel [9] have shown that the number of maximum area triangles in the plane is at most n (which is tight). In 3-space we show that this number is at least $\Omega(n^{4/3})$ in the worst case. In contrast, for minimum-area triangles, we prove that the quadratic upper bound from the planar case remains in effect for 3-space, with a different constant of proportionality.

As mentioned earlier, Erdős and Purdy [19] showed that a suitable n-element section of the integer lattice determines $\Omega(n^2\log\log n)$ triangles of the same area. Clearly, this bound is also valid in 3-space. In the same paper, via a forbidden graph argument applied to the incidence graph between points and cylinders whose axes pass through the origin, Erdős and Purdy deduced an $O(n^{5/3})$ upper bound on the number of unit-area triangles incident to a common point, and thereby an $O(n^{8/3})$ upper bound on the number of unit-area triangles determined by n points in 3-space. Here, applying a careful (and somewhat involved) analysis of the structure of point-cylinder incidences in $\mathbb{R}^3$, we prove a new upper bound of $O(n^{17/7}\beta(n)) = O(n^{2.4286})$, for $\beta(n) = \exp(\alpha(n)^{O(1)})$, where $\alpha(n)$ is the extremely slowly growing inverse Ackermann function.

It is conjectured [8,10,22] that n points in $\mathbb{R}^3$, not all on a line, determine at least $\lfloor (n-1)/2\rfloor$ distinct triangle areas. This bound has recently been established in the plane [32], but the question is still wide open in $\mathbb{R}^3$. It is attained by n equally spaced points distributed evenly on two parallel lines (which is in fact a planar construction). We obtain a first result on this question and show that n points in $\mathbb{R}^3$, not all on a line, determine at least $n^{2/3}\exp(-\alpha(n)^{O(1)}) = \Omega(n^{.666})$ triangles of distinct areas. Moreover, all these triangles share a common side.

2. Unit-area triangles in the plane

The general case. We establish a new upper bound on the maximum number of unit-area triangles determined by n points the plane.

Theorem 1. *The number of unit-area triangles spanned by n points in the plane is $O(n^{2+6/19}) = O(n^{2.3158})$.*

Proof. Let S be a set of n points in the plane. Consider a triangle Δabc spanned by S. We call the three lines containing the three sides of Δabc, *base lines* of Δ, and the three lines parallel to the base lines and incident to the third vertex, *top lines* of Δ. For a parameter k, $1 \leqslant k \leqslant \sqrt{n}$, to be optimized later, we partition the set of unit-area triangles as follows.

- U_1 denotes the set of unit-area triangles where one of the top lines is incident to fewer than k points of S.
- U_2 denotes the set of unit-area triangles where all three top lines are *k-rich* (i.e., each contains at least k points of S).

We derive different upper bounds for each of these types of unit-area triangles.

Bound for $|U_1|$. For any two distinct points, $a, b \in \mathbb{R}^2$, let ℓ_{ab} denote the line through a and b. The points c for which the triangle Δabc has unit area lie on two lines ℓ_{ab}^-, ℓ_{ab}^+ parallel to ℓ_{ab} at distances $2/|ab|$ on either side of ℓ_{ab}. The $\binom{n}{2}$ segments determined by S generate at most $2\binom{n}{2}$ such lines (counted with multiplicity). If $\Delta abc \in U_1$ and its top line incident to the fewest points of S is $\ell'_{ab} \in \{\ell_{ab}^-, \ell_{ab}^+\}$, then ℓ'_{ab} is incident to at most k points, so the segment ab is the base of at most k triangles $\Delta abc \in U_1$ (with $c \in \ell'_{ab}$). This gives the upper bound

$$|U_1| \leqslant 2\binom{n}{2}\cdot k = O\big(n^2 k\big).$$

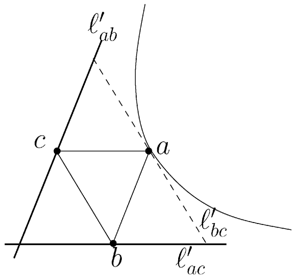

Fig. 1. One of the hyperbolas defined by the triangle Δabc.

Bound for $|U_2|$. Let L be the set of k-rich lines, and let $m = |L|$. By the Szemerédi–Trotter theorem [37], we have $m = O(n^2/k^3)$ for any $k \leqslant \sqrt{n}$. Furthermore, the cardinality of the set $I(S, L)$ of point-line incidences between S and L is $|I(S, L)| = O(n^2/k^2)$.

For any pair of nonparallel lines $\ell_1, \ell_2 \in L$, let $\gamma(\ell_1, \ell_2)$ denote the locus of points $p \in \mathbb{R}^2$, $p \notin \ell_1 \cup \ell_2$, such that the parallelogram that has a vertex at p and two sides along ℓ_1 and ℓ_2, respectively, has area 2. The set $\gamma(\ell_1, \ell_2)$ consists of two hyperbolas with ℓ_1 and ℓ_2 as asymptotes. See Fig. 1. For instance, if $\ell_1 : y = 0$ and $\ell_2 : y = ax$, then $\gamma(\ell_1, \ell_2) = \{(x, y) \in \mathbb{R}^2\colon\ xy = y^2/a + 2\} \cup \{(x, y) \in \mathbb{R}^2\colon\ xy = y^2/a - 2\}$. Any two nonparallel lines uniquely determine two such hyperbolas. Let Γ denote the set of these hyperbolas. Note that $|\Gamma| = O(m^2)$. The family of such hyperbolas for all pairs of nonparallel lines form a 4-parameter family of quadratic curves (where the parameters are the coefficients of the defining lines).

For any triangle $\Delta abc \in U_2$, any pair of its top lines, say, ℓ'_{ab} and ℓ'_{ac}, determine a hyperbola passing through a, which is incident to the third top line ℓ'_{bc}; furthermore ℓ'_{bc} is tangent[4] to the hyperbola at a. See Fig. 1. Any hyperbola in this 4-parameter family is uniquely determined by two incident points and the two respective tangent lines at those points.

We define a topological graph G as follows. For each point $p \in S$, which is incident to d_p lines of L, we create $2d_p$ vertices in G, as follows (refer to Fig. 2). Draw a circle $C_\varepsilon(p)$ centered at p with a sufficiently small radius $\varepsilon > 0$, and place a vertex at every intersection point of the circle $C_\varepsilon(p)$ with the d_p lines incident to p. The number of vertices is $v_G = 2|I(S, L)| = O(n^2/k^2)$. Next, we define the edges of G. For each connected branch γ of every hyperbola in Γ, consider the set $S(\gamma)$ of points $p \in S$ that are (i) incident to γ and (ii) some line of L is tangent to γ at p. For any two consecutive points $p, q \in S(\gamma)$, draw an edge along γ between the two vertices of G that (i) correspond to the incidences (p, ℓ_p) and (q, ℓ_q), where ℓ_p and ℓ_q are the tangents of γ at p and q, respectively, and (ii) are closest to each other along γ. Specifically, the edge follows γ between the circles $C_{2\varepsilon}(p)$ and $C_{2\varepsilon}(q)$ and follows straight line segments in the interiors of those circles. Choose $\varepsilon > 0$ sufficiently small so that the circles $C_{2\varepsilon}(p)$ have disjoint interiors and the portions of the hyperbolas in the interiors of the circles $C_{2\varepsilon}(p)$, for every $p \in S$, meet at p only. This guarantees that the edges of G cross only at intersection points of the hyperbolas. The graph G is *simple* because two points and two tangent lines uniquely determine a hyperbola in Γ. The number of edges is at least $3|U_2| - 2m^2$, since every triangle in U_2 corresponds to three point-hyperbola incidences in $I(S, \Gamma)$ (satisfying the additional condition of tangency with the respective top lines); and along each of the $2m^2$ hyperbola branches, each of its incidences with the points of S (of the special kind under consideration), except for one, contributes one edge to G.

Thus G is a simple topological graph with $v_G = 2I(S, L) = O(n^2/k^2)$ vertices and $e_G \geqslant 3|U_2| - 2m^2$ edges. Since in this drawing of G, every crossing is an intersection of two hyperbolas, the crossing

[4] For a quick proof, let $\mathbf{u}$ (resp., $\mathbf{v}$) be a unit vector along ℓ'_{ac} (resp., ℓ'_{ab}). The point a can be parametrized as $\mathbf{x} = t\mathbf{u} + \frac{\kappa}{t}\mathbf{v}$, where $\kappa = 2/\sin\theta$, and θ is the angle between ℓ'_{ac} and ℓ'_{ab}. Hence the tangent to the hyperbola at a is given by $\dot{\mathbf{x}} = \mathbf{u} - \frac{\kappa}{t^2}\mathbf{v} \parallel t\mathbf{u} - \frac{\kappa}{t}\mathbf{v} = \vec{cb}$.

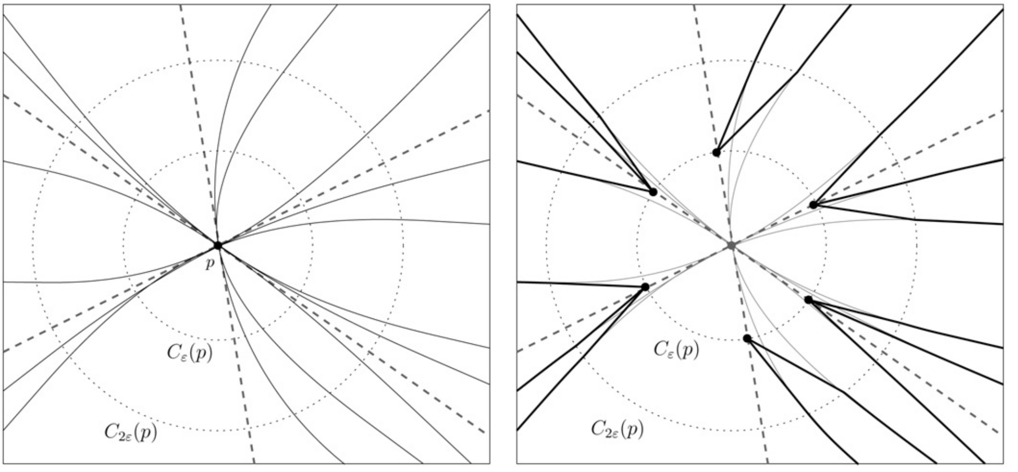

Fig. 2. (Left) A point $p \in S$ incident to three lines of L (dashed) and 8 hyperbolas, each tangent to one of those lines. (Right) The 6 vertices of G corresponding to the 3 point-line incidences at p, and the drawings of the edges along the hyperbolas.

number of G is upper bounded by $\operatorname{cr}(G) = O(|\Gamma|^2) = O(m^4)$. We can also bound the crossing number of G from below via the Crossing Lemma of Ajtai et al. [5] and Leighton [27]. It follows that

$$\Omega\left(\frac{e_G^3}{v_G^2}\right) - 4v_G \leqslant \operatorname{cr}(G) \leqslant O\left(m^4\right).$$

Rearranging this chain of inequalities, we obtain $e_G^3 = O(m^4 v_G^2 + v_G^3)$, or $e_G = O(m^{4/3} v_G^{2/3} + v_G)$. Comparing this bound with our lower bound $e_G \geqslant 3|U_2| - 2m^2$, we have $|U_2| = O(m^{4/3} v_G^{2/3} + v_G + m^2)$. Hence, for $k \leqslant \sqrt{n}$, we have

$$|U_2| = O\left(\left(\frac{n^2}{k^3}\right)^{4/3}\left(\frac{n^2}{k^2}\right)^{2/3} + \frac{n^2}{k^2} + \left(\frac{n^2}{k^3}\right)^2\right) = O\left(\frac{n^4}{k^{16/3}} + \frac{n^2}{k^2}\right) = O\left(\frac{n^4}{k^{16/3}}\right).$$

The total number of unit-area triangles is $|U_1| + |U_2| = O(n^2 k + n^4/k^{16/3})$. This expression is minimized for $k = n^{6/19}$, and we get $|U_1| + |U_2| = O(n^{44/19})$. $\square$

2.1. Convex position

The construction of Erdős and Purdy [19] with many triangles of the same area, the $\sqrt{\log n} \times (n/\sqrt{\log n})$ section of the integer lattice, also contains many collinear triples. Here we consider the unit-area triangle problem in the special case of point sets in strictly convex position, so no three points are collinear. We show that n points in convex position in the plane can determine a superlinear number of unit-area triangles. On the other hand, we do not know of any subquadratic upper bound.

Theorem 2. *For all $n \geqslant 3$, there exist n-element point sets in convex position in the plane that span $\Omega(n \log n)$ unit-area triangles.*

Proof. We recursively construct a set S_i of $n_i = 3^i$ points on the unit circle that determine $t_i = i3^{i-1}$ unit-area triangles, for $i = 1, 2, \ldots$. Take a circle C of unit radius centered at the origin o. We start with a set S_1 of 3 points along the circle forming a unit-area triangle, so we have $n_1 = 3$ points and $t_1 = 1$ unit-area triangles. In each step, we triple the number of points, i.e., $n_{i+1} = 3n_i$, and create new unit-area triangles, so that $t_{i+1} = 3t_i + n_i$. This implies $n_i = 3^i$, and $t_i = i3^{i-1}$, yielding the desired lower bound. The ith step, $i \geqslant 2$, goes as follows. Choose a generic angle value α_i, close to $\pi/2$, say, and let β_i be the angle such that the three unit vectors at direction 0, α_i, and β_i from the origin determine a unit-area triangle, which we denote by D_i (note that β_i lies in the third quadrant). Rotate D_i around the origin to each position where its 0 vertex coincides with one of the n_i points of S_i, and add the other two vertices of D_i in these positions to the point set. (With appropriate choices of S_1 and the angles α_i, β_i, one can guarantee that no two points of any S_i coincide.) For each point of S_i, we added two new points, so $n_{i+1} = 3n_i$. Also, we have n_i new unit-area triangles from rotated copies of D_i; and each of the t_i previous triangles have now two new copies rotated by α_i and β_i. This gives $t_{i+1} = 3t_i + n_i$. □

3. Minimum-area triangles in the plane

The general case. We first present a simple but effective charging scheme that gives an upper bound of $n^2 - n$ on the number of minimum (nonzero) area triangles spanned by n points in the plane (Proposition 1). This technique yields a very short proof of the minimum area result from [9], with a much better constant of proportionality. Moreover, its higher-dimensional variants lead to an asymptotically tight bound of $O(n^k)$ on the maximum number of minimum-volume k-dimensional simplices determined by n points in $\mathbb{R}^d$, for any $1 \leqslant k \leqslant d$, presented in [16].

Proposition 1. *The number of triangles of minimum (nonzero) area spanned by n points in the plane is at most $n^2 - n$.*

Proof. Consider a set S of n points in the plane. Assign every triangle of minimum area to one of its longest sides. For a segment ab, with $a, b \in S$, let R^+_{ab} and R^-_{ab} denote the two rectangles of extents $|ab|$ and $2/|ab|$ with ab as a common side. If a minimum-area triangle Δabc is assigned to ab, then c must lie in the relative interior of the side parallel to ab in either R^+_{ab} or R^-_{ab}. If there were two points, c_1 and c_2, on one of these sides, then the area of Δac_1c_2 would be smaller than that of Δabc, a contradiction. Therefore, at most two triangles are assigned to each of the $\binom{n}{2}$ segments (at most one on each side of the segments), and so there are at most $n^2 - n$ minimum-area triangles. □

We now refine our analysis and establish a $\frac{2}{3}(n^2 - n)$ upper bound, which leaves only a small gap from our lower bound $(\frac{6}{\pi^2} - o(1))n^2$; both bounds are presented in Theorem 3 below. Let us point out again that here we allow collinear triples of points. The maximum number of collinear triples is clearly $\binom{n}{3} = \Theta(n^3)$. The bounds below, however, consider only nondegenerate triangles of *positive* areas.

Theorem 3. *The number of triangles of minimum (nonzero) area spanned by n points in the plane is at most $\frac{2}{3}(n^2 - n)$. The points in the $\lfloor\sqrt{n}\rfloor \times \lfloor\sqrt{n}\rfloor$ integer grid span $(\frac{6}{\pi^2} - o(1))n^2 \gtrapprox .6079n^2$ minimum-area triangles.*

Proof. We start with the upper bound. Consider a set S of n points in the plane, and let L be the set of connecting lines determined by S. Assume, without loss of generality, that none of the lines in L is vertical. Let T be the set of minimum (nonzero) area triangles spanned by S, and put $t = |T|$. There are $3t$ pairs (ab, c) where $\Delta abc \in T$, and we may assume, without loss of generality, that for at least half of these pairs (i.e., for at least $\frac{3}{2}t$ pairs) Δabc lies above the line spanned by a and b.

For each line $\ell \in L$, let ℓ' denote the line parallel to ℓ, lying above ℓ, passing through some point(s) of S, and closest to ℓ among these lines. Clearly, if $c \in S$ generates with $a, b \in \ell$ a minimum-area triangle which lies above ab then (i) a and b are a closest pair among the pairs of points in $\ell \cap S$, and (ii) $c \in \ell'_{ab}$ (the converse does not necessarily hold).

Now fix a line $\ell \in L$; set $k_1 = |\ell \cap S| \geqslant 2$, and $k_2 = |\ell' \cap S| \geqslant 1$, where ℓ' is as defined above. The number of minimum-area triangles determined by a pair of points in ℓ and lying above ℓ is at most

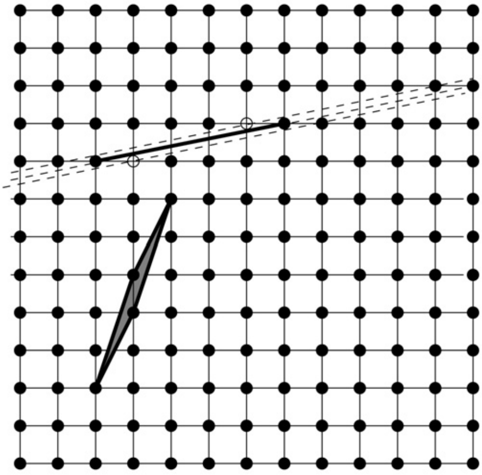

Fig. 3. In an integer lattice section, every visibility segment which is not axis-parallel is the longest side of two triangles of minimum area.

$(k_1 - 1)k_2$. We have

$$\binom{k_1}{2} + \binom{k_2}{2} \geqslant (k_1 - 1)k_2. \tag{1}$$

Indeed, multiplying by 2 and subtracting the right-hand side from the left-hand side gives

$$k_1^2 - k_1 + k_2^2 - k_2 - 2k_1k_2 + 2k_2 = (k_1 - k_2)^2 - (k_1 - k_2) \geqslant 0,$$

which holds for any $k_1, k_2 \in \mathbb{Z}$.

We now sum (1) over all lines $\ell \in L$. The sum of the terms $\binom{k_1}{2}$ is $\binom{n}{2}$, and the sum of the terms $\binom{k_2}{2}$ is at most $\binom{n}{2}$, because a line $\lambda \in L$ spanned by at least two points of S can arise as the line ℓ' for at most one line $\ell \in L$. Hence we obtain

$$\frac{3}{2}t \leqslant \sum_{\ell \in L}(k_1 - 1)k_2 \leqslant 2\binom{n}{2} = n(n-1),$$

thus $t \leqslant \frac{2}{3}(n^2 - n)$, as asserted.

We now prove the lower bound. Consider the set S of points in the $\lfloor\sqrt{n}\rfloor \times \lfloor\sqrt{n}\rfloor$ section of the integer lattice. Clearly $|S| \leqslant n$. The minimum nonzero area of triangles in S is $1/2$ (by Pick's theorem). Recall that the charging scheme used in the proof of Proposition 1 assigns each triangle of minimum area to one of its longest sides, which is necessarily a *visibility segment* (a segment not containing any point of S in its relative interior). We show that every visibility segment ab which is not axis-parallel is assigned to exactly two triangles of minimum area.

Draw parallel lines to ab through all points of the integer lattice. Every line parallel to ab and incident to a point of S contains equally spaced points of the (infinite) integer lattice. The distance between consecutive points along each line is exactly $|ab|$. This implies that each of the two lines parallel to ab and closest to it contains a lattice point on the side of the respective rectangle R_{ab}^- or R_{ab}^+, opposite to ab, and this lattice point is in S. Finally, observe that there are no empty acute triangles in the integer lattice. It follows that our charging scheme uniquely assigns empty triangles to visibility segments. An illustration is provided in Fig. 3.

A non-axis-parallel segment ab is a visibility segment if and only if the coordinates of the vector $\overrightarrow{ab}$ are relatively prime. It is well known that $6/\pi^2$ is the limit of the probability that a pair of integers (i, j) with $1 \leqslant i, j \leqslant m$ are relatively prime, as m tends to infinity [39]. Hence, a fraction of about $6/\pi^2$ of the $\binom{|S|}{2} \leqslant \binom{n}{2}$ segments spanned by S are visibility segments which are not axis-parallel.

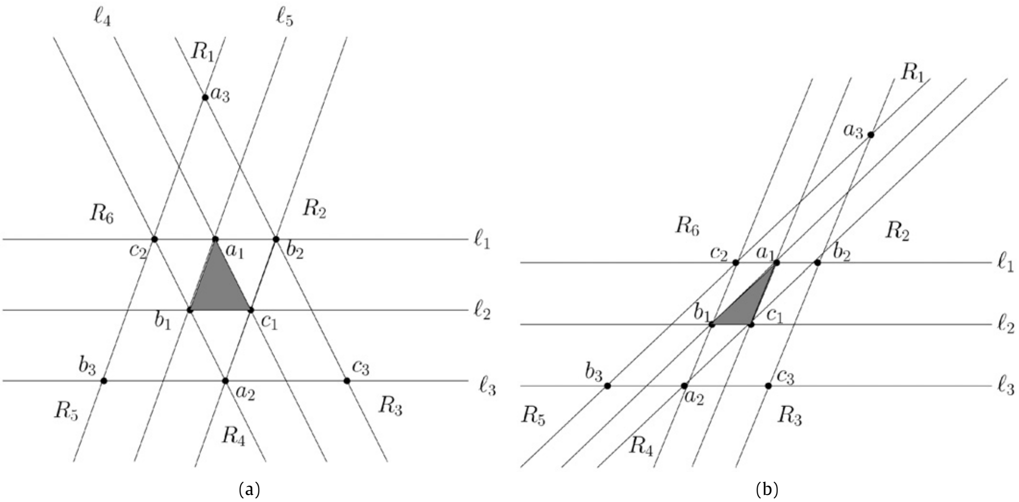

Fig. 4. (a) Acute triangles: the graph G is planar. (b) Convex position: the graph G is quasi-planar.

Each of these $(\frac{6}{\pi^2} - o(1))\binom{n}{2}$ segments corresponds to two unique triangles of minimum area, so S determines at least $(\frac{6}{\pi^2} - o(1))n^2$ minimum-area triangles. □

3.1. Special cases

In this subsection we consider some new variants of the minimum-area triangle problem for the two special cases (i) where no three points are collinear, and (ii) where the points are in convex position. We also show that the maximum number of *acute* triangles of minimum area, for any point set, is only linear.

Acute triangles. We have seen that n points in an integer grid may span $\Omega(n^2)$ triangles of minimum area. However, in that construction, all these triangles are obtuse (or right-angled). Here we prove that for any n-element point set in the plane, the number of *acute* triangles of minimum area is only linear. This bound is attained in the following simple example. Take two groups of about $n/2$ equally spaced points on two parallel lines: the first group consist of the points $(i, 0)$, for $i = 0, \dots, \lceil n/2 \rceil - 1$, and the second group of the points $(i + 1/2, \sqrt{3}/2)$, for $i = 0, \dots, \lfloor n/2 \rfloor - 1$. This point set determines $n - 2$ acute triangles of minimum area.

Theorem 4. *The maximum number of acute triangles of minimum area determined by n points in the plane is $O(n)$. This bound is asymptotically tight.*

Proof. Let S be a set of n points in the plane, and let T denote the set of acute minimum-area triangles determined by S. Define a geometric graph $G = (V, E)$ on $V = S$, where $uv \in E$ if and only if uv is a shortest side of a triangle in T. We first argue that every segment uv is a shortest edge of at most two triangles in T, and then we complete the proof by showing that G is planar and so it has only $O(n)$ edges.

Let $\Delta a_1b_1c_1 \in T$ and assume that b_1c_1 is a shortest side of $\Delta a_1b_1c_1$. Let $\Delta a_2b_2c_2$ be the triangle such that the midpoints of its sides are a_1, b_1, c_1; and let $\Delta a_3b_3c_3$ be the triangle such that the midpoints of its sides are a_2, b_2, c_2. Refer to Fig. 4(a). Since $\Delta a_1b_1c_1$ has minimum area, then, in the notation of the figure, each point of $S \setminus \{a_1, b_1, c_1\}$ lies in one of the (closed) regions R_1 through R_6 or on one of the lines ℓ_2, ℓ_4 or ℓ_5; also, no point of $S \setminus \{a_1, b_1, c_1\}$ lies in the interior of $\Delta a_3b_3c_3$. Similarly, any point $a \in S$ of a triangle $\Delta ab_1c_1 \in T$ must lie on ℓ_1 or ℓ_3. Thus $a = a_1$ and $a = a_2$ are the only possible positions of a. This follows from the fact that the triangles of T are acute: any point on, say, $\ell_1 \cap \partial R_2$ or $\ell_1 \cap \partial R_6$ forms an *obtuse* triangle with b_1c_1.

Consider two acute triangles $\Delta a_1b_1c_1, \Delta xyz \in T$ of minimum area with shortest sides $b_1c_1 \in E$ and $xy \in E$, respectively. Assume that edges b_1c_1 and xy cross each other. We have the following four possibilities: (i) x and y lie in two opposite regions R_iR_{i+3}, for some $i \in \{1,2,3\}$; (ii) $x = a_1$ and $y \in R_4$; (iii) $x \in \ell_4$ and $y \in R_4$; (iv) $x \in \ell_5$ and $y \in R_4$. Since xy is a shortest side of Δxyz, the distance from z to the line through x and y is at least $\sqrt{3}/2|xy|$. But then, in all four cases Δxyz cannot be an acute triangle of minimum area, since it contains one of the vertices of $\Delta a_1b_1c_1$ in its interior, a contradiction. (For instance if $x \in R_1$ and $y \in R_4$, Δxyc_1 would be obtuse and Δxyz contains c_1 in its interior, or if $x = a_1$ and $y \in R_4$, Δxyz contains either b_1 or c_1 in its interior.) □

Convex position. For points in strictly convex position we prove a tight $\Theta(n)$ bound on the maximum possible number of minimum-area triangles. Note that a regular n-gon has n such triangles, so it remains to show an $O(n)$ upper bound. Also, n points equally distributed on two parallel lines (at equal distances) give a well-known quadratic lower bound, so the requirement that the points be in strictly convex position is essential for the bound to hold.

Theorem 5. *The maximum number of minimum-area triangles determined by n points in (strictly) convex position in the plane is $O(n)$. This bound is asymptotically tight.*

Proof. The argument below is similar to that in the proof of Theorem 4. Since there can be only $O(n)$ acute triangles of minimum area, it is sufficient to consider right-angled and obtuse triangles (for simplicity, we refer to both types as obtuse), even though the argument also works for acute triangles. We use a similar notation: now T denotes the set of obtuse triangles of minimum area. We define a geometric graph $G = (V, E)$ on $V = S$, where $uv \in E$ if and only if uv is a shortest side of a triangle in T. See Fig. 4(b).

Let $\Delta a_1b_1c_1 \in T$ with b_1c_1 a shortest side. By convexity, at most four triangles in T can have a common shortest side b_1c_1: at most two such triangles have a third vertex on ℓ_1 and at most another two of them have a third vertex on ℓ_3. A graph drawn in the plane is said to be *quasi-planar* if it has no three edges which are pairwise crossing; it is known [3] (see also [2]) that any quasi-planar graph with n vertices has at most $O(n)$ edges. We now show that G is quasi-planar, which will complete the proof of the theorem.

Consider the triangles $\Delta a_2b_2c_2$ and $\Delta a_3b_3c_3$, defined as in the proof of Theorem 4. Each point of $S \setminus \{a_1, b_1, c_1\}$ lies in one of the (closed) regions R_1 through R_6; in particular no such point lies in the interior of $\Delta a_3b_3c_3$. (Here, unlike the previous analysis, strict convexity rules out points on any of the three middle lines, such as ℓ_2.) In addition, by convexity, the regions R_1, R_3 and R_5 are empty of points. Assume now that b_1c_1, xy, uv form a triplet of pairwise crossing edges, where xy and uv are distinct shortest sides of two triangles $\Delta xyz \in T$ and $\Delta uvw \in T$. It follows that each of the two edges xy and uv must have one endpoint at a_1 and the other in R_4 (since each crosses b_1c_1). Thus two edges in this triplet have a common endpoint, and so they do not cross, which is a contradiction. □

No three collinear points. We conjecture that if no three points are collinear, then the maximum number of triangles of minimum area is close to linear. It is not linear, though: It has been proved recently [14] that there exist n-element point sets in the plane that span $\Omega(n \log n)$ empty congruent triangles. Here, we show that one can repeat this construction such that there is no collinear triples of points and that the $\Omega(n \log n)$ empty congruent triangles have minimum (nonzero) area. However, we do not know of any sub-quadratic upper bound.

Theorem 6. *For all $n \geqslant 3$, there exist n-element point sets in the plane that have no three collinear points and span $\Omega(n \log n)$ triangles of minimum (nonzero) area.*

Proof. The construction is essentially the one given in [14], and we provide here only a brief description. We then specify the additional modifications needed for our purposes. First, a point set S is constructed with many, i.e., $\Omega(n \log n)$, pairwise congruent triples of collinear points, which can be also viewed as degenerate empty congruent triangles. Then this construction is slightly perturbed to obtain a set of points S with no collinear triples, so that these degenerate triangles become nondegenerate empty congruent triangles of minimum (nonzero) area. The details are as follows (see [14]).

Let $n = 3^k$ for some $k \in \mathbb{N}$. Consider k unit vectors $b_1, \dots, b_k$, and for $1 \leqslant i \leqslant k$, let β_i be the counterclockwise angle from the x-axis to b_i. Let $\lambda \in (0, 1)$ be fixed and let $a_i = \lambda b_i$. Consider now all 3^k possible sums of these $2k$ vectors, a_i and b_i, $1 \leqslant i \leqslant k$, with coefficients 0 or 1, satisfying the condition that for each i, at least one of a_i or b_i has coefficient 0. Let S be the set of 3^k points determined by these vectors. Clearly, each triple of the form $(v, v + a_i, v + b_i)$, where v is a subset sum that does not involve a_i or b_i, consists of collinear points. For such a triple, denote by $s_i(v)$ the segment whose endpoints are v and $v + b_i$. We say that the collinear triple $(v, v + a_i, v + b_i)$ is of type i, $i = 1, \dots, k$. For each i there are exactly 3^{k-1} triples of type i, therefore a total of $k3^{k-1} = (n \log n)/(3 \log 3) = \Omega(n \log n)$ triples of collinear points. Clearly, all these triples form degenerate congruent triangles in S. Denote by $\ell_i(v)$ the line supporting the segment $s_i(v)$, and by L the set of lines corresponding to these triples.

We need the following slightly stronger version of Lemma 1 in [14]. The proof is very similar to the proof of Proposition 1 in [14], and we omit the details.

Lemma 1. *There exist angles $\beta_1, \dots, \beta_k$, and $\lambda \in (0, 1)$, such that* (i) *S consists of n distinct points*; (ii) *if $u, v, w \in S$ are collinear* (*in this order*)*, then $v = u + a_i$ and $w = u + b_i$.*

Let ε be the minimum distance between points $p \in S \setminus \{v, v + a_i, v + b_i\}$ and lines $\ell_i(v) \in L$, over all pairs (v, i). By Lemma 1, we have $\varepsilon > 0$. Now instead of choosing a_i to be collinear with b_i, slightly rotate λb_i counterclockwise from b_i through a sufficiently small angle δ about their common origin, so the collinearity disappears. This modification is carried out at the same time for all vectors a_i, $i = 1, \dots, k$, that participate in the construction. By continuity, there exists a sufficiently small $\delta = \delta(\varepsilon) > 0$, so that (i) each of the triangles $\Delta(v, v + a_i, v + b_i)$ remains empty throughout this small perturbation, (ii) the point set S is in general position after the perturbation, and (iii) the congruent triangles $\Delta(v, v + a_i, v + b_i)$ have minimum area. This completes the proof. □

4. Unit-area triangles in 3-space

Erdős and Purdy [19] showed that a $\sqrt{\log n} \times (n/\sqrt{\log n})$ section of the integer lattice determines $\Omega(n^2 \log \log n)$ triangles of the same area. Clearly, this bound is also valid in 3-space. They have also derived an upper bound of $O(n^{8/3})$ on the number of unit-area triangles in $\mathbb{R}^3$. Here we improve this bound to $O(n^{17/7}\beta(n)) = O(n^{2.4286})$. We use $\beta(n)$ to denote any function of the form $\exp(\alpha(n)^{O(1)})$, where $\alpha(n)$ is the extremely slowly growing inverse Ackermann function. Any such function $\beta(n)$ is also extremely slowly growing.

Theorem 7. *The number of unit-area triangles spanned by n points in $\mathbb{R}^3$ is $O(n^{17/7}\beta(n)) = O(n^{2.4286})$.*

The proof of the theorem is quite long, and involves several technical steps. Let S be a set of n points in $\mathbb{R}^3$. For each pair a, b of distinct points in S, let ℓ_{ab} denote the line passing through a and b, and let C_{ab} denote the cylinder whose axis is ℓ_{ab} and whose radius is $2/|ab|$. Clearly, any point $c \in S$ that forms with ab a unit-area triangle, must lie on C_{ab}. The problem is thus to bound the number of incidences between $\binom{n}{2}$ cylinders and n points, but it is complicated for two reasons: (i) The cylinders need not be distinct. (ii) Many distinct cylinders can share a common generator line, which may contain many points of S.

Cylinders with large multiplicity. Let $\mathcal{C}$ denote the multiset of the $\binom{n}{2}$ cylinders C_{ab}, for $a, b \in S$. Since the cylinders in $\mathcal{C}$ may appear with multiplicity, we fix a parameter $\mu = 2^j$, $j = 0, 1, \dots$, and consider separately incidences with each of the sets $\mathcal{C}_\mu$, of all the cylinders whose multiplicity is between μ and $2\mu - 1$. Write $c_\mu = |\mathcal{C}_\mu|$. We regard $\mathcal{C}_\mu$ as a set (of distinct cylinders), and will multiply the bound that we get for the cylinders in $\mathcal{C}_\mu$ by 2μ, to get an upper bound on the number of incidences that we seek to estimate. We will then sum up the resulting bounds over μ to get an overall bound.

Let C be a cylinder in $\mathcal{C}_\mu$. Then its axis ℓ must contain μ pairs of points of P at a fixed distance apart (equal to $2/r$, where r is the radius of C). That is, ℓ contains $t > \mu$ points of S. Let us now fix t to be a power of 2, and consider the subset $\mathcal{C}_{\mu,t} \subset \mathcal{C}_\mu$ of those cylinders in $\mathcal{C}_\mu$ that have at least t and at most $2t-1$ points on their axis. By the Szemerédi–Trotter theorem [37] (or, rather, its obvious extension to 3-space), the number of lines containing at least t points of S is $O(n^2/t^3 + n/t)$. Any such line ℓ can be the axis of many cylinders in $\mathcal{C}_\mu$ (of different radii). Any such cylinder "charges" $\Theta(\mu)$ pairs of points out of the $O(t^2)$ pairs along ℓ, and no pair is charged more than once. Hence, for a given line ℓ incident to at least $t > \mu$ and at most $2t-1$ points of S, the number of distinct cylinders in $\mathcal{C}_\mu$ that have ℓ as axis is $O(t^2/\mu)$. Summing over all axes incident to at least t and at most $2t-1$ points yields that the number of distinct cylinders in $\mathcal{C}_{\mu,t}$ is

$$c_{\mu,t} = O\left(\left(\frac{n^2}{t^3} + \frac{n}{t}\right)\frac{t^2}{\mu}\right) = O\left(\frac{n^2}{t\mu} + \frac{nt}{\mu}\right). \tag{2}$$

We next sum this over t, a power of 2 between μ and ν, and conclude that the number of distinct cylinders in $\mathcal{C}_\mu$ having at most ν points on their axis is

$$c_{\mu,\leqslant\nu} = O\left(\frac{n^2}{\mu^2} + \frac{n\nu}{\mu}\right). \tag{3}$$

Restricted incidences between points and cylinders. We distinguish two *type*s of incidences, which we count separately. An incidence between a point p and a cylinder C is of *type* 1 if the generator of C passing through p contains at least one additional point of S; otherwise it is of *type* 2. We begin with the following subproblem, in which we bound the number of incidences between the cylinders of $\mathcal{C}$, counted with multiplicity, and *multiple* points that lie on their generator lines, as well as incidences with cylinders with "rich" axes. Specifically, we have the following lemma.

Lemma 2. *Let S be a set of n points and $\mathcal{C}$ be the multiset of the $\binom{n}{2}$ cylinders C_{ab}, for $a, b \in S$ (counted with multiplicity). The total number of all incidences of type* 1 *and all incidences involving cylinders having at least $n^{14/45}$ points on their axis is bounded by $O(n^{107/45}\,\mathrm{polylog}(n)) = O(n^{2.378})$.*

Proof. Let L denote the set of lines spanned by the points of S. Fix a parameter $k = 2^i$, $i = 1, \ldots,$ and consider the set L_k of all lines that contain at least k and at most $2k-1$ points of S. We bound the number of incidences between cylinders in $\mathcal{C}$ that contain lines in L_k as generators and points that lie on those lines. Formally, we bound the number of triples (p, ℓ, C), where $p \in S$, $\ell \in L_k$, and $C \in \mathcal{C}$, such that $p \in \ell$ and $\ell \subset C$. Summing these bounds over k will give us a bound for the number of incidences of type 1. Along the way, we will also dispose of incidences with cylinders whose axes contain many points.

As already noted, the Szemerédi–Trotter theorem [37] implies that $\lambda_k := |L_k| = O(\frac{n^2}{k^3} + \frac{n}{k})$.

Line-cylinder incidences. Consider the subproblem of bounding the number of incidences between lines in L_k and cylinders in $\mathcal{C}$, where a line ℓ is said to be incident to cylinder C if ℓ is a generator of C. We will then multiply the resulting bound by $2k$ to get an upper bound on the number of point-line-cylinder incidences involving L_k, and then sum the resulting bounds over k.

Generator lines with many points. Let us first dispose of the case $k > n^{1/3}$. Any line $\ell \in L_k$ can be a generator of at most n cylinders (counted with multiplicity), because, having fixed $a \in S$, the point $b \in S$ such that C_{ab} contains ℓ is determined (up to multiplicity 2). Hence the number of incidences between the points that lie on ℓ and the cylinders of $\mathcal{C}$ is $O(nk)$. Summing over $k = 2^i > n^{1/3}$ yields the overall bound

$$O\left(\sum_k nk\lambda_k\right) = O\left(\sum_k \left(\frac{n^3}{k^2} + n^2\right)\right) = O\left(n^{7/3}\right).$$

Hence, in what follows, we may assume that $k \leqslant n^{1/3}$. In this range of k we have

$$\lambda_k = O\left(\frac{n^2}{k^3}\right). \tag{4}$$

Axes with many points. Let us also fix the multiplicity μ of the cylinders under consideration (up to a factor of 2, as above). The number of distinct cylinders in $\mathcal{C}_\mu$ having between $t > \mu$ and $2t-1$ points on their axes, is $O(n^2/(t\mu) + nt/\mu)$; see (2). While the first term is sufficiently small for our purpose, the second term may be too large when t is large. To avoid this difficulty, we fix another threshold exponent $z < 1/2$ that we will optimize later, and handle separately the cases $t \geqslant n^z$ and $t < n^z$. That is, in the first case, for $t \geqslant n^z$ a power of 2, we seek an upper bound on the overall number of incidences between the points of S and the cylinders in $\mathcal{C}$ whose axis contains between t and $2t-1$ points of S. (For this case, we combine all the multiplicities $\mu < t$ together.) By the Szemerédi–Trotter theorem [37], the number of such axes is $O(n^2/t^3 + n/t)$.

Fix such an axis α. It defines $\Theta(t^2)$ cylinders, and the multiplicity of any of these cylinders is at most $O(t)$. Since no two distinct cylinders in this collection can pass through the same point of S, it follows that the total number of incidences between the points of S and these cylinders is $O(nt)$. Hence the overall number of incidences under consideration is $O(n^2/t^3 + n/t) \cdot O(nt) = O(n^3/t^2 + n^2)$. Summing over all $t \geqslant n^z$, a power of 2, we get the overall bound $O(n^{3-2z})$.

Note that this bound takes care of *all* the incidences between the points of S and the cylinders having at least $t \geqslant n^z$ points along their axes, not just those of type 1 (involving multiple points on generator lines).

Cylinders with low multiplicity. We now confine the analysis to cylinders having fewer than n^z points on their axis, and go back to fixing the multiplicity μ, which we may assume to be at most n^z. We thus want to bound the number of incidences between λ_k distinct lines and $c_{\mu,\leqslant n^z}$ distinct cylinders in $\mathcal{C}_\mu$, for given $k \leqslant n^{1/3}$, $\mu \leqslant n^z$. Note that a cylinder can contain a line if and only if it is parallel to the axis of the cylinder, so we can split the problem into subproblems, each associated with some direction θ, so that in the θ-subproblem we have a set of some $c_\mu^{(\theta)}$ cylinders and a set of some $\lambda_k^{(\theta)}$ lines, so that the lines and the cylinder axes are all parallel (and have direction θ); we have $\sum_\theta c_\mu^{(\theta)} = c_{\mu,\leqslant n^z}$, and $\sum_\theta \lambda_k^{(\theta)} = \lambda_k$.

For a fixed θ, we project the cylinders and lines in the θ-subproblem onto a plane with normal direction θ, and obtain a set of $c_\mu^{(\theta)}$ circles and a set of $\lambda_k^{(\theta)}$ points, so that the number of line-cylinder incidences is equal to the number of point-circle incidences. By [4,6,28],[5] the number of point-circle incidences between N points and M circles in the plane is $O(N^{2/3}M^{2/3} + N^{6/11}M^{9/11}\log^{2/11}(N^3/M) + N + M)$. It follows that the number of such line-cylinder incidences is

$$O\left(\left(\lambda_k^{(\theta)}\right)^{2/3}\left(c_\mu^{(\theta)}\right)^{2/3} + \left(\lambda_k^{(\theta)}\right)^{6/11}\left(c_\mu^{(\theta)}\right)^{9/11}\log^{2/11}\left(\left(\lambda_k^{(\theta)}\right)^3/c_\mu^{(\theta)}\right) + \lambda_k^{(\theta)} + c_\mu^{(\theta)}\right). \tag{5}$$

Note that, for any fixed θ, we have $\lambda_k^{(\theta)} \leqslant n/k$ and $c_\mu^{(\theta)} \leqslant n^{1+z}/\mu$. The former inequality is trivial. To see the latter inequality, note that an axis with $t < n^z$ points defines $\binom{t}{2}$ cylinders. Since we only consider cylinders with multiplicity $\Theta(\mu)$, the number of distinct such cylinders is $O(t^2/\mu)$, and the number of lines (of direction θ) with about t points on them is at most n/t, for a total of at most $O(nt/\mu)$ distinct cylinders. Partitioning the range $\mu < t \leqslant n^z$ by powers of 2, as above, and summing up the resulting bounds, the bound $c_\mu^{(\theta)} \leqslant n^{1+z}/\mu$ follows.

Summing over θ, and using Hölder's inequality, we have (here x is a parameter between 2/11 and 6/11 that we will fix shortly)

$$\sum_\theta \left(\lambda_k^{(\theta)}\right)^{6/11}\left(c_\mu^{(\theta)}\right)^{9/11} \leqslant \left(\frac{n}{k}\right)^{6/11-x}\left(\frac{n^{1+z}}{\mu}\right)^{x-2/11}\sum_\theta \left(\lambda_k^{(\theta)}\right)^x\left(c_\mu^{(\theta)}\right)^{1-x}$$

[5] The bound that we use, from [28], is slightly better than the previous ones.

$$\leqslant \frac{n^{(4-2z)/11+xz}}{k^{6/11-x}\mu^{x-2/11}}\left(\sum_\theta \lambda_k^{(\theta)}\right)^x\left(\sum_\theta c_\mu^{(\theta)}\right)^{1-x} = \frac{n^{(4-2z)/11+xz}}{k^{6/11-x}\mu^{x-2/11}}\lambda_k^x c_{\mu,\leqslant n^z}^{1-x}.$$

We need to multiply this bound by $\Theta(k\mu)$. Substituting the bounds $\lambda_k = O(n^2/k^3)$ from (4), and $c_{\mu,\leqslant n^z} = O(n^2/\mu^2 + n^{1+z}/\mu)$ from (3), we get the bound

$$O\left(n^{(4-2z)/11+xz}k^{5/11+x}\mu^{13/11-x}\left(\frac{n^2}{k^3}\right)^x\left(\frac{n^2}{\mu^2}+\frac{n^{1+z}}{\mu}\right)^{1-x}\log^{2/11} n\right)$$
$$= O\left(k^{5/11-2x}\left(n^{2+(4-2z)/11+xz}\mu^{x-9/11}+n^{(15+9z)/11+x}\mu^{2/11}\right)\log^{2/11} n\right).$$

Choosing $x = 5/22$ (the smallest value for which the exponent of k is nonpositive), the first term becomes $O(n^{2+4/11+z/22}\log^{2/11} n)$, which we need to balance with $O(n^{3-2z})$; for this, we choose $z = 14/45$ and obtain the bound $O(n^{107/45}\log^{2/11} n) = O(n^{2.378})$; for this choice of z, recalling that $\mu < n^z$, the second term is dominated by the first. Summing over k, μ only adds logarithmic factors, for a resulting overall bound $O(n^{2.378})$.

Similarly, we have (with a different choice of x, soon to be made)

$$\sum_\theta\left(\lambda_k^{(\theta)}\right)^{2/3}\left(c_\mu^{(\theta)}\right)^{2/3} \leqslant \left(\frac{n}{k}\right)^{2/3-x}\left(\frac{n^{1+z}}{\mu}\right)^{x-1/3}\sum_\theta\left(\lambda_k^{(\theta)}\right)^x\left(c_\mu^{(\theta)}\right)^{1-x}$$
$$\leqslant \frac{n^{(1-z)/3+xz}}{k^{2/3-x}\mu^{x-1/3}}\left(\sum_\theta \lambda_k^{(\theta)}\right)^x\left(\sum_\theta c_\mu^{(\theta)}\right)^{1-x} = \frac{n^{(1-z)/3+xz}}{k^{2/3-x}\mu^{x-1/3}}\lambda_k^x c_{\mu,\leqslant n^z}^{1-x}.$$

Multiplying by $k\mu$ and arguing as above, we get

$$O\left(n^{(1-z)/3+xz}k^{1/3+x}\mu^{4/3-x}\left(\frac{n^2}{k^3}\right)^x\left(\frac{n^2}{\mu^2}+\frac{n^{1+z}}{\mu}\right)^{1-x}\log^{2/11} n\right)$$
$$= O\left(k^{1/3-2x}\left(n^{2+(1-z)/3+xz}\mu^{x-2/3}+n^{1+(1+2z)/3+x}\mu^{1/3}\right)\log^{2/11} n\right).$$

We choose here $x = 1/6$ and note that, for $z = 14/45$ and $\mu < n^z$, the bound is smaller than $O(n^{7/3})$, which is dominated by the preceding bound $O(n^{2.378})$.

Finally, the linear terms in (5), multiplied by $k\mu$, add up to

$$k\mu\sum_\theta O\left(\lambda_k^{(\theta)}+c_\mu^{(\theta)}\right) = O\left(k\mu(\lambda_k + c_{\mu,\leqslant n^z})\right) = O\left(\frac{n^2\mu}{k^2}+\frac{n^2 k}{\mu}+n^{1+z}k\right),$$

which, by our assumptions on k, μ, and z is also dominated by $O(n^{2.378})$. Summing over k, μ only add logarithmic factors, for a resulting overall bound $O(n^{2.378})$. This completes the proof of Lemma 2. □

It therefore remains to count point-cylinder incidences of type 2, involving cylinders having at most $n^{14/45}$ points on their axes.

The intersection pattern of three cylinders. We need the following consequence of Bézout's theorem [23].

Lemma 3. *Let C, C_1, C_2 be three cylinders with no pair of parallel axes. Then $C \cap C_1 \cap C_2$ consists of at most* 8 *points.*

Point-cylinder incidences. Using the partition technique [11,31] for disjoint cylinders in $\mathbb{R}^3$, we show the following:

Lemma 4. *For any parameter r, $1 \leqslant r \leqslant \min\{m, n^{1/3}\}$, the maximum number of incidences of type* 2 *between n points and m cylinders in* 3*-space satisfies the following recurrence*:

$$I(n,m) = O\big(n + mr^2\beta(r)\big) + O\big(r^3\beta(r)\big) \cdot I\left(\frac{n}{r^3}, \frac{m}{r}\right), \tag{6}$$

for some slowly growing function $\beta(n)$, as above.

Proof. Let $\mathcal{C}$ be a set of m cylinders, and S be a set of n points. Construct a $(1/r)$-cutting of the arrangement $\mathcal{A}(\mathcal{C})$. The cutting has $O(r^3\beta(r))$ relatively open pairwise disjoint cells, each crossed by at most m/r cylinders and containing at most n/r^3 points of S [12] (see also [33, p. 271]); the first property is by definition of $(1/r)$-cuttings, and the second is enforced by subdividing cells with too many points. The number of incidences between points and cylinders *crossing* their cells is thus

$$O\big(r^3\beta(r)\big) \cdot I\left(\frac{n}{r^3}, \frac{m}{r}\right).$$

(Note that any incidence of type 2 remains an incidence of type 2 in the subproblem it is passed to.)

It remains to bound the number of incidences between the points of S and the cylinders that *contain* their cells. Let τ be a (relatively open) lower-dimensional cell of the cutting. If $\dim(\tau) = 2$ then we can assign any point p in τ to one of the two neighboring full-dimensional cells, and count all but at most one of the incidences with p within that cell. Hence, this increases the count by at most n.

If $\dim(\tau) = 0$, i.e., τ is a vertex of the cutting, then any cylinder containing τ must cross or define one of the full-dimensional cells adjacent to τ. Since each cell has at most $O(1)$ vertices, it follows that the total number of such incidences is $O(r^3\beta(r)) \cdot (m/r) = O(mr^2\beta(r))$.

Suppose then that $\dim(\tau) = 1$, i.e., τ is an edge of the cutting. An immediate implication of Lemma 3 is that only $O(1)$ cylinders can contain τ, unless τ is a line, which can then be a generator of arbitrarily many cylinders.

Since we are only counting incidences of type 2, this implies that any straight-edge 1-dimensional cell τ of the cutting generates at most one such incidence with any cylinder that fully contains τ. Nonstraight edges of the cutting are contained in only $O(1)$ cylinders, as just argued, and thus the points on such edges generate a total of only $O(n)$ incidences with the cylinders. Thus the overall number of incidences in this subcase is only $O(n + r^3\beta(r))$. Since $r \leqslant m$, this completes the proof of the lemma. $\square$

Lemma 5. *The number of incidences of type 2 between n points and m cylinders in $\mathbb{R}^3$ is*

$$O\big((m^{6/7}n^{5/7} + m + n)\beta(n)\big). \tag{7}$$

Proof. Let $\mathcal{C}$ be a set of m cylinders, and S be a set of n points. We first derive an upper bound of $O(n^5 + m)$ on the number of incidences of type 2 between $\mathcal{C}$ and S. We represent the cylinders as points in a dual 5-space, so that each cylinder C is mapped to a point C^*, whose coordinates are the five degrees of freedom of C (four specifying its axis and the fifth specifying its radius). A point $q \in \mathbb{R}^3$ is mapped to a surface q^* in $\mathbb{R}^5$, which is the locus of all points dual to cylinders that are incident to q. With an appropriate choice of parameters, each surface q^* is semi-algebraic of constant description complexity. By definition, this duality preserves incidences.

After dualization, we have an incidence problem involving m points and n surfaces in $\mathbb{R}^5$. We construct the arrangement $\mathcal{A}$ of the n dual surfaces, and bound the number of their incidences with the m dual points as follows. The arrangement $\mathcal{A}$ consists of $O(n^5)$ relatively open cells of dimensions $0, 1, \ldots, 5$. Let τ be a cell of $\mathcal{A}$. We may assume that $\dim(\tau) \leqslant 4$, because no point in a full-dimensional cell can be incident to any surface.

If τ is a vertex, consider any surface φ that passes through τ. Then τ is a vertex of the arrangement restricted to φ, which is a 4-dimensional arrangement with $O(n^4)$ vertices. This implies that the number of incidences at vertices of $\mathcal{A}$ is at most $n \cdot O(n^4) = O(n^5)$.

Let then τ be a cell of $\mathcal{A}$ of dimension $\geqslant 1$, and let u denote the number of surfaces that contain τ. If $u \leqslant 8$ then each point in τ (dual to a cylinder) has at most $O(1)$ incidences of this kind, for a total of $O(m)$.

Otherwise, $u \geqslant 9$. Since $\dim(\tau) \geqslant 1$, it contains infinitely many points dual to cylinders (not necessarily in $\mathcal{C}$). By Lemma 3, back in the primal 3-space, if three cylinders contain the same nine points, then the axes of at least two of them are parallel. Hence all u points lie on one line or on two parallel lines, which are common generators of these pair of cylinders. In this case, all cylinders whose dual points lie in τ contain these generator(s). But then, by definition, the incidences between these points and the cylinders of $\mathcal{C}$ whose dual points lie on τ are of type 1, and are therefore not counted at all by the current analysis. Since τ is a face of $\mathcal{A}$, no other point lies on any of these cylinders, so we may ignore them completely.

Hence, the overall number of incidences under consideration is $O(n^5 + m)$.

If $m > n^5$, this bound is $O(m)$. If $m < n^{1/3}$, we apply Lemma 4 with $r = m$, which then yields that each recursive subproblem has at most one cylinder, so each point in a subproblem generates at most one incidence, for a total of $O(n)$ incidences. Hence, in this case (6) implies that the number of incidences between $\mathcal{C}$ and S is $O(n + m^3\beta(m)) = O(n\beta(n))$.

Otherwise we have $n^{1/3} \leqslant m \leqslant n^5$, so we can apply Lemma 4 with parameter $r = (n^5/m)^{1/14}$; observe that $1 \leqslant r \leqslant \min\{m, n^{1/3}\}$ in this case. Using the above bound for each of the subproblems in the recurrence, we obtain $I(n/r^3, m/r) = O((n/r^3)^5 + m/r)$, and thus the total number of incidences of type 2 in this case is

$$O\left(n + mr^2\beta(r)\right) + O\left(r^3\beta(r)\right) \cdot O\left(\left(\frac{n}{r^3}\right)^5 + \frac{m}{r}\right) = O\left(\frac{n^5}{r^{12}} + mr^2\right)\beta(r).$$

The choice $r = (n^5/m)^{1/14}$ yields the bound (7). Combining this with the other cases, the bound in the lemma follows. □

We are now in position to complete the proof of Theorem 7.

Proof of Theorem 7. We now return to our original setup, where the cylinders in $\mathcal{C}$ may have multiplicities. We fix some parameter μ and consider, as above, all cylinders in C_μ, and recall our choice of $z = 14/45$. The case $\mu \geqslant n^z$ is taken care of by Lemma 2, accounting for at most $O(n^{107/45}\,\mathrm{polylog}(n))$ incidences. In fact, Lemma 2 takes care of all cylinders that contain at least n^z points on their axes. Assume then that $\mu < n^z$, and consider only those cylinders in $\mathcal{C}_\mu$ containing fewer than n^z points on their axes. By (3), we have $c_{\mu,\leqslant n^z} = O(n^2/\mu^2)$. Consequently, the number of incidences with the remaining cylinders in C_μ, counted with multiplicity, but excluding multiple points on the same generator line, is

$$O\left(\mu\beta(n) \cdot \left(\left(\frac{n^2}{\mu^2}\right)^{6/7} \cdot n^{5/7} + \frac{n^2}{\mu^2} + n\right)\right) = O\left(\left(\frac{n^{17/7}}{\mu^{5/7}} + \frac{n^2}{\mu} + n\mu\right)\beta(n)\right).$$

Summing over all $\mu \leqslant n^z$ (powers of 2), and adding the bound $O(n^{107/45}\,\mathrm{polylog}(n)) = O(n^{2.378})$ from Lemma 2 on the other kinds of incidences, we get the desired overall bound of $O(n^{17/7}\beta(n)) = O(n^{2.4286})$. □

Remark. In a nutshell, the "bottleneck" in the analysis is the case where μ is small (say, a constant) and we count incidences of type 2. The rest of the analysis, involved as it is, just shows that all the other cases contribute fewer (in fact, much fewer) incidences. One could probably simplify some parts of the analysis, at the cost of weakening the other bounds, but we leave these parts as they are, in the hope that the bottleneck case could be improved, in which case these bounds might become the dominant ones.

5. Minimum-area triangles in 3-space

Place n equally spaced points on the three parallel edges of a right prism whose base is an equilateral triangle, such that inter-point distances are small along each edge. This construction yields $\frac{2}{3}n^2 - O(n)$ minimum-area triangles, a slight improvement over the lower bound construction in the plane. Here is yet another construction with the same constant 2/3 in the leading term: Form a rhombus in the xy-plane from two equilateral triangles with a common side, extend it to a prism in 3-space, and place $n/3$ equally spaced points on each of the lines passing through the vertices of the shorter diagonal of the rhombus, and $n/6$ equally spaced points on each of the two other lines, where again the inter-point distances along these lines are all equal and small. The number of minimum-area triangles is

$$2\left(\frac{1}{3\cdot 3}+\frac{4}{3\cdot 6}\right)n^2 - O(n) = \frac{2}{3}n^2 - O(n).$$

An $O(n^2)$ upper bound has been shown in [16], which is optimal up to constant factors. The following theorem significantly improves the constant factor. Similarly to the planar case, we assign each triangle in T to one of its longest sides. However, here we distinguish between *fat* and *thin* triangles (defined below). We show that the number N_1 of thin triangles of minimum area is at most $2\binom{n}{2} = n^2 - n$, and that the number N_2 of fat triangles of minimum area is only $O(n)$.

Theorem 8. *The number of triangles of minimum (nonzero) area spanned by n points in* $\mathbb{R}^3$ *is at most* $n^2 + O(n)$.

Proof. Consider a set S of n points in $\mathbb{R}^3$, and let T be the set of triangles of minimum (nonzero) area spanned by S. Without loss of generality, assume the minimum area to be 1. Consider a segment ab, with $a, b \in S$, and let $h = |ab|$. Every point $c \in S \setminus \{a, b\}$ for which the triangle Δabc has minimum (unit) area must lie on a bounded cylinder C with axis ab, radius $r = 2/h$, and bases that lie in the planes π_a and π_b, incident to a and b, respectively, and orthogonal to ab. In fact, if Δabc is assigned to ab (that is, ab is the longest side), then c must lie on a smaller portion C' of C, bounded by bases that intersect ab at points at distance $h - \sqrt{h^2 - r^2}$ from a and b, respectively. Assume for convenience that ab is vertical, a is the origin and $b = (0, 0, h)$. Since ab is the longest side of Δabc, the side of the isosceles triangle with base ab and height r must be no larger than h, i.e., $\frac{1}{4}h^2 + r^2 \leqslant h^2$, or $r^2 \leqslant \frac{3}{4}h^2$. Notice that the triangle formed by any two points of S lying on C' with either a or b is nondegenerate.

We next derive a simple formula that relates the area of any (slanted) triangle to the area of its xy-projection. Consider a triangle Δ that is spanned by two vectors u, v, and let Δ_0, u_0, and v_0 denote the xy-projections of Δ, u, and v, respectively. Write (where $\mathbf{k}$ denotes, as usual, the vector $(0, 0, 1)$)

$$u = u_0 + x\mathbf{k} \quad \text{and} \quad v = v_0 + y\mathbf{k},$$

and put $A = \text{area}(\Delta)$, $A_0 = \text{area}(\Delta_0)$. Then

$$A^2 = \frac{1}{4}\|u \times v\|^2 = \frac{1}{4}\big\|(u_0 + x\mathbf{k}) \times (v_0 + y\mathbf{k})\big\| = \frac{1}{4}\big(\|u_0 \times v_0\|^2 + \|yu_0 - xv_0\|^2\big)$$

or

$$A^2 = A_0^2 + \frac{1}{4}\|yu_0 - xv_0\|^2. \tag{8}$$

We distinguish between the cases in which the minimum-area triangles charged to the segment ab are "thin" or "fat." A triangle is called *fat* (resp., *thin*) if the length of the height corresponding to its longest side is at least (resp., less than) half of the length of the longest side.

(a) $r < \frac{1}{2}h$ (thin triangles). We claim that in this case at most two triangles can be assigned to ab. Indeed, suppose to the contrary that at least three triangles are assigned to ab, so their third vertices, $c, d, e \in S$ lie on $C' \subset C$. Write the z-coordinates of c, d, e as z_1h, z_2h, z_3h, respectively, and assume, without loss of generality, that $0 < z_1 \leqslant z_2 \leqslant z_3 < 1$, and $z_2 \leqslant 1/2$. Consider the triangle Δacd, and let A denote its area. As before, write, without loss of generality,

$$c = (r, 0, z_1h) \quad \text{and} \quad d = (r\cos\alpha, r\sin\alpha, z_2h),$$

for some $0 \leqslant \alpha \leqslant 180°$. Using (8), we get

$$A^2 = \frac{1}{4} r^4 \sin^2\alpha + \frac{1}{4} r^2 h^2 \left(z_1^2 + z_2^2 - 2 z_1 z_2 \cos\alpha\right).$$

Thus, recalling that $r < \frac{1}{2}h$ and that $h^2 r^2 = 4$, we get

$$A^2 < \frac{1}{4} r^2 h^2 \left(\frac{1}{4}\sin^2\alpha + z_1^2 + z_2^2 - 2 z_1 z_2 \cos\alpha\right) = \frac{1}{4}\sin^2\alpha + z_1^2 + z_2^2 - 2 z_1 z_2 \cos\alpha. \tag{9}$$

Let us fix z_1, z_2 and vary only α. Write

$$f(\alpha) = \frac{1}{4}\sin^2\alpha + z_1^2 + z_2^2 - 2 z_1 z_2 \cos\alpha \quad \text{and} \quad f'(\alpha) = \frac{1}{2}\sin\alpha\cos\alpha + 2 z_1 z_2 \sin\alpha.$$

f attains its maximum at the zero of its derivative, namely at α_0 that satisfies

$$\cos\alpha_0 = -4 z_1 z_2.$$

(Note that since $z_1 \leqslant z_2 \leqslant \frac{1}{2}$, we always have $4 z_1 z_2 \leqslant 1$. Also, at the other zero $\alpha = 0$, f attains its minimum $(z_1 - z_2)^2$.)

Substituting α_0 into (9), and using $z_1 \leqslant z_2 \leqslant \frac{1}{2}$, we get

$$A^2 < \frac{1 - 16 z_1^2 z_2^2}{4} + z_1^2 + z_2^2 + 8 z_1^2 z_2^2 = \frac{1}{4} + z_1^2 + z_2^2 + 4 z_1^2 z_2^2 = \left(\frac{1}{2} + 2 z_1^2\right)\left(\frac{1}{2} + 2 z_2^2\right) \leqslant 1,$$

which contradicts the minimality of the area of Δabc (recall that Δacd is nondegenerate).

We have thus shown that at most two thin triangles of minimum area can be assigned to any segment ab, so $N_1 \leqslant 2\binom{n}{2} = n^2 - n$.

(b) $r \geqslant \frac{1}{2}h$ (fat triangles). Recall that we always have $r \leqslant \frac{\sqrt{3}}{2}h$. Multiplying these two inequalities by $h/2$, we get

$$\frac{h^2}{4} \leqslant 1 \leqslant \frac{h^2\sqrt{3}}{4} \quad \text{or} \quad \frac{2}{3^{1/4}} \leqslant h \leqslant 2.$$

Let E denote the set of all segments ab such that the minimum-area triangles charged to ab are fat. Note that the length of each edge in E is in the interval $[2/3^{1/4}, 2]$.

We next claim that, for any pair of points $p, q \in S$ with $|pq| < 1$, neither p nor q can be an endpoint of an edge in E. Indeed, suppose to the contrary that p, q is such a pair and that pa is an edge of E, for some $a \in S$; by construction, $a \neq q$. Let Δpab be a fat minimum-area triangle charged to pa. If q is collinear with pa, then Δpqb is a nondegenerate triangle of area strictly smaller than that of Δpab (recall that $|pq| < 1 < |pa|$), a contradiction. If q is not collinear with pa, Δpaq is a nondegenerate triangle of area $\leqslant \frac{|pa| \cdot |pq|}{2} < \frac{2 \cdot 1}{2} = 1$, again a contradiction.

Let $S' \subseteq S$ be the set obtained by repeatedly removing the points of S whose nearest neighbor in S is at distance smaller than 1. Clearly, the minimum inter-point distance in S' is at least 1, and the endpoints of each edge in E lie in S'. This implies, via an easy packing argument, that the number of edges of E incident to any fixed point in S' (all of length at most 2) is only $O(1)$. Hence $|E| = O(n)$. Since each edge in E determines at most $O(1)$ minimum-area triangles, as shown in [16], we conclude that $N_2 = O(n)$, as claimed. Hence there are at most $2\binom{n}{2} + O(n) = n^2 + O(n)$ minimum-area triangles in total. □

6. Maximum-area triangles in 3-space

Ábrego and Fernández-Merchant [1] showed that one can place n points on the unit sphere in $\mathbb{R}^3$ so that they determine $\Omega(n^{4/3})$ pairwise distances of $\sqrt{2}$ (see also [29, p. 191] and [8, p. 261]). This implies the following result:

Theorem 9. *For any integer n, there exists an n-element point set in* $\mathbb{R}^3$ *that spans* $\Omega(n^{4/3})$ *triangles of maximum area, all incident to a common point.*

Proof. Denote the origin by o, and consider a unit sphere centered at o. The construction in [1] consists of a set $S = \{o\} \cup S_1 \cup S_2$ of n points, where $S_1 \cup S_2$ lies on the unit sphere, $|S_1| = \lfloor (n-1)/2 \rfloor$, $|S_2| = \lceil (n-1)/2 \rceil$, and there are $\Omega(n^{4/3})$ pairs of orthogonal segments of the form (os_i, os_j) with $s_i \in S_1$ and $s_j \in S_2$.

Moreover, this construction can be realized in such a way that S_1 lies in a small neighborhood of $(1,0,0)$, and S_2 lies in a small neighborhood of $(0,1,0)$, say. The area of every right-angled isosceles triangle Δos_is_j with $s_i \in S_1$ and $s_j \in S_2$ is $1/2$. All other triangles have smaller area: this is clear if at least two vertices of a triangle are from S_1 or from S_2; otherwise the area is given by $\frac{1}{2}\sin\alpha$, where α is the angle of the two sides incident to the origin, so the area is less than $1/2$ if these sides are not orthogonal. □

We next show that the construction in Theorem 9 is almost tight, in the sense that at most $O(n^{4/3+\varepsilon})$ maximum-area triangles can be incident to any point of an n-element point set in $\mathbb{R}^3$, for any $\varepsilon > 0$.

Theorem 10. *The number of triangles of maximum area spanned by a set S of n points in* $\mathbb{R}^3$ *and incident to a fixed point* $a \in S$ *is* $O(n^{4/3+\varepsilon})$*, for any* $\varepsilon > 0$.

Assume, without loss of generality, that the maximum area is 1. Similarly to the proof of Theorem 7, we map maximum-area triangles to point-cylinder incidences. Specifically, if Δabc is a maximum-area triangle spanned by a point set S, then every point of S lies on, or in the interior of, the cylinder with axis ab and radius $2/|ab|$ (c itself lies on the cylinder). The following lemma gives upper bounds on the number of point-cylinder incidences in this setting.

Lemma 6. *Let S be a set of n points, and* $\mathcal{C}$ *a set of m cylinders in* $\mathbb{R}^3$*, such that the axis of each cylinder passes through the origin, and no point lies in the exterior of any cylinder. Then the number of point-cylinder incidences is* $O((n^{2/3}m^{2/3} + n + m)^{1+\varepsilon})$*, for any* $\varepsilon > 0$.

The proof is omitted—it is almost identical to an argument of Edelsbrunner and Sharir [17], where it is shown that the number of point-sphere incidences between n points and m spheres in $\mathbb{R}^3$ is $O(n^{2/3}m^{2/3} + n + m)$, provided that no point lies in the exterior of any sphere. Their argument uses the fact that the complexity of the intersection of n balls in $\mathbb{R}^3$ is $O(n^2)$. We use instead a result of Halperin and Sharir [24], that the complexity of a single cell in the arrangement of n constant degree algebraic surfaces (cylinders in our case) in $\mathbb{R}^3$ is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$.

Proof of Theorem 10. Let A denote the maximum triangle area determined by a set S of n points in $\mathbb{R}^3$. For every point $a \in S$, consider the system of $n-1$ points in $S \setminus \{a\}$ and $n-1$ cylinders, each defined by a point $b \in S \setminus \{a\}$, and has axis ab and radius $2A/|ab|$. Every point-cylinder incidence corresponds to a triangle of area A spanned by S and incident to a. Since A is the maximum area, no point of S may lie in the exterior of any cylinder. By Lemma 6, the number of such triangles is $O(n^{4/3+\varepsilon})$, for any $\varepsilon > 0$. □

Theorems 9 and 10 imply the following bounds on the number of maximum-area triangles in $\mathbb{R}^3$:

Theorem 11. *The number of triangles of maximum area spanned by n points in* $\mathbb{R}^3$ *is* $O(n^{7/3+\varepsilon})$*, for any* $\varepsilon > 0$*. For all* $n \geqslant 3$*, there exist n-element point sets in* $\mathbb{R}^3$ *that span* $\Omega(n^{4/3})$ *triangles of maximum area.*

7. Distinct triangle areas in 3-space

Following earlier work by Erdős and Purdy [20], Burton and Purdy [10], and Dumitrescu and Tóth [15], Pinchasi [32] has recently proved that n noncollinear points in the plane always determine at least $\lfloor \frac{n-1}{2} \rfloor$ distinct triangle areas, which is attained by n equally spaced points distributed evenly on two parallel lines. No linear lower bound is known in 3-space, and the best we can show is the following:

Theorem 12. *Any set S of n points in $\mathbb{R}^3$, not all on a line, determines at least $\Omega(n^{2/3}/\beta(n))$ triangles of distinct areas, for some extremely slowly growing function $\beta(n)$. Moreover, all these triangles share a common side.*

For the proof, we first derive a new upper bound (Lemma 7) on the number of point-cylinder incidences in $\mathbb{R}^3$, for the special case where the axes of the cylinders pass through the origin (but without the additional requirement that no point lies outside any cylinder). Consider a set $\mathcal{C}$ of m such cylinders. These cylinders have only three degrees of freedom, and we can dualize them to points in 3-space. Specifically, we fix some generic halfspace H whose bounding plane passes through the origin, say, the halfspace $z > 0$. We then map each cylinder with axis ℓ and radius ϱ to the point on $\ell \cap H$ at distance $1/\varrho$ from the origin; and we map each point $p \in H$ to the cylinder whose axis is the line spanned by op and whose radius is $1/|op|$. As argued above, this duality preserves point-cylinder incidences.

By (a dual version of) Lemma 3, any three points can be mutually incident to at most eight cylinders whose axes pass through the origin. That is, the bipartite incidence graph (whose two classes of vertices correspond to the points of S and the cylinders of $\mathcal{C}$, respectively, and an edge represents a point-cylinder incidence) is $K_{3,9}$-free. It follows from the theorem of Kővári, Sós and Turán [26] (see also [29, p. 121]) that the number of point-cylinder incidences is $O(nm^{2/3} + m)$. We then combine this bound with the partition technique of Clarkson et al. [13], to prove a sharper upper bound on the number of point-cylinder incidences of this kind. Specifically, we have

Lemma 7. *Given n points and m cylinders, whose axes pass through the origin, in 3-space, the number of point-cylinder incidences is $O(n^{3/4}m^{3/4}\beta(n) + n + m)$.*

Proof. Let $\mathcal{C}$ be the set of the m given cylinders, and S be the set of the n given points. Let h be a plane containing the origin, but no point of S, and assume, without loss of generality, that the subset S' of points lying in the positive halfspace h^+ contributes at least half of the incidences with $\mathcal{C}$. If $m > n^3$, then the Kővári–Sós–Turán theorem yields an upper bound of $I(S', \mathcal{C}) = O(nm^{2/3} + m) = O(m)$. Similarly, if $m < n^{1/3}$, the duality mentioned above leads to the bound $I(S', \mathcal{C}) = O(mn^{2/3} + n) = O(n)$. For these two cases we have then $I(S, \mathcal{C}) \leqslant 2I(S', \mathcal{C}) = O(m + n)$. Assume henceforth that $n^{1/3} \leqslant m \leqslant n^3$.

We apply Lemma 4 with parameter $r = \lfloor n^{3/8}/m^{1/8} \rfloor$, and use the Kővári–Sós–Turán theorem to bound the number of incidences between the at most n/r^3 points and m/r cylinders in each subproblem. Note that $1 \leqslant r \leqslant m$ in the above range of m. The total number of incidences is thus

$$\begin{aligned} I(S, \mathcal{C}) &= O\left(n + mr^2\beta(r)\right) + O\left(r^3\beta(r)\right) \cdot O\left(\frac{n}{r^3} \cdot \left(\frac{m}{r}\right)^{2/3} + \frac{m}{r}\right) \\ &= O\left(n + \frac{m^{2/3}n}{r^{2/3}}\beta(n) + mr^2\beta(r)\right) = O\left(n + n^{3/4}m^{3/4}\beta(n)\right). \end{aligned}$$

Putting all three cases together gives the bound in the lemma. □

Proof of Theorem 12. If there are $n/100$ points in a plane but not all on a line, then the points in this plane already determine $\Omega(n)$ triangles of distinct areas [10]. We thus assume, in the remainder of the proof, that there are at most $n/100$ points on any plane.

According to a result of Beck [7], there is an absolute constant $k \in \mathbb{N}$ such that if no line is incident to $n/100$ points of S, then S spans $\Theta(n^2)$ distinct lines, each of which is incident to at most k points of S. Since each point of S is incident to at most $n-1$ of these lines, there is a point $a \in S$ incident to $\Theta(n)$ such lines. Select a point of $S \setminus \{a\}$ on each of these lines, to obtain a set P of $\Theta(n)$ points.

Let t denote the number of distinct triangle areas determined by S, and let $\alpha_1, \alpha_2, \ldots, \alpha_t$ denote these areas. For each point $b \in P$ and $i = 1, 2, \ldots, t$, we define a cylinder $C(ab, \alpha_i)$ with axis (the line spanned by) ab and radius $2\alpha_i/|ab|$. Every point $c \in S$ for which the area of the triangle Δabc is α_i must lie on the cylinder $C(ab, \alpha_i)$. Let $\mathcal{C}$ denote the set of the $O(nt)$ cylinders $C(ab, \alpha_i)$, for $b \in P$ and $i = 1, 2, \ldots, t$. For each point $b \in P$, there are $n - k = \Theta(n)$ points off the line through ab, each of which must lie on a cylinder $C(ab, \alpha_i)$ for some $i = 1, 2, \ldots, t$. Therefore, the number $I(S, \mathcal{C})$ of point-cylinder incidences between S and $\mathcal{C}$ is $\Omega(n^2)$. On the other hand, by Lemma 7, we have

$$\Omega\left(n^2\right) \leqslant I(S, \mathcal{C}) \leqslant O\left(n^{3/4}(nt)^{3/4}\beta(n) + n + nt\right) = O\left(n^{3/2}t^{3/4}\beta(n)\right),$$

which gives $t = \Omega(n^{2/3}/\beta^{4/3}(n)) = \Omega(n^{2/3}/\beta'(n))$, for another function $\beta'(n)$ of the same slowly growing type, as required. □

8. Conclusion

We have presented many results on the number of triangles of specific areas determined by n points in the plane or in three dimensions. Our results improve upon the previous bounds, but, most likely, many of them are not asymptotically tight. This leaves many open problems of closing the respective gaps. Even in cases where the bounds are asymptotically tight, such as those involving minimum-area triangles in two and three dimensions, determining the correct constants of proportionality still offers challenges.

Here is yet another problem on triangle areas, of a slightly different kind, with triangles determined by lines, not points (motivated in fact by the question of bounding $|U_2|$ in the proof of Theorem 1). Any three nonconcurrent, and pairwise nonparallel lines in the plane determine a triangle of positive area. What is the maximum number of unit area triangles determined by n lines in the plane?

Theorem 13. *The maximum number of unit-area triangles determined by n lines in the plane is $O(n^{7/3})$, and for any $n \geqslant 3$, there are n lines that determine $\Omega(n^2)$ unit-area triangles.*

Proof. *Lower bound*: Place $n/3$ equidistant parallel lines at angles 0, $\pi/3$, and $2\pi/3$, through the points of an appropriate section of the triangular lattice, and observe that there are $\Omega(n^2)$ equilateral triangles of unit side (i.e., of the same area) in this construction.

Upper bound: Let L be a set of n lines in the plane. We define a variant of the hyperbolas used in the proof of Theorem 1: For any pair of nonparallel lines $\ell_1, \ell_2 \in L$, let $\gamma(\ell_1, \ell_2)$ denote the locus of points $p \in \mathbb{R}^2$, $p \notin \ell_1 \cup \ell_2$, such that the parallelogram that has a vertex at p and two sides along ℓ_1 and ℓ_2, respectively, has area $1/2$. The set $\gamma(\ell_1, \ell_2)$ is the union of two hyperbolas with ℓ_1 and ℓ_2 as asymptotes (four connected branches in total). Any two nonparallel lines uniquely determine two such hyperbolas. Let Γ denote the set of the branches of these hyperbolas, and note that $|\Gamma| = O(n^2)$. Observe now that, if ℓ_1, ℓ_2, and ℓ_3 determine a unit area triangle, then ℓ_3 is tangent to one of the two hyperbolas in $\gamma(\ell_1, \ell_2)$.

We first derive a weaker bound. Construct two bipartite graphs $G_1, G_2 \subseteq L \times \Gamma$. We put an edge (ℓ, γ) in G_1 (resp., G_2) if ℓ is tangent to γ and ℓ lies below (resp., above) γ. The edges of G_1 and G_2 account for all line-curve tangencies. Observe that neither graph contains a $K_{5,2}$, that is, there cannot be five distinct lines in L tangent to two branches of hyperbolas from above (or from below). Indeed, this would force the two branches to intersect at five points, which is impossible for a pair of distinct quadrics. It thus follows from the theorem of Kővári, Sós and Turán [26] (see also [29, p. 121]) that the number of line-hyperbola tangencies between any n_0 lines in L and any m_0 hyperbolas in Γ is $O(n_0 m_0^{4/5} + m_0)$. With $n_0 = n$ and $m_0 = O(n^2)$, this already gives a bound of $O(n \cdot n^{8/5} + n^2) = O(n^{13/5})$

on the number of unit-area triangles determined by n lines in the plane. We next derive an improved bound.

Let L be the given set of n lines, and let Γ be the corresponding set of $m = O(n^2)$ hyperbola branches. We can assume that no line in L is vertical, and apply a standard duality which maps each line $\ell \in L$ to a point ℓ^*. A hyperbolic branch γ is then mapped to a curve γ^*, which is the locus of all points dual to lines tangent to γ; it is easily checked that each γ^* is a quadric. Let L^* denote the set of the n dual points, and let Γ^* denote the set of $m = O(n^2)$ dual curves. A line-hyperbola tangency in the primal plane is then mapped to a point-curve incidence in the dual plane.

We next construct a $(1/r)$-cutting for Γ^*, partitioning the plane into $O(r^2)$ relatively open cells of bounded description complexity, each of which contains at most n/r^2 points and is crossed by at most m/r curves. By using the previous bound for each cell, the total number of incidences involving points in the interior of these cells is

$$O\left(r^2\left(\frac{n}{r^2}\left(\frac{m}{r}\right)^{4/5}+\frac{m}{r}\right)\right)=O\left(n\left(\frac{m}{r}\right)^{4/5}+mr\right).$$

We balance the two terms by setting $r = n^{5/9}/m^{1/9}$, and observe that $1 \leqslant r \leqslant m$ if $m \leqslant n^5$ and $n \leqslant m^2$; since $m = \Theta(n^2)$, both inequalities do hold in our case. Hence, the total number of incidences under consideration is $O(m^{8/9}n^{5/9}) = O(n^{7/3})$.

It remains to bound the overall number of incidences involving points lying on the boundaries of at least two cells. A standard argument, which we omit, shows that the number of these incidences is also $O(n^{7/3})$, and thereby completes the proof of the theorem. □

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