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Algebraic independence in $SL(3, \mathbb{C})$ character varieties of free groups

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ABSTRACT

Let \mathfrak{X}_r be the moduli space of $SL(3, \mathbb{C})$ representations of a free group of rank r . In this paper we describe maximal algebraically independent subsets of certain minimal sets of coordinate functions on \mathfrak{X}_r . These subsets locally parametrize the moduli space.

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1. Introduction

It is the purpose of this paper to describe local parameters (maximal independent coordinates) for the moduli space of $SL(3, \mathbb{C})$ -representations of free groups of arbitrary rank. We begin by reviewing a recent description of global coordinates of this moduli space (universal parameter space), and then apply the method of [ATZ94] to prove that certain subsets of these coordinate functions admit no (polynomial) relations. This establishes that these subsets are algebraically independent. They are maximal with this property and at generic non-singular points provide local parameters that determine linearly independent tangent vectors.

Let $F_r = \langle g_1, \dots, g_r \rangle$ be a rank r free group (non-abelian). Any homomorphism $\rho : F_r \rightarrow SL(m, \mathbb{C}) \subset GL(m, \mathbb{C})$ is a complex m -dimensional representation of F_r . We call the set $\mathfrak{R}_r = \text{Hom}(F_r, SL(m, \mathbb{C}))$ the $SL(m, \mathbb{C})$ -representation variety of F_r . The evaluation map,

$$\rho \mapsto (\rho(g_1), \dots, \rho(g_r)),$$

gives a bijection between \mathfrak{R}_r and $SL(m, \mathbb{C})^{\times r}$. Since $SL(m, \mathbb{C})$ is a smooth affine variety (irreducible, non-singular, algebraic set), \mathfrak{R}_r is likewise a smooth affine variety.

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Let $\mathfrak{G} = \text{SL}(m, \mathbb{C})$. The conjugation action of \mathfrak{G} on \mathfrak{X}_r is rational; that is, $\mathfrak{G} \times \mathfrak{X}_r \rightarrow \mathfrak{X}_r$ is regular, or the mapping is by polynomials in the matrix entries of \mathfrak{G} in the coordinates of \mathfrak{X}_r . In particular, this action is either $(g, \rho) \mapsto g\rho g^{-1}$ or

$$(g, (\rho(g_1), \dots, \rho(g_r))) \mapsto (g\rho(g_1)g^{-1}, \dots, g\rho(g_r)g^{-1})$$

depending on whether we are working with $\text{Hom}(F_r, \mathfrak{G})$ or $\mathfrak{G}^{\times r}$, respectively. We often switch back and forth as is convenient.

Let K be a compact Lie group. Then K is a real algebraic group which embeds in $O(n, \mathbb{R})$ for some n . Since K is algebraic there is an ideal \mathfrak{J} in the real coordinate ring $\mathbb{R}[O(n, \mathbb{R})]$ defining its points. Let G be the complex zeros of \mathfrak{J} . Then G is a complex algebraic subgroup of $O(n, \mathbb{C})$ (denoted by $K_{\mathbb{C}}$ and called the *complexification* of K) with coordinate ring $\mathbb{C}[G] = \mathbb{R}[K] \otimes_{\mathbb{R}} \mathbb{C}$. Any complex algebraic group G which arises in this fashion is called reductive. The “unitary trick” shows $\text{SL}(m, \mathbb{C})$ is reductive. We note that this definition, although not the most general, coincides with all more general notions of reductivity when the algebraic group is complex. In particular, another equivalent definition is that a complex algebraic group G is reductive if for every finite dimensional representation of G all subrepresentations have invariant complements. The important observation is that such groups act like and have the algebraic structure of compact groups. See [Sch01].

A theorem of Nagata [Nag64] says that if a reductive group acts on a finitely generated domain A , the subdomain of invariants $A^G = \{a \in A \mid g \cdot a = a\}$ is likewise finitely generated. This is one answer to Hilbert’s fourteenth problem.

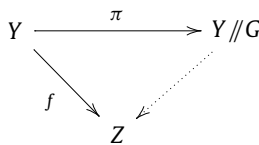
Since \mathfrak{X}_r is an affine variety, its coordinate ring $\mathbb{C}[\mathfrak{X}_r]$ is a finitely generated domain, and since \mathfrak{G} acts on \mathfrak{X}_r it also acts on $\mathbb{C}[\mathfrak{X}_r]$ by $(g, f(\rho)) \mapsto f(g^{-1}\rho g)$. Since $\text{SU}(m)_{\mathbb{C}} = \text{SL}(m, \mathbb{C})$, \mathfrak{G} is reductive. Thus $\mathbb{C}[\mathfrak{X}_r]^{\mathfrak{G}}$ is a finitely generated domain, and consequently we define

$$\mathfrak{X}_r = \text{Spec}_{\max}(\mathbb{C}[\mathfrak{X}_r]^{\mathfrak{G}}),$$

the set of maximal ideals, to be the \mathfrak{G} -character variety of F_r .

It can be shown that \mathfrak{X}_r is the categorical quotient $\mathfrak{X}_r // \mathfrak{G}$ in the category of affine varieties (or Hausdorff spaces or complex analytic varieties [Lun75, Lun76]). We recall the definition of a categorical quotient to be concrete.

Definition 1. A categorical quotient of a variety Y with an algebraic group G acting rationally is an object $Y // G$ and a G -invariant morphism $\pi : Y \rightarrow Y // G$ such that the following commutative diagram exists uniquely for all invariant morphisms $f : Y \rightarrow Z$:



It is a *good* categorical quotient if the following conditions also hold:

- (i) for open subsets $U \subset Y // G$, $\mathbb{C}[U] \approx \mathbb{C}[\pi^{-1}(U)]^G$;
- (ii) π maps closed invariant sets to closed sets;
- (iii) π separates closed invariant sets.

When G is reductive and Y is an affine variety, then $Y \rightarrow \text{Spec}_{\max}(\mathbb{C}[Y]^G)$ is a good categorical quotient. See [Dol03] for details.

It is worth giving a more down to earth description of this quotient. There is a one-to-one correspondence between the points of \mathfrak{X}_r and the orbits of *completely reducible* representations (representations that are sums of irreducibles); these are the points whose orbits are closed. Any representation can be continuously and conjugate-invariantly deformed to one that is completely reducible, so the points of \mathfrak{X}_r are unions of orbits of representations that are deformable in this way. Such a union is called an *extended orbit equivalence class*. The character variety \mathfrak{X}_r may be accurately thought of as either the usual orbit space of \mathfrak{R}_r with the non-completely reducible representations removed, or as the usual orbit space with extended orbit equivalences. Either way, the resulting space is, or is in one-to-one correspondence with, an affine algebraic set, irreducible and singular, that satisfies the diagrammatic requirements needed to be a categorical quotient.

Any such reductive quotient has an affine lift (see [MFK94]). In other words, there is an affine space \mathbb{A}^N for some potentially large N where $\mathfrak{R}_r \subset \mathbb{A}^N$ and where the action of \mathfrak{G} extends. Then

$$\Pi : \mathbb{C}[\mathbb{A}^N] \longrightarrow \mathbb{C}[\mathfrak{R}_r]$$

and more importantly

$$\Pi_{\mathfrak{G}} : \mathbb{C}[\mathbb{A}^N // \mathfrak{G}] \longrightarrow \mathbb{C}[\mathfrak{R}_r // \mathfrak{G}]$$

are surjective morphisms. We may take \mathbb{A}^N in our case to be $\mathfrak{gl}(m, \mathbb{C})^{\times r}$ and the action of \mathfrak{G} to be, as it is on $\mathfrak{G}^{\times r}$, diagonal conjugation.

The coordinate ring of this affine space is

$$\mathbb{C}[\mathfrak{gl}(m, \mathbb{C})^{\times r}] = \mathbb{C}[x_{ij}^k \mid 1 \leq i, j \leq m, 1 \leq k \leq r],$$

the complex polynomial ring in rm^2 variables.

Let

$$\mathbf{X}_k = \begin{pmatrix} x_{11}^k & x_{12}^k & \cdots & x_{1m}^k \\ x_{21}^k & x_{22}^k & \cdots & x_{2m}^k \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1}^k & x_{m2}^k & \cdots & x_{mm}^k \end{pmatrix}$$

be a *generic matrix* of size $m \times m$.

In 1976 Procesi [Pro76] proves

Theorem 2 (1-st fundamental theorem of invariants of $m \times m$ matrices).

$$\mathbb{C}[\mathfrak{gl}(m, \mathbb{C})^{\times r} // \mathfrak{G}] = \mathbb{C}[\text{tr}(\mathbf{X}_{i_1} \mathbf{X}_{i_2} \cdots \mathbf{X}_{i_l}) \mid 1 \leq l \leq d(m)],$$

where $d(m)$ is a fixed positive integer dependent only on m .

However, according to [For02] the above result had been discovered earlier in [Kir67].

The number $d(m)$ is called the *degree of nilpotency*. The only values known are $d(2) = 3$, $d(3) = 6$ and $d(4) = 10$, but it is known that $d(m) \leq m^2$ (see [Raz74]). Therefore, $\mathbb{C}[\mathfrak{R}_m // \mathfrak{G}]$ is generated by $\Pi(\text{tr}(\mathbf{X}_{i_1} \mathbf{X}_{i_2} \cdots \mathbf{X}_{i_l})) = \text{tr}(\widehat{\mathbf{X}}_{i_1} \widehat{\mathbf{X}}_{i_2} \cdots \widehat{\mathbf{X}}_{i_l})$ where $\widehat{\mathbf{X}}_k = (x_{ij}^k)$ and $x_{ij}^k = \Pi(x_{ij}^k) \in \mathbb{C}[\mathfrak{R}_m]$.

In [Law08] minimal generators for $\mathbb{C}[\text{SL}(3, \mathbb{C})^{\times r} // \text{SL}(3, \mathbb{C})]$ are constructed, providing global coordinates for its spectrum of maximal ideals. The main theorem from [Law08] is

Theorem 3. $\mathbb{C}[\mathrm{SL}(3, \mathbb{C})^{\times r} // \mathrm{SL}(3, \mathbb{C})]$ is minimally generated by $\binom{r}{1}$ invariants of the form $\mathrm{tr}(\mathbf{X})$, $\binom{r}{1}$ invariants of the form $\mathrm{tr}(\mathbf{X}^{-1})$, $\binom{r}{2}$ invariants of the form $\mathrm{tr}(\mathbf{XY})$, $2\binom{r}{2}$ invariants of the form $\mathrm{tr}(\mathbf{XY}^{-1})$, $\binom{r}{2}$ invariants of the form $\mathrm{tr}(\mathbf{X}^{-1}\mathbf{Y}^{-1})$, $\binom{r}{2}$ invariants of the form $\mathrm{tr}(\mathbf{XYX}^{-1}\mathbf{Y}^{-1})$, $2\binom{r}{3}$ invariants of the form $\mathrm{tr}(\mathbf{XYZ})$, $6\binom{r}{3}$ invariants of the form $\mathrm{tr}(\mathbf{XYZ}^{-1})$, $3\binom{r}{3}$ invariants of the form $\mathrm{tr}(\mathbf{XYZY}^{-1})$, $6\binom{r}{3}$ invariants of the form $\mathrm{tr}(\mathbf{XY}^{-1}\mathbf{Z}^{-1})$, $6\binom{r}{3}$ invariants of the form $\mathrm{tr}(\mathbf{XYZ}^{-1}\mathbf{Y}^{-1})$, $\binom{r}{3}$ invariants of the form $\mathrm{tr}(\mathbf{X}^{-1}\mathbf{Y}^{-1}\mathbf{Z}^{-1})$, $5\binom{r}{4}$ invariants of the form $\mathrm{tr}(\mathbf{WXYZ})$, $20\binom{r}{4}$ invariants of the form $\mathrm{tr}(\mathbf{WXYZ}^{-1})$, $18\binom{r}{4}$ invariants of the form $\mathrm{tr}(\mathbf{WXY}^{-1}\mathbf{Z}^{-1})$, $8\binom{r}{4}$ invariants of the form $\mathrm{tr}(\mathbf{WXYZ}^{-1})$, $12\binom{r}{5}$ invariants of the form $\mathrm{tr}(\mathbf{UVWXY})$, $35\binom{r}{5}$ invariants of the form $\mathrm{tr}(\mathbf{VWXYZ}^{-1})$, and $15\binom{r}{6}$ invariants of the form $\mathrm{tr}(\mathbf{UVWXYZ})$.

Counting the various generator types we conclude

Corollary 4. The number of minimal generators for $\mathbb{C}[\mathrm{SL}(3, \mathbb{C})^{\times r}]^{\mathrm{SL}(3, \mathbb{C})}$ is

$$N_r = \frac{r}{240} (396 + 65r^2 - 5r^3 + 19r^4 + 5r^5),$$

and enumerating the set of generators from Theorem 3 by $\{t_1, \dots, t_{N_r}\}$ we have

$$\mathcal{T}_{N_r} = (t_1, \dots, t_{N_r}) : \mathfrak{X}_r \hookrightarrow \mathbb{C}^{N_r}$$

is an affine embedding where N_r is minimal among all affine embeddings $\mathfrak{X}_r \rightarrow \mathbb{C}^N$.

2. Algebraic independence

Using methods from [ATZ94] (see also [Ter88]) we pick out maximal subsets of these minimal generators that are algebraically independent.

Definition 5. Let $k \subset K$ be fields and $B \subset K$ be a set of elements. Then the set B is algebraically independent over k if for any positive integer n , any non-zero polynomial $f(x_1, \dots, x_n)$ with coefficients in k , and any set of n distinct elements in B , denoted by b_1, \dots, b_n , we have $f(b_1, \dots, b_n) \neq 0$.

Any maximal set of algebraically independent elements has the same cardinality (the *transcendence degree*) and such a maximal set is called a *transcendence basis*.

For an affine variety X over \mathbb{C} , the dimension of X (called the *Krull dimension*) is equal to the maximal number of independent rational functions on X ; that is, the transcendence degree of the quotient field $\mathbb{C}(X)$ over \mathbb{C} . In this case, it is also equal to the common length of all maximal chains of prime ideals in the coordinate ring $\mathbb{C}[X]$, and also to the dimension of a tangent space at a smooth point.

Suppose the transcendence basis $\{f_1, \dots, f_d\}$ is in fact in the coordinate ring $\mathbb{C}[X] \subset \mathbb{C}(X)$. In general a set of functions whose differentials give a basis for the cotangent space at a point is called a set of *local parameters*. In [Sha94] it is shown that local parameters $\{x_1, \dots, x_d\}$ exist and that a set of functions $\{f_1, \dots, f_d\}$ in the local coordinate ring can be written in their terms. The Jacobian is the determinant of the $d \times d$ matrix $(\partial f_i / \partial x_j)$. If the Jacobian is not identically zero on X , then at a generic smooth point p (themselves generic in X) the differentials $d(f_i - f_i(p))$ of these coordinate functions, translated to p , give a basis for $T_p^*(X)$ and by duality for $T_p(X)$. Any such set will necessarily be algebraically independent.

Since irreducible representations $\mathbb{F}_r \rightarrow \mathfrak{G}$ are generic in \mathfrak{X}_r (for $r \geq 2$), it follows that generic orbits have dimension equal to $\dim \mathfrak{G} = m^2 - 1$. Thus the Krull dimension of \mathfrak{X}_r is $(m^2 - 1)(r - 1)$, as long as $r \geq 2$. It then follows that at a non-singular point $[\rho]$ of \mathfrak{X}_r (these are exactly the irreducibles) we wish to obtain $(m^2 - 1)(r - 1)$ generators of $\mathbb{C}[\mathfrak{X}_r]$ whose Jacobian is non-zero on \mathfrak{X}_r . This generically provides $(m^2 - 1)(r - 1)$ linearly independent vectors in $T_{[\rho]}(\mathfrak{X}_r)$ and establishes these invariants are algebraically independent.

For the remainder of this section let $\mathfrak{G} = \text{SL}(3, \mathbb{C})$.
 Our main theorem is

Theorem 6. Let $\mathcal{A} = \{\text{tr}(\mathbf{X}_1), \text{tr}(\mathbf{X}_2), \text{tr}(\mathbf{X}_1^{-1}), \text{tr}(\mathbf{X}_2^{-1}), \text{tr}(\mathbf{X}_1\mathbf{X}_2), \text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_2), \text{tr}(\mathbf{X}_2^{-1}\mathbf{X}_1), \text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_2^{-1})\}$, and $\mathcal{B} = \{\text{tr}(\mathbf{X}_3), \text{tr}(\mathbf{X}_3^{-1}), \text{tr}(\mathbf{X}_1\mathbf{X}_3), \text{tr}(\mathbf{X}_2\mathbf{X}_3), \text{tr}(\mathbf{X}_1\mathbf{X}_3^{-1}), \dots, \text{tr}(\mathbf{X}_r), \text{tr}(\mathbf{X}_r^{-1}), \text{tr}(\mathbf{X}_1\mathbf{X}_r), \text{tr}(\mathbf{X}_2\mathbf{X}_r), \text{tr}(\mathbf{X}_1\mathbf{X}_r^{-1})\}$. Then $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ are local parameters from the minimal generators from Theorem 3, where \mathcal{C} is any one of:

$$\begin{aligned} & \{\text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_3^{-1}), \text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_3), \text{tr}(\mathbf{X}_2^{-1}\mathbf{X}_3^{-1}), \dots, \text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_r^{-1}), \text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_r), \text{tr}(\mathbf{X}_2^{-1}\mathbf{X}_r^{-1})\}, \\ & \{\text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_3^{-1}), \text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_3), \text{tr}(\mathbf{X}_2^{-1}\mathbf{X}_3), \dots, \text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_r^{-1}), \text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_r), \text{tr}(\mathbf{X}_2^{-1}\mathbf{X}_r)\}, \\ & \{\text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_3^{-1}), \text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_3), \text{tr}(\mathbf{X}_2\mathbf{X}_3^{-1}), \dots, \text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_r^{-1}), \text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_r), \text{tr}(\mathbf{X}_2\mathbf{X}_r^{-1})\}, \\ & \{\text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_3), \text{tr}(\mathbf{X}_2^{-1}\mathbf{X}_3), \text{tr}(\mathbf{X}_2\mathbf{X}_3^{-1}), \dots, \text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_r), \text{tr}(\mathbf{X}_2^{-1}\mathbf{X}_r), \text{tr}(\mathbf{X}_2\mathbf{X}_r^{-1})\}, \\ & \{\text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_3^{-1}), \text{tr}(\mathbf{X}_2\mathbf{X}_3^{-1}), \text{tr}(\mathbf{X}_2^{-1}\mathbf{X}_3^{-1}), \dots, \text{tr}(\mathbf{X}_1^{-1}\mathbf{X}_r^{-1}), \text{tr}(\mathbf{X}_2\mathbf{X}_r^{-1}), \text{tr}(\mathbf{X}_2^{-1}\mathbf{X}_r^{-1})\}. \end{aligned}$$

In all cases these sets number the Krull dimension of \mathfrak{X}_r which is $8r - 8$. Consequently, they are maximally algebraically independent in $\mathbb{C}[\mathfrak{X}_r]$.

Proof. The outline of the proof is as follows. We will proceed by induction. For $r = 1$ the number of minimal generators equals the dimension of \mathfrak{X}_1 so there cannot be any relations at all, and for $r = 2$ Theorem 6 was shown earlier in [Law07]. For $r \geq 3$ we calculate the Jacobian matrix of the $8r - 8$ trace functions is the statement of Theorem 6 in the following $8r - 8$ independent variables:

- (a) x_{11}^1, x_{22}^1 from \mathbf{X}_1 ,
- (b) $x_{11}^2, x_{21}^2, x_{13}^2, x_{22}^2, x_{23}^2, x_{33}^2$ from \mathbf{X}_2 ,
- (c) $x_{11}^k, x_{12}^k, x_{13}^k, x_{21}^k, x_{22}^k, x_{23}^k, x_{32}^k, x_{33}^k$ from \mathbf{X}_k for $3 \leq k \leq r$.

Call the set of matrix elements from the above list \mathcal{V} . We then show the determinant of the Jacobian is generically non-zero. This will establish independence.

We now proceed with the proof. First, we justify our choice of matrix elements $\mathcal{V} \subset \{x_{ij}^k\}$ by showing them to be independent. Generically, we can assume that \mathbf{X}_1 is diagonalizable and conjugate it into diagonal form. Since its determinant is 1, we may write $x_{33}^1 = 1/(x_{22}^1 x_{11}^1)$. Therefore, we need only x_{11}^1 and x_{22}^1 from \mathbf{X}_1 .

We may still conjugate by any matrix that preserves this normal form, such as by diagonal matrices. Doing so we will show we can assume the conjugation orbit of \mathbf{X}_2 is independent of its lower diagonal. The entry x_{31}^2 is generically a function of the other matrix elements from \mathbf{X}_2 since we may solve for it in the expression $\det(\mathbf{X}_2) = 1$. In any event we can always choose either this entry or an element other than the lower diagonal to solve the determinant. Now we are free to show that the lower diagonal may be assumed to be parameters that vary only in the conjugation orbit. Indeed, conjugate by

$$\mathbf{D} = \begin{pmatrix} s & 0 & 0 \\ 0 & x_{21}^2 & 0 \\ 0 & 0 & \frac{x_{21}^2 x_{32}^2}{t} \end{pmatrix}.$$

This matrix has non-zero determinant in the free \mathbb{C}^* parameters s and t as long as the lower diagonal of \mathbf{X}_2 is non-zero (generically the case). Otherwise, the lower diagonal is fixed or we can slightly change the matrix \mathbf{D} to take advantage of one of the zeros on the lower diagonal (when there is only one zero).

Then

$$D^{-1}X_2D = \begin{pmatrix} x_{11}^2 & \frac{x_{12}^2 x_{21}^2}{s} & \frac{x_{13}^2 x_{21}^2 x_{32}^2}{st} \\ s & x_{22}^2 & \frac{x_{23}^2 x_{32}^2}{t} \\ \frac{x_{31}^2 st}{x_{21}^2 x_{32}^2} & t & x_{33}^2 \end{pmatrix}.$$

As s and t vary, X_1 is fixed (recall it is now diagonal) and the orbit of X_2 is fixed.

Thus performing this change of variables shows that we may assume without loss of generality that $x_{11}^1, x_{22}^1, x_{ij}^2 (1 \leq i \leq j \leq 3)$ are independent variables for \mathfrak{X}_2 . Now for $k \geq 3$ we can likewise solve $\det(X_k) = 1$ for x_{31}^k (generically). We thus conclude that x_{ij}^k , where $1 \leq i, j \leq 3$ and $(i, j) \neq (3, 1)$ for $k \geq 3$, provide the additional independent parameters for \mathfrak{X}_r when $r \geq 3$.

Heuristically, their independence follows since they total $8r - 8 = \dim \mathfrak{X}_r$ and the conjugation action (generically having orbits of dimension 8) has been entirely accounted for (6 degrees of freedom used on X_1 and 2 degrees of freedom used on X_2). A more precise way to say this is that the order $8r - 8$ subset of polynomial indeterminates $\mathcal{V} \subset \{x_{ij}^k\} \subset \mathbb{C}[\mathfrak{X}]$ are independent in $\mathbb{C}[\mathfrak{X}]$, and so are independent in $\mathbb{C}[\mathfrak{X}] = \mathbb{C}[\mathfrak{X}]^{\mathfrak{G}}$ as long as they (generically) distinguish orbits. Since we just (constructively) showed that this set does generically distinguish all orbits (their closures to be precise), then we have shown independence. The set \mathcal{V} is a set of parameters we can now work with.

We now say a word or two about the Jacobian matrix. The functions $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ are generically functions of the variables \mathcal{V} . Letting f_1, \dots, f_{8r-8} be the trace functions from $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ and z_1, \dots, z_{8r-8} be the variables \mathcal{V} , the Jacobian is the $(8r - 8) \times (8r - 8)$ matrix of partial derivatives $(\frac{\partial f_i}{\partial z_j})$. If there is a dependence relation among the f_i 's then locally (that is on an open affine subset) the Jacobian will have zero determinant since locally these functions cannot give a full dimensional tangent space. So computing this determinant and finding it non-zero at a general point in a Zariski open set shows the functions are algebraically independent.

We now proceed with the induction. The cases $r = 1, 2$ are done. Suppose now that $r_0 \geq 3$ and that for all $r < r_0$ the Jacobian is non-singular.

Putting the 8 trace functions which are in terms of X_{r_0} in the last 8 rows and the 8 variables $x_{ij}^{r_0}$ in the last 8 columns we get a block diagonal matrix $\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$ and so by induction M is non-singular. The $8(r - 2) \times 8$ block of zeros arises since the first $8(r - 2)$ trace functions are constant with respect to X_{r_0} and the last 8 columns come from differentiating with respect to the indeterminates $x_{ij}^{r_0}$. The block form of the matrix implies that its determinant is the product of the determinant of M and that of N . It remains to show that the last eight traces are independent in the variables from the last matrix X_r ; that is, the 8×8 matrix N is non-singular (generically).

For all choices of \mathcal{C} , using *Mathematica* (code is available upon request), we calculate this subdeterminant and evaluate at random unimodular matrices (those with determinant 1); finding it non-zero. If there was a relation the determinant would be identically zero and so any non-zero evaluation shows independence. We note that we only need to test the $r = 3$ case since all 8×8 lower right blocks N are identical excepting labels. \square

Remark 7. Using the same method as above it is not hard to prove that the natural affine lifts (replace exponents of -1 with exponents of 2) of the generators in the main theorem with $\{\text{tr}(X_1^2), \dots, \text{tr}(X_r^2)\}$ added to the sets form a maximal algebraically independent set for $\mathfrak{gl}(3, \mathbb{C})^{\times r} // \mathfrak{G}$. For instance,

$$\begin{aligned} & \{ \text{tr}(X_1), \text{tr}(X_2), \text{tr}(X_1^2), \text{tr}(X_2^2), \text{tr}(X_1^3), \text{tr}(X_2^3), \text{tr}(X_1 X_2), \text{tr}(X_1^2 X_2), \text{tr}(X_2^2 X_1), \\ & \text{tr}(X_1^2 X_2^2), \text{tr}(X_3), \text{tr}(X_3^2), \text{tr}(X_3^3), \text{tr}(X_1 X_3), \text{tr}(X_2 X_3), \text{tr}(X_1 X_3^2), \text{tr}(X_1^2 X_3^2), \\ & \text{tr}(X_1^2 X_3), \text{tr}(X_2^2 X_3^2), \dots, \text{tr}(X_r), \text{tr}(X_r^2), \text{tr}(X_r^3), \text{tr}(X_1 X_r), \text{tr}(X_2 X_r), \text{tr}(X_1 X_r^2), \\ & \text{tr}(X_1^2 X_r^2), \text{tr}(X_1^2 X_r), \text{tr}(X_2^2 X_r^2) \}, \end{aligned}$$

are independent and maximal.

3. Pulling back parameters and the Magnus trace map

In this section, let $\mathfrak{Y}_r = \mathfrak{gl}(m, \mathbb{C})^r // \mathrm{SL}(m, \mathbb{C})$, $\mathfrak{X}_r = \mathrm{SL}(m, \mathbb{C})^r // \mathrm{SL}(m, \mathbb{C})$, and

$$\Pi_{\mathfrak{G}} : \mathbb{C}[\mathfrak{Y}_r] \rightarrow \mathbb{C}[\mathfrak{X}_r] \approx \mathbb{C}[\mathfrak{Y}_r] / (\det(\mathbf{X}_1) - 1, \dots, \det(\mathbf{X}_r) - 1)$$

be the projection discussed in the introduction.

A set of generators for $\mathbb{C}[\mathfrak{Y}_r]$ or $\mathbb{C}[\mathfrak{X}_r]$ of the form $\{\mathrm{tr}(A_{i_1} A_{i_2} \cdots A_{i_j})\}$ are called *Procesi generators*, if additionally no generator has the form $\mathrm{tr}(\mathbf{W}_1 \mathbf{X}^m \mathbf{W}_2)$ where at least one of the words \mathbf{W}_i is not the identity (for $\mathbb{C}[\mathfrak{X}_r]$ we additionally require this if both $\mathbf{W}_i = \mathbf{I}$). Using the characteristic polynomial $\sum c_k(\mathbf{X}) \mathbf{X}^{n-k} = 0$ one can always arrange for any set of minimal generators of $\mathbb{C}[\mathfrak{Y}_r]$ or $\mathbb{C}[\mathfrak{X}_r]$ to be Procesi generators. We will call a maximal set of algebraically independent Procesi generators *Procesi parameters*. Such a maximal set always has order equal to the Krull dimension of \mathfrak{Y}_r or \mathfrak{X}_r , respectively.

In [Law08] we show that minimal Procesi generators of $\mathbb{C}[\mathfrak{gl}(m, \mathbb{C})^{\times r} // \mathrm{SL}(m, \mathbb{C})]$ project by $\Pi_{\mathfrak{G}}$ to minimal generators of $\mathbb{C}[\mathrm{SL}(m, \mathbb{C})^{\times r} // \mathrm{SL}(m, \mathbb{C})]$. One may show that a set of algebraically independent Procesi generators of $\mathrm{SL}(2, \mathbb{C})^{\times r} // \mathrm{SL}(2, \mathbb{C})$ pull back by $\Pi_{\mathrm{SL}(2, \mathbb{C})}$ to algebraically independent generators of $\mathfrak{gl}(2, \mathbb{C})^{\times r} // \mathrm{SL}(2, \mathbb{C})$, and the above work and remark from the last section shows the same is true for $\mathrm{SL}(3, \mathbb{C})$.

We now prove

Theorem 8. *Let $B \subset \mathbb{C}[\mathfrak{X}_r]$ be a set of Procesi parameters for \mathfrak{X}_r . Then there is a collection of elements $\bar{B} \subset \Pi_{\mathfrak{G}}^{-1}(B)$ so that $\{\mathrm{tr}(\mathbf{X}_1^m), \dots, \mathrm{tr}(\mathbf{X}_r^m)\} \cup \bar{B} \subset \mathbb{C}[\mathfrak{Y}_r]$ is a set of Procesi parameters for \mathfrak{Y}_r . Said shortly, π pulls back Procesi parameters.*

Proof. By assumption the parameters for \mathfrak{X}_r are given by traces of words in generic matrices where all letters have positive exponents; that is in the form $\mathrm{tr}(\mathbf{X}_{i_1} \mathbf{X}_{i_2} \cdots \mathbf{X}_{i_j})$, and the collection B does not include any parameter of the form $\mathrm{tr}(\mathbf{X}^m)$. Then these invariants extend to invariants of \mathfrak{Y}_r ; that is, for such generators, $\Pi_{\mathfrak{G}}(\mathrm{tr}(\mathbf{W})) = \mathrm{tr}(\mathbf{W})|_{\mathfrak{X}_r}$. If there was a dependence relation among these extensions to all of \mathfrak{Y}_r , then restricting would provide a dependence relation on all of \mathfrak{X}_r , which cannot exist by assumption. Thus B pulls back to an independent set \bar{B} .

Let $\mathbb{C}[\bar{B}]$ be the subring of $\mathbb{C}[\mathfrak{Y}_r]$ generated by \bar{B} . The collection $\{\mathrm{tr}(\mathbf{X}_k^{m-j})\}$ for $1 \leq j \leq m - 1$ is included in \bar{B} , since they must be included in any set of Procesi parameters for \mathfrak{X}_r ; the latter following since the Procesi parameters for \mathfrak{X}_1 are unique, and setting $\mathbf{X}_2 = \cdots = \mathbf{X}_r = \mathbf{I}$ recovers $\mathbb{C}[\mathfrak{X}_1]$ as a subring of $\mathbb{C}[\mathfrak{X}_r]$. Note that they are independent since $\mathfrak{gl}(m, \mathbb{C}) // \mathrm{SL}(m, \mathbb{C}) = \mathbb{C}^m$. Suppose $\mathbb{C}[\bar{B}][\mathrm{tr}(\mathbf{X}_1^m), \dots, \mathrm{tr}(\mathbf{X}_r^m)] \subset \mathbb{C}[\mathfrak{Y}_r]$ is an algebraic extension of $\mathbb{C}[\bar{B}]$. Then locally, using the characteristic polynomial, we can express the determinant in terms of the other coefficients of the characteristic polynomial, which we know to be impossible generally, since the coefficients of the characteristic polynomial are already independent. For example, by letting all elements \mathbf{X}_j be the identity except for one, say \mathbf{X}_{i_0} , we would be able to locally ascertain $\det(\mathbf{X}_{i_0})$ from $\mathrm{tr}(\mathbf{X}_{i_0}^{m-k})$ for $1 \leq k \leq m - 1$; which we cannot. Thus, it is a transcendental extension. Therefore, $\bar{B} \cup \{\mathrm{tr}(\mathbf{X}_1^m), \dots, \mathrm{tr}(\mathbf{X}_r^m)\}$ are algebraically independent and since their number $(m^2 - 1)(r - 1) + r = (r - 1)m^2 + 1 = \dim \mathfrak{Y}_r$, they are a set of parameters for \mathfrak{Y}_r . \square

We also take a moment to contrast the fact in [Law08] that N_r minimal generators provide an affine embedding $\mathfrak{X}_r \hookrightarrow \mathbb{C}^{N_r}$, where N_r is minimal among all such embeddings.

Given a set of independent generators \mathcal{I} of order $|\mathcal{I}|$ one can construct a morphism $\mathrm{MagTr}_{\mathcal{I}} : \mathfrak{X}_r \rightarrow \mathbb{C}^{|\mathcal{I}|}$, called the *Magnus trace map* in [Flo06]. This map factors through the affine embedding $\mathfrak{X}_r \rightarrow \mathbb{C}^{N_r}$ with a projection when \mathcal{I} is a subset (as it can always be taken to be) of a minimal set of generators defining the embedding.

It is not hard to prove

Proposition 9. *$\mathrm{MagTr}_{\mathcal{I}}$ is dominant and generically submersive.*

Proof. Since the algebraically independent generators induce an injection $\mathbb{C}[B] \hookrightarrow \mathbb{C}[\mathfrak{X}_r]$ we automatically get dominance [Har77, p. 81], and then this immediately implies there exists an open dense set $U \subset \mathfrak{X}_r$ where $\text{MagTr}_{\mathcal{I}}$ is a smooth submersion [Har77, p. 271]. \square

Suppose $\mathcal{I} \subset \{\text{tr}(\mathbf{X}_{i_1} \mathbf{X}_{i_2} \cdots \mathbf{X}_{i_r})\}$ is taken to be a maximal. For $\text{SL}(m, \mathbb{C})$ and $r = 1$ it is surjective since it defines an isomorphism. Surprisingly, we also have an isomorphism for $\text{SL}(2, \mathbb{C})$ and $r = 2$ (see [Gol08] for more about the Fricke–Klein–Vogt Theorem). For $\text{SL}(3, \mathbb{C})$ and $r = 2$ and for $\text{SL}(2, \mathbb{C})$ and $r = 3$ it is a branched double cover. It is natural to conjecture that the map is surjective in general. However, for $\mathfrak{G} = \text{SL}(2, \mathbb{C})$, Florentino [Flo06] recently showed that it is not surjective for $r \geq 4$.

We conjecture the same is true for $\text{SL}(m, \mathbb{C})$ and $r \geq 2$ for $m \geq 3$ excepting only $(m, r) = (3, 2)$. In other words, we now make

Conjecture 10. *Let $\mathcal{I} \subset \{\text{tr}(\mathbf{X}_{i_1} \mathbf{X}_{i_2} \cdots \mathbf{X}_{i_r})\}$ be a set of parameters for \mathfrak{X}_r . Then $\text{MagTr}_{\mathcal{I}}$ is only surjective for the cases $(m, r) = (1, r), (m, 1), (2, 2), (2, 3), (3, 2)$.*

Remark 11. Additionally, in [Flo06] it is shown that a representation $\rho : \pi \rightarrow \text{SL}(2, \mathbb{C})$ where π is a finitely generated group with generators $\{g_1, \dots, g_r\}$ is reducible if and only if all triples $(\rho(g_{i_1}), \rho(g_{i_2}), \rho(g_{i_3}))$ are simultaneously reducible by conjugation. If we replace $\text{SL}(2, \mathbb{C})$ by $\text{SL}(m, \mathbb{C})$ and i_3 by $i_{d(m)}$ where $d(m)$ is the degree of nilpotency discussed in the introduction, then Florentino’s statement remains valid. This follows since globally all invariants are $\text{tr}(\mathbf{X}_{i_1} \mathbf{X}_{i_2} \cdots \mathbf{X}_{i_{d(m)}})$, and the points of \mathfrak{X}_r are distinguished by such traces. It is important to observe that it does not matter that the points in \mathfrak{X}_r correspond to unions of orbits, since such a union contains either one and only one orbit of an irreducible representation or it contain only reducible representations (in different orbits, but all reducible).

4. Closing remarks

In this section we take the opportunity to briefly describe our general outlook on the project that this paper, in part, contributes.

In [Law07] we showed the following theorem.

Theorem 12. *Let $\mathfrak{X} = \text{SL}(3, \mathbb{C})^{\times 2} // \text{SL}(3, \mathbb{C})$. Then the following hold:*

- (i) $\mathbb{C}[\mathfrak{X}]$ is minimally generated by the nine affine coordinate functions

$$\mathcal{G} = \{ \text{tr}(\mathbf{X}_1), \text{tr}(\mathbf{X}_2), \text{tr}(\mathbf{X}_1 \mathbf{X}_2), \text{tr}(\mathbf{X}_1^{-1}), \text{tr}(\mathbf{X}_2^{-1}), \text{tr}(\mathbf{X}_1 \mathbf{X}_2^{-1}), \\ \text{tr}(\mathbf{X}_2 \mathbf{X}_1^{-1}), \text{tr}(\mathbf{X}_1^{-1} \mathbf{X}_2^{-1}), \text{tr}(\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_1^{-1} \mathbf{X}_2^{-1}) \}.$$

- (ii) *The eight elements in $\mathcal{G} \setminus \{\text{tr}(\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_1^{-1} \mathbf{X}_2^{-1})\}$ are a maximal algebraically independent subset and are local parameters.*
- (iii) *$\text{tr}(\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_1^{-1} \mathbf{X}_2^{-1})$ satisfies a monic (degree 2) relation over the algebraically independent generators. It generates the ideal.*
- (iv) *$\text{Out}(\mathbb{F}_2)$ acts on $\mathbb{C}[\mathfrak{X}]$ and has an order 8 subgroup which acts as a permutation group on the independent generators; as such distinguishes them.*

In [Law08] we generalize part (i) of this theorem to \mathfrak{X}_r by describing minimal generators for any value of r . In this paper we generalize part (ii) of this theorem again to \mathfrak{X}_r for any value of r . We are currently exploring a generalization of part (iv) to \mathfrak{X}_r . Generalizing part (iii) seems to be a very hard problem.

Recently, exciting new results using methods similar to those in [AP89] were established in [BD07] concerning the ideal of relations for generic 3×3 matrices. In particular, the minimal degree of

generators of the ideal of relations was found to be 7 and the degree 7 relations were then classified in general. Using this we can get like results for \mathcal{X}_r , but this is just the beginning.

It would seem that either new ideas are needed to solve the relations problem in general, or a massive computational project. On the other hand, to solve the problem in general, one would like to know the minimal generators to work with (part (i) above and its generalization), the subsets of generators that do not admit relations (part (ii) above and its generalization), and group actions which can simplify the form of relations (part (iv) above and its generalization).

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