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A theoretical construction of wormhole supported by phantom energy

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Abstract

A new solution has been presented for the spherically symmetric space–time describing wormholes with phantom energy. The model suggests that the existence of wormhole is supported by arbitrarily small quantity of phantom energy.

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Wormholes are geometrical structures connecting two distinct space–times. It is very interesting object in modern science as because if traversable wormhole exists, the time machine can be constructed [1]. Wormhole is the solution of Einstein equation and shared by the violation of null energy condition. The matter that characterized the above stress energy tensor is known as exotic matter. There has been a fairly large amount of discussions [2–12] on wormholes beginning with the work of Morris and Throne [13]. Several authors have discussed wormholes in scalar tensor theory of gravity in which scalar field may play the role of exotic matter [14–19]. Recently, researcher are interested to know how much exotic matter is needed to get a traversable wormhole [20–23]. Recent astrophysical observations indicate that the Universe at present is accelerating. There are different ways of evading these unexpected behavior. Most of these attempts focus on alternative gravity theories or the supposition of existence of a hypothetical dark energy with a positive energy density and a negative pressure [24–31]. The matter with the property, energy density, $\rho > 0$ but pressure $p < -\rho < 0$ is known as phantom energy. The phantom energy violates the null energy condition what is needed to support traversable wormhole. Several authors have recently discussed the physical properties and characteristics of traversable wormholes by taking phantom energy as source [32–34].

Recently, Zaslavskii [35] have found a particular exact solution describing wormhole with a linear equation of state, $p_r < -\rho < 0$, $p_{tr} > 0$ (p_r and p_{tr} are radial and transverse pressures, respectively). In this Letter, we try to carry out the most general study of the existence of possible static spherical symmetric solution describing wormholes with phantom energy as source.

A static spherically symmetric Lorentzian wormhole can be described by a manifold R^2XS^2 endowed with the general metric in Schwarzschild coordinates (t, r, θ, ϕ) as

$$ds^2 = -e^{2f(r)}\,dt^2 + \frac{1}{[1 - \frac{b(r)}{r}]}dr^2 + r^2\,d\Omega_2^2, \qquad (1)$$

where, $r \epsilon (-\infty, +\infty)$.

To represent a wormhole, one must impose the following conditions on the metric (1) as [1]:

(1) The redshift function, $f(r)$ must be finite for all values of r. This means no horizon exists in the space–time.

(2) The shape function, $b(r)$ must obey the following conditions at the throat $r = r_0$: $b(r_0) = r_0$ and $b'(r_0) < 1$ (these are known as Flare-out conditions).

(3) $\frac{b(r)}{r} < 1$ for $r > r_0$, i.e., out of throat.

(4) The space–time is asymptotically flat, i.e., $\frac{b(r)}{r} \to 0$ as $|r| \to \infty$.

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Using the Einstein field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$, in orthonormal reference frame (with $c = G = 1$), we obtain the following stress energy scenario,

$$\rho(r) = \frac{b'}{8\pi r^2}, \tag{2}$$

$$p(r) = \frac{1}{8\pi}\left[-\frac{b}{r^3} + 2\frac{f'}{r}\left(1 - \frac{b}{r}\right)\right], \tag{3}$$

$$p_{\mathrm{tr}}(r) = \frac{1}{8\pi}\left(1 - \frac{b}{r}\right)\left[f'' - \frac{(b'r - b)}{2r(r-b)}f' + f'^2 + \frac{f'}{r} - \frac{(b'r-b)}{2r^2(r-b)}\right], \tag{4}$$

where $\rho(r)$ is the energy density, $p(r)$ is the radial pressure and $p_{\mathrm{tr}}(r)$ is the transverse pressure.

Using the conservation of stress energy tensor $T^{\mu\nu}_{;\nu} = 0$, one can obtain the following equation

$$p' + f'\rho + \left(f' + \frac{2}{r}\right)p - \frac{2}{r}p_{\mathrm{tr}} = 0. \tag{5}$$

From now on, we assume that our source is characterized by the phantom energy with equation of state that contains a radial pressure

$$p = -k\rho \tag{6}$$

we suppose also that pressures are isotropic and

$$p_{\mathrm{tr}} = k\rho \tag{7}$$

(here k is either > 1 or < -1).

Since only two equations of the system (2)–(4) are independent, it is convenient to represent them as follows:

$$b' = 8\pi r^2 \rho(r), \tag{8}$$

$$f' = \frac{(8\pi p r^3 + b)}{2r(r-b)}. \tag{9}$$

From (5) by using (6) and (7), one can obtain

$$\rho(r) e^{(1-\frac{1}{k})f} = \frac{\rho_0}{r^4}, \tag{10}$$

where ρ_0 is an integration constant.

Taking into account Eqs. (8)–(10), we have the following equation containing 'b' as

$$2Ab''r^2 - 2Abb''r + 4rAb' - 4Abb' + krb'^2 - bb' = 0, \tag{11}$$

where $\frac{k}{1-k} = A$.

Now to investigate whether there exists physically meaningful solutions consistent with the boundary requirements (conditions (1) to (4)), we take a general functional form of $b(r)$. We can generally express it in the form

$$b(r) = \sum_{n=1}^{\infty} b_n r^n + \sum_{m=0}^{\infty} a_m r^{-m} \tag{12}$$

since $\frac{b(r)}{r} \to 0$ as $r \to \infty$, Eq. (12) is consistent only when all the b_n's in $b(r)$ vanish, i.e.,

$$b(r) = \sum_{m=0}^{\infty} \frac{a_m}{r^m}. \tag{13}$$

Plugging this in Eq. (11) and matching the coefficients of equal powers of r from both sides, we get, for '$k \neq 0$'

$$b(r) = a_0 + \frac{a_1}{r} - \frac{a_0 a_1}{4Ar^2} + \frac{1}{12Ar^3}\left[a_0^2 a_1\left(\frac{1}{2A} - 1\right) - (k+1)a_1^2\right] + \frac{1}{24Ar^4}\left[a_0^3 a_1\left(\frac{1}{2A} - 1\right)\left(1 - \frac{1}{4A}\right) - a_0 a_1^2\left((k+2) - \frac{4+5k}{4A}\right)\right] + \cdots.$$

Thus we get two parameters family of solutions.

Now the expression for ρ can be obtained from Eq. (8) as

$$\rho = \frac{1}{8\pi}\left[-\frac{a_1}{r^4} + \frac{a_0 a_1}{2Ar^5} - \frac{1}{4Ar^6}\left[a_0^2 a_1\left(\frac{1}{2A} - 1\right) - (k+1)a_1^2\right] + \cdots\right]. \tag{14}$$

Since k is either > 1 or < -1, so A lies in $(-1, -\frac{1}{2})$. Thus $\rho > 0$ implies a_1 should be negative.

Eq. (10) gives the following expression for 'f' as

$$e^{2f} = \left[\frac{\rho_0}{8\pi[-a_1 + \frac{a_0 a_1}{2Ar} - \frac{1}{4Ar^2}[a_0^2 a_1(\frac{1}{2A} - 1) - (k+1)a_1^2] + \cdots]}\right]^{\frac{2k}{k-1}}. \tag{15}$$

The throat of the wormhole occurs at $r = r_0$ where r_0 is the solution of the equation $b(r) = r$. Suppose $\frac{1}{r} = y$, then $b(r) = r$ implies

$$g(y) = a_0 y + a_1 y^2 - \frac{a_0 a_1}{4A} y^3 + \cdots - 1 = 0. \tag{16}$$

This is a polynomial equation with negative last term. Then this equation must have at least one positive root, say, $y = \frac{1}{r_0}$. Since $\frac{1}{r_0}$ is a root of Eq. (16), then by standard theorem of algebra, either $g(y) > 0$ for $y > \frac{1}{r_0}$ and $g(y) < 0$ for $y < \frac{1}{r_0}$ or $g(y) < 0$ for $y > \frac{1}{r_0}$ and $g(y) > 0$ for $y < \frac{1}{r_0}$. Let us take the first possibility and one can note that for $y = \frac{1}{r} < \frac{1}{r_0}$, i.e., $r > r_0$, $g(y) < 0$, in other words, $b(r) < r$. But when $y = \frac{1}{r} > \frac{1}{r_0}$, i.e., $r < r_0$, $g(y) > 0$, this means, $b(r) > r$, which violates the wormhole structure given in Eq. (1).

One can note that the redshift function $f(r)$ always finite for $r \geqslant r_0 > 0$, i.e., no horizon exists in the space–time. Thus our solution describing a static spherically symmetric wormhole supported by the phantom energy.

The asymptotical wormhole mass reads

$$M = \lim_{r \to \infty} \frac{1}{2} b(r) = \frac{a_0}{2}. \tag{17}$$

The axially symmetric embedded surface $z = z(r)$ shaping the wormhole's spatial geometry is a solution of

$$\frac{dz}{dr} = \pm\frac{1}{\sqrt{\frac{r}{b(r)} - 1}}. \tag{18}$$

By the definition of wormhole, we can note that at the value $r = r_0$ (the wormhole throat) Eq. (18) is divergent, i.e., embedded surface is vertical there.

According to Morris and Throne [13], the 'r' coordinate is ill-behaved near the throat, but proper radial distance

$$l(r) = \pm\int_{r_0^+}^{r} \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}} \tag{19}$$

must be well behaved everywhere, i.e., we must require that $l(r)$ is finite throughout the space–time.

For our model,

$$l(r) = \pm\int_{r_0^+}^{r} \frac{dr}{\sqrt{1 - \frac{1}{r}[a_0 + \frac{a_1}{r} - \frac{a_0 a_1}{4Ar^2} + \cdots]}}. \tag{20}$$

Though we cannot find the explicit form of the integral but one can see that the above integral is a convergent integral, i.e., proper length should be finite.

To summarize, we have constructed exact solution describing static symmetric wormholes supported by the phantom energy. The resulting line element represents a two parameter family of geometries which contains wormholes. Hence this shows clearly that the phantom energy can support the existence of static wormholes.

For $a_1 = 0$, we obtain the standard Schwarzschild solution, viz., $e^{2f(r)} = [1 - \frac{b(r)}{r}] = 1 - \frac{r_s}{r}$, provided $a_0 = r_s = 2GM$ (the Schwarzschild radius).

One of the most striking feature of our model is that if we choose k is very close to unity, then $p + \rho = (-k+1)\rho = \frac{(-k+1)}{8\pi}[-\frac{a_1}{r^4} + \frac{a_0 a_1}{2Ar^5} - \frac{1}{4Ar^6}[a_0^2 a_1(\frac{1}{2A} - 1) - (k+1)a_1^2] + \cdots]$ can be made arbitrarily small. This reveals the fact that there is possible to construct a wormhole by an arbitrarily small amount of phantom energy.

Finally, if we take the parameter $a_0 > 0$, then asymptotic mass 'M' of the phantom energy wormhole is positive, i.e., a distant observer could not see any difference of gravitational nature between wormhole and a compact mass 'M'.

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