A classification of regular $t$-balanced Cayley maps on dihedral groups

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Abstract

In their study of regular $t$-balanced Cayley maps on a group, Conder et al. [M. Conder, R. Jajcay, T. Tucker, Regular $t$-balanced Cayley maps, J. Combin. Theory Ser. B (in press)] recently classified the regular anti-balanced Cayley maps on an abelian group. In this paper, we classify the regular $t$-balanced Cayley maps on a dihedral group for any $t$.

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1. Introduction

Let $\Gamma$ be a simple graph. A dart of $\Gamma$ is an ordered pair of adjacent vertices of $\Gamma$. By a map with an underlying graph $\Gamma$, we mean a triple $\mathcal{M} = (\Gamma; R, T)$, where $R$ is a permutation of the dart set $D = D(\Gamma)$ of $\Gamma$ whose orbits coincide with the sets of darts initiated from the same vertex and $T$ is an involution of $D$ whose orbits coincide with the sets of darts based on the same edge. The permutations $R$ and $T$ are called the rotation and the dart-reversing involution of $\mathcal{M}$, respectively. Given two maps $\mathcal{M}_1 = (\Gamma_1; R_1, T_1)$ and $\mathcal{M}_2 = (\Gamma_2; R_2, T_2)$, a map isomorphism $\Phi: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a
bijection $\Phi : D(\Gamma_1) \rightarrow D(\Gamma_2)$ such that $\Phi R_1 = R_2 \Phi$ and $\Phi T_1 = T_2 \Phi$. In fact, such a map isomorphism becomes a graph isomorphism between the underlying graphs. In particular, if $M_1 = M_2 = M$ then $\Phi$ is called a map automorphism of $M$. It is well known [1] that the automorphism group $\text{Aut}(M)$ acts semi-regularly on the dart set $D$. If $\text{Aut}(M)$ acts transitively on $D$ then the map $M$ is called regular. For more information on regular maps, we refer the reader to Nedela’s recent survey paper [6]. In the following, a map $M = (\Gamma; R, T)$ will be written as a pair $M = (\Gamma; R)$ because the involution $T$ becomes clear in a context.

Let $G$ be a group and let $X$ be a generating set of $G$ such that $X = X^{-1}$ and $1 \notin X$. A Cayley graph $\Gamma = \text{Cay}(G, X)$ is a graph with the vertex set $G$ and two vertices $g$ and $h$ are adjacent if and only if $g^{-1}h \in X$. So, the dart set of the Cayley graph $\text{Cay}(G, X)$ is $\{(g, gx) \mid g \in G, \ x \in X\}$. The set of left translations $L(G) = \{L_g \mid g \in G\}$, defined by $L_g(h) = gh$ for any $h \in G$, forms a subgroup of $\text{Aut}(\Gamma)$ which acts regularly on the vertex set. Let $p$ be a cyclic permutation of $X$. A Cayley map $M = \text{CM}(G, X, p)$ is the map $M = (\Gamma, R)$ with the underlying graph $\Gamma = \text{Cay}(G, X)$ and the rotation $R$ defined by $R(g, gx) = (g, gp(x))$ for $g \in G$ and $x \in X$. It is easy to see that for every $g \in G$, $L_g R = RL_g$ and $L_g T = TL_g$, so $L(G)$ is also a subgroup of $\text{Aut}(M)$ acting regularly on the vertex set. A Cayley map $M = \text{CM}(G, X, p)$ is regular if and only if there exists an automorphism $\rho$ in the stabilizer $(\text{Aut}(M))_g$ of a vertex $g$ which permutes cyclically the darts initiated from $g$. In this case, $\text{Aut}(M)$ is a product of $L(G)$ with a cyclic group $\langle \rho \rangle \cong \mathbb{Z}_{|X|}$ (see [3,4]). It is well known [3] that a map $M$ can be represented as a Cayley map if and only if there exists a subgroup of $\text{Aut}(M)$ which acts regularly on the vertex set of $M$.

Two Cayley maps $\text{CM}(G_1, X_1, p_1)$ and $\text{CM}(G_2, X_2, p_2)$ are said to be equivalent if there exists a group isomorphism $\phi : G_1 \rightarrow G_2$ mapping $X_1$ to $X_2$ such that $\phi p_1 = p_2 \phi$. Equivalent Cayley maps are isomorphic as maps. On the other hand, isomorphic Cayley maps may not be equivalent as Cayley maps. For more information on regular Cayley maps, the reader is referred to [1,4,8,9].

A Cayley map $M = \text{CM}(G, X, p)$ is called $t$-balanced if $p(x)^{-1} = p'(x^{-1})$ for every $x \in X$. In particular, if $t = 1$, namely, $p(x)^{-1} = p(x^{-1})$ for every $x \in X$, we call $M$ balanced. It is known [8] that a regular Cayley map $\text{CM}(G, X, p)$ is balanced if and only if there exists a group automorphism $\phi$ of $G$ whose restriction on $X$ is equal to $p$. In this case, the group $L(G)$ is a normal subgroup of $\text{Aut}(M)$ and $\text{Aut}(M) \cong L(G) \rtimes \langle \phi \rangle$, a semidirect product of $L(G)$ by $\langle \phi \rangle \cong \mathbb{Z}_{|X|}$. And, if $t = -1$, namely, $p(x)^{-1} = p^{-1}(x^{-1})$ for every $x \in X$ then the Cayley map is called anti-balanced.

Recently, Conder et al. [2] developed a general theory of $t$-balanced Cayley maps and classified the regular anti-balanced Cayley maps on an abelian group. In this paper, we classify the regular $t$-balanced Cayley maps on a dihedral group for any $t$. This paper is organized as follows. In the next section, we review some known results on skew-morphisms of finite groups. In Section 3, some regular $t$-balanced Cayley maps on dihedral groups are constructed. In Section 4, it is shown that those Cayley maps constructed in Section 3 are all the regular $t$-balanced ones on dihedral groups up to isomorphism.
2. Skew-morphisms of finite groups

For a finite group \(G\), let \(\psi : G \rightarrow G\) be a permutation of the underlying set of \(G\) of order \(r\) (in the full symmetric group \(\text{Sym}(G)\)) and let \(\pi : G \rightarrow \mathbb{Z}_r\) be a function from \(G\) into the cyclic group \(\mathbb{Z}_r\). The permutation \(\psi\) is called a skew-morphism [4] with respect to the power function \(\pi\) if \(\psi(1_G) = 1_G\) and \(\psi(gh) = \psi(g)\psi(\pi(g))(h)\) for all \(g, h \in G\), where \(\psi(\pi(g))(h)\) is the image of \(h\) under \(\psi\) applied \(\pi(g)\) times and \(1_G\) is the identity element in \(G\).

The following theorem gives a characterization of a regular Cayley map \(CM(G, X, p)\) in terms of a skew-morphism.

**Theorem 2.1** ([4]). A Cayley map \(CM(G, X, p)\) is regular if and only if there exists a skew-morphism \(\psi\) of \(G\) such that \(\psi(x) = p(x)\) for all \(x \in X\).

The skew-morphism and its power function associated with a regular Cayley map are uniquely determined by the map. More precisely, for a given regular Cayley map \(CM(G, X, p)\), the associated skew-morphism \(\psi\) is of order \(|X|\) as an extension of \(p\) and its power function \(\pi(x)\) is defined as follows [4]:

\[
\pi(x) \equiv \chi(\psi(x)) - \chi(x) + 1 \pmod{|X|},
\]

(1) for any \(x \in X\), where \(\chi(x)\) is the smallest nonnegative integer such that \(p^{\chi(x)}(x) = x^{-1}\) for each \(x \in X\). The following lemma gives some properties of skew-morphisms.

**Lemma 2.2** ([4]). Let \(\psi\) be a skew-morphism of \(G\) and let \(\pi\) be the power function of \(\psi\). Then

1. \(\text{Ker}(\pi) := \{g \in G \mid \pi(g) = 1\}\) is a subgroup of \(G\).
2. \(\pi(g) = \pi(h)\) if and only if \(g\) and \(h\) are contained in the same coset of the subgroup \(\text{Ker}(\pi)\).

For a Cayley map \(M = CM(G, X, p)\), let \(G^+\) denote the subgroup of \(G\) whose elements can be expressed as a product of even number of elements in \(X\). Then \(G^+\) is a subgroup of \(G\) of index 1 or 2.

Skoviera and Sirány [8] showed that the skew-morphism \(\psi\) associated with a regular balanced Cayley map \(CM(G, X, p)\) is a group automorphism of \(G\) and its power function is simply the constant function whose images are 1. A similar description of a regular \(t\)-balanced Cayley map \(CM(G, X, p)\) in terms of the skew-morphism and its power function was given by Conder et al. in [2]. It can be restated as follows.

**Lemma 2.3.** Let \(M = CM(G, X, p)\) be a Cayley map. Then

1. The map \(M\) is regular balanced if and only if there exists a group automorphism \(\psi\) whose restriction on \(X\) is \(p\).
2. The map \(M\) is regular \(t\)-balanced with \(t > 1\) if and only if \(t\) is a square root of 1 in \(\mathbb{Z}_{|X|}\) (other than 1 itself) and there exist a skew-morphism \(\psi\) and its power function \(\pi\) satisfying the following conditions:
   i. The restriction of \(\psi\) on \(X\) is \(p\).
   ii. \(\psi\) has only two values 1 and \(t\), and \(\text{Ker}(\pi) = G^+\) with \(|G : G^+| = 2\).
   iii. \(\psi\) preserves \(\text{Ker}(\pi)\) setwise and the restriction of \(\psi\) on \(\text{Ker}(\pi)\) is a group automorphism of \(\text{Ker}(\pi)\).
Two Cayley maps $\mathcal{M}_1 = CM(G, X, p)$ and $\mathcal{M}_2 = CM(G, Y, q)$ with $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_r\}$ are said to have the *same rotation type* if the two cyclic permutations $p$ and $q$ can be expressed as $p = (x_1 x_2 \cdots x_r)$ and $q = (y_1 y_2 \cdots y_r)$ such that $x_i^{-1} = p^j(x_i)$ if and only if $y_i^{-1} = q^j(y_i)$ for any $i, j \in \{1, 2, \ldots, r\}$.

In [7], Richter et al. characterized two isomorphic regular balanced Cayley maps of the same rotation type in terms of a group isomorphism between the underlying groups. It can be generalized to any two regular Cayley maps as follows.

**Lemma 2.4.** Let $\mathcal{M}_1 = CM(G, X, p)$ and $\mathcal{M}_2 = CM(G, Y, q)$ be two regular Cayley maps of the same rotation type. Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic if and only if there exists a group automorphism $\phi$ of $G$ such that $\phi(X) = Y$ and $\phi p = q \phi$, i.e., they are equivalent.

**Proof.** The sufficiency is trivial. For the necessity, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be isomorphic as maps. Since $\mathcal{M}_1 = CM(G, X, p) = (Cay(G, X), R_1)$ and $\mathcal{M}_2 = CM(G, Y, q) = (Cay(G, Y), R_2)$ have the same rotation type and both of them are regular, there exists a map isomorphism $\varphi$ from $\mathcal{M}_1$ to $\mathcal{M}_2$ satisfying $\varphi(1_G) = 1_G$, $\varphi(X) = Y$, $\varphi p = q \varphi$ on $X$ and $\varphi(x_i^{-1}) = \varphi(x_i)^{-1}$ for any $x_i \in X$. Now $\varphi$ is considered as a bijection of the dart set as well as of the vertex set. Without any loss of generality, one can assume that $\varphi(x_i) = y_i$, by relabelling the generators in $Y$ if necessary. To complete the proof, it suffices to show that $\varphi$ is a group automorphism of $G$. So it is enough to show $\varphi(x_{i_1} x_{i_2} \cdots x_{i_d}) = \varphi(x_{i_1}) \varphi(x_{i_2}) \cdots \varphi(x_{i_d})$ for any $d$ elements $x_{i_1}, x_{i_2}, \ldots, x_{i_d} \in X$ (not necessarily distinct). We prove it by induction on $d$. It is trivial for $d = 1$. Suppose that for any $d \leq k$ elements $x_{i_1}, x_{i_2}, \ldots, x_{i_d} \in X$, we have

$$\varphi(x_{i_1} x_{i_2} \cdots x_{i_d}) = \varphi(x_{i_1}) \varphi(x_{i_2}) \cdots \varphi(x_{i_d}) = y_{i_1} y_{i_2} \cdots y_{i_d}.$$

Then for any $k + 1$ elements $x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_{i_{k+1}} \in X$, there exists a positive integer $j$ such that $x_{i_{k+1}} = p^j(x_{i_k}^{-1})$, which implies that $y_{i_{k+1}} = q^j(y_{i_k}^{-1})$. So, as a function of the dart set,

$$\varphi(x_{i_1} \cdots x_{i_k}, x_{i_1} \cdots x_{i_k} x_{i_{k+1}}) = \varphi(x_{i_1} \cdots x_{i_k} x_{i_1} \cdots x_{i_k} p^j(x_{i_k}^{-1}))$$

$$= \varphi R^j \varphi(x_{i_1} \cdots x_{i_k} x_{i_1} \cdots x_{i_k} x_{i_{k-1}})$$

$$= R^j \varphi(x_{i_1} \cdots x_{i_k} x_{i_1} \cdots x_{i_k} x_{i_{k-1}})$$

$$= R^j (y_{i_1} \cdots y_{i_k} y_{i_1} \cdots y_{i_k-1})$$

$$= (y_{i_1} \cdots y_{i_k}, y_{i_1} \cdots y_{i_k} y_{i_{k+1}}).$$

Thus

$$\varphi(x_{i_1} x_{i_2} \cdots x_{i_{k+1}}) = y_{i_1} y_{i_2} \cdots y_{i_{k+1}} = \varphi(x_{i_1}) \varphi(x_{i_2}) \cdots \varphi(x_{i_{k+1}}).$$

This proves that $\varphi$ is a group automorphism of $G$. □

3. Construction of regular $t$-balanced Cayley maps

For a regular $t$-balanced Cayley map $CM(G, X, p)$, the images of generators under its power function are $t$ by Lemma 2.3. Hence the choices of generating set $X$ and its cyclic
permutation \( p \) are very restrictive. But, for a general regular Cayley map, there might be so many possible choices for the generating set \( X \) and its cyclic permutation. It makes a difficulty to classify all regular Cayley maps on a given family of groups, like the dihedral groups \( D_n \). In fact, we do not know whether there exists a regular Cayley map on \( D_n \) which is not \( t \)-balanced for any \( t \). It means that the full classification of regular Cayley maps on \( D_n \) is still open. Even for cyclic group \( \mathbb{Z}_n \), the classification of regular Cayley maps on \( \mathbb{Z}_n \) is not yet determined.

We restrict our discussion within the \( t \)-balanced case in this section and construct the class of regular \( t \)-balanced Cayley maps on dihedral groups.

For each \( n \), let \( D_n = \langle a, b \mid a^n = b^2 = 1, ab = ba^{-1} \rangle \) be the dihedral group of order \( 2n \). The automorphism group of \( D_n \) is

\[
\text{Aut}(D_n) = \{ \sigma_{i,j} \mid \sigma_{i,j}(a) = a^i, \sigma_{i,j}(b) = a^j b, i, j \in \{1, 2, \ldots, n\} \}
\]

and \( \gcd(i, n) = 1 \).

By Lemma 2.3(1), a Cayley map \( \mathcal{M} = CM(D_n, X, p) \) is regular and balanced if and only if there exists a group automorphism \( \psi \) of \( D_n \) whose restriction on \( X \) is \( p \). For each integer \( \ell \) with \( \gcd(\ell, n) = 1 \), let \( k \) be the smallest positive integer such that

\[
\ell^{k-1} + \ell^{k-2} + \cdots + \ell + 1 \equiv 0 \pmod{n},
\]

and let

\[
X = \{ b, ab, a^{\ell+1} b, \ldots, a^{\ell^{k-2} + \ell^{k-3} + \cdots + \ell + 1} b \},
\]

\[
p = (b \; ab \; a^{\ell+1} b \; \cdots \; a^{\ell^{k-2} + \ell^{k-3} + \cdots + \ell + 1} b).
\]

Kwak et al. [5] showed that for any positive integers \( \ell, k \geq 2 \) except \( (\ell, k) = (2, 3) \) and \((2, 4)\), the Cayley graph \( \text{Cay}(D_n, X) \) where \( n = \sum_{j=0}^{k-1} \ell^j \) is one-regular and has a cyclic vertex stabilizer. Wang and Feng [10] showed that any regular balanced Cayley map on \( D_n \) can be described as a Cayley map \( CM(D_n, X, p) \) for some \( \ell \) with \( \gcd(\ell, n) = 1 \). In this case, \( k = |X| \) is the valency of the Cayley map \( CM(D_n, X, p) \). Furthermore, they showed that any two such maps on \( D_n \) with \( \ell_1 \) and \( \ell_2 \) such that \( \gcd(\ell_1, n) = \gcd(\ell_2, n) = 1 \) are isomorphic if and only if \( \ell_1 = \ell_2 \). So the number of non-isomorphic regular balanced Cayley maps on \( D_n \) is equal to \( \phi(n) \), where \( \phi \) is the Euler \( \phi \)-function.

Any regular Cayley map on \( D_n \) of valency 2 is isomorphic to \( CM(D_n, \{ b, ab \}, (b, ab)) \). It is regular, balanced and also anti-balanced. From now on, we consider only \( r \)-valent maps with \( r \geq 3 \).

We first construct regular \( t \)-balanced Cayley maps on the dihedral group \( D_n \) with \( t > 1 \).

**Construction of the Cayley map** \( CM(n, \ell, k, m) \). Let \( \mathfrak{T} \) be the set of quadruples \((n, \ell, k, m)\) of positive integers satisfying the following three conditions.

(i) \( n \) is even, \( 1 \leq \ell < n \) with \( \gcd(\ell, n) = 1 \) and \( 1 \leq k, m < n/2 \).

(ii) \( m \) is the smallest positive integer such that

\[
\ell^m \equiv -1 \pmod{n}.
\]

(2)
Theorem 3.2. Let $t$ any two quadruples $F$ or any quadruple $\Xi$.

Corollary 3.3. (1) For any even $t$, no regular $t$-balanced Cayley maps on $D_n$ exist.

(2) For any odd $t > 1$, the number of non-isomorphic regular $t$-balanced Cayley maps

(3) For any $t > 1$, the valency of any regular $t$-balanced Cayley map on $D_n$ is a multiple of 4.

The case of odd $n$ in Corollary 3.3(2) will be proved later independently in Lemma 4.1.

Let $n = 2n'$ be even. By Theorems 3.1 and 3.2, one can show that a regular Cayley map $CM(n, \ell, k, m)$ is anti-balanced if and only if $4m - 1 = 2m + 1$, i.e., $m = 1$ or equivalently, $t = 3$. In this case, $X = \{b, a, a^{-1}\}$ and $p = (b a a^{-1} a^{-1})$. Moreover, Eqs. (2) and (3) can be rewritten as $n = 1$ and $k^2 \equiv -1 (\text{mod } n')$, respectively. Thus the number of non-isomorphic regular anti-balanced Cayley maps on the dihedral group $D_n$ equals the number of solutions of the congruence equation $x^2 \equiv -1 (\text{mod } n')$. Now let $n' = 2^{\alpha'} p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$ be the prime factorization of $n'$, where $p_1, p_2, \ldots, p_s$ are distinct odd primes, $\alpha' \geq 0$ and $a_i \geq 1$ for each $i = 1, 2, \ldots, s$. By the Chinese Remainder Theorem, one can show that the number of solutions of the congruence $x^2 \equiv -1 (\text{mod } n')$ is $2^{\alpha'}$ if $\alpha' \leq 1$ and $p_i \equiv 1 (\text{mod } 4)$ for all $1 \leq i \leq s$, and is 0 otherwise. Therefore we have the following corollary.
Corollary 3.4. Any regular anti-balanced Cayley map on the dihedral group $D_n$ has valency either 2 or 4.

1. If it has valency 2, it is isomorphic to the Cayley map $CM(D_n, \{b, ab\})$.
2. If it has valency 4, it is isomorphic to the Cayley map $CM(D_n, X, p)$, where $X = \{b, a, a^{2^k}b, a^{-1}\}$ and $p = (b \ a \ a^{2^k}b \ a^{-1})$ for some integer $k$ with $k^2 \equiv -1 \mod{4}$.

Furthermore, for an $n = 2^{a_0}p_{a_1}^{a_2} \cdots p_{a_s}^{a_t}$ where $p_1, p_2, \ldots, p_s$ are distinct odd primes, $a_0 \geq 0$ and $a_i \geq 1$ for $1 \leq i \leq s$, the number of non-isomorphic regular anti-balanced Cayley maps on the dihedral group $D_n$ of valency 4 is $2^s$ if $1 \leq a_0 \leq 2$ and $p_1 \equiv 1 \mod{4}$ for all $1 \leq i \leq s$, and is 0 otherwise.

Remark. If $n \not\equiv 0 \mod{4}$, Eq. (3) can be simplified as

$$\ell^2 - (4k^2 + 2)\ell + 1 \equiv 0 \mod{n}. \quad (3')$$

In fact, from Eq. (2), we have

$$(\ell - 1)(2k^2(\ell^m + \ell^{m-1} + \cdots + \ell) + (\ell - 1)) \equiv \ell^2 - (4k^2 + 2)\ell + 1 \mod{n}. \quad (4)$$

Thus Eq. (3) implies Eq. (3'). On the other hand, from $\ell^m - 1 \equiv -2 \mod{n}$ and $(\ell - 1) \mid (\ell^m - 1)$, one can get $\gcd(n, \ell - 1) \mid \gcd(n, \ell^m - 1) = 2$. But, $\ell$ must be odd by Eq. (2), and then $\gcd(n, \ell - 1) = 2$. So, when $n \not\equiv 0 \mod{4}$, i.e., when $n'$ is odd, $\gcd(n', \ell - 1) = 1$. If Eq. (3') holds, then one can get by Eq. (4) that

$$2k^2(\ell^m + \ell^{m-1} + \cdots + \ell) + (\ell - 1) \equiv 0 \mod{n'}.$$ 

Noting that $2k^2(\ell^m + \ell^{m-1} + \cdots + \ell) + (\ell - 1)$ is even while $n'$ is odd, Eq. (3) is obtained immediately.

If $m$ is even, then $\ell^m \equiv 1 \mod{4}$ and $n \not\equiv 0 \mod{4}$ by Eq. (2). Therefore Eqs. (3) and (3') are equivalent. For example, if $m = 2$ then $(n, \ell, k, 2) \in \Sigma$ if and only if $n = 2n'$, $1 \leq \ell < n$ with $\gcd(\ell, n) = 1$, $1 \leq k < n'$, $\ell^2 \equiv -1 \mod{n}$ and $\ell^2 - (4k^2 + 2)\ell + 1 \equiv -4k^2 + 2 \ell \equiv 0 \mod{n}$. Because $\gcd(n, \ell) = 1$, the last congruence is equivalent to $4k^2 + 2 \equiv 0 \mod{n}$, or equivalently to $2k^2 + 1 \equiv 0 \mod{n'}$. So we have the following.

Corollary 3.5. The number of non-isomorphic regular 5-balanced Cayley maps on $D_n$ is 0 if $n$ is odd. If $n = 2n'$ is even, then it is the number of pairs $(\ell, k)$ such that $1 \leq \ell < n$ with $\gcd(\ell, n) = 1$, $1 \leq k < n'$, $\ell^2 \equiv -1 \mod{n}$ and $2k^2 + 1 \equiv 0 \mod{n'}$.

4. Proofs of Theorems 3.1 and 3.2

The elements of the dihedral group $D_n$ can be partitioned into two disjoint subsets

$$\langle a \rangle = \{e, a, a^2, \ldots, a^{n-1}\} \quad \text{and} \quad D_n - \langle a \rangle = \{b, ab, a^2b, \ldots, a^{n-1}b\}.$$ 

For convenience, we call an element in the subset $\langle a \rangle$ an $a$-type element and one in the subset $D_n - \langle a \rangle$ a $b$-type element.

Let $CM(D_n, X, p)$ be a regular $t$-balanced Cayley map with $t > 1$ and let $\psi$ and $\pi$ be the associated skew-morphism and the power function, respectively. Then, by Lemma 2.3(2), $\ker(\pi)$ is the index 2 subgroup $D_n^+$ whose elements can be expressed as
a product of even number of elements in \( X \). In fact, \( D_n \) has only one index 2 subgroup \( \langle a \rangle \) if \( n \) is odd, and \( D_n \) has three index 2 subgroups \( \langle a \rangle, \langle a^2, b \rangle \) and \( \langle a^2, ab \rangle \) if \( n \) is even. However, the last two index 2 subgroups \( \langle a^2, b \rangle \) and \( \langle a^2, ab \rangle \) are isomorphic. Note that \( X \) has at least one \( b \)-type element since \( X \) generates \( D_n \). If \( \text{Ker}(\pi) = D_n^+ = \langle a \rangle \), then there is no \( a \)-type element in \( X \). In this case, the Cayley map \( CM(D_n, X, p) \) is regular balanced (1-balanced) since all elements in \( X \) are involutions, which is contradictory to the assumption. Therefore \( n \) must be even and \( \text{Ker}(\pi) = D_n^+ = \langle a^2, b \rangle \) or \( \langle a^2, ab \rangle \). Because \( \langle a^2, b \rangle \) and \( \langle a^2, ab \rangle \) are isomorphic, one can assume that

\[
\text{Ker}(\pi) = D_n^+ = \langle a^2, ab \rangle
\]

\[
= \{a^i, a^j b \mid 0 \leq i \leq n - 2 \text{ is even and } 1 \leq j \leq n - 1 \text{ is odd}\}.
\]

**Lemma 4.1.** (1) If \( n \) is odd, no regular \( t \)-balanced Cayley maps on \( D_n \) with \( t > 1 \) exist.

(2) For any even \( n \), let \( CM(D_n, X, p) \) be an \( r \)-valent regular \( t \)-balanced Cayley map with \( t > 1 \). Then \( t \) is odd and \( r = 2(t - 1) \). Moreover, \( p \) permutes an \( a \)-type element to a \( b \)-type element and vice versa.

**Proof.** Since (1) is mentioned already, we need to prove only (2). Let \( \psi \) and \( \pi \) be the skew-morphism and the power function associated to the map \( CM(D_n, X, p) \), respectively. Assume that \( X = \{x_0, x_1, \ldots, x_{r-1}\} \) and \( p = (x_0 \ x_1 \ x_2 \ \cdots \ x_{r-1}) \). For any \( x_i \in X \), since \( x_i \notin \text{Ker}(\pi) \), we have \( \pi(x_i) = t \) by Lemma 2.3(2). Also it is easy to see that \( \chi(x) = 0 \) for any \( b \)-type element \( x \in X \), where \( \chi(x) \) is the number defined in Section 2.

Now let \( x_i \in X \) be a \( b \)-type element. If \( x_{i+1} = p(x_i) \) is of \( b \)-type too, then \( \pi(x_i) = \chi(x_{i+1}) - \chi(x_i) + 1 = 0 + 0 + 1 = 1 \) by Eq. (1) in Section 2, a contradiction. This shows that \( p \) maps a \( b \)-type element to an \( a \)-type element.

Let \( x_j \in X \) be an \( a \)-type element. Since \( X \) has at least one \( b \)-type element, one can assume that \( x_0 \) is of \( b \)-type without any loss of generality. Then \( x_1 \) is an \( a \)-type element.

Now let \( x_1 = a^s \) for some integer \( s \). Then

\[
x_t = p^t(x_0) = p^t(x_0^{-1}) = p(x_0)^{-1} = x_{1}^{-1} = a^{-s}.
\]

Therefore

\[
\psi(x_j x_0) = \psi(x_j) \psi(x_j)(x_0) = \psi(x_j) \psi(x_0) = x_{j+1} x_t = x_{j+1} a^{-s}.
\]

Since \( x_j x_0 \in D_n^+ = \text{Ker}(\pi) \), \( x_j x_0 \) is a \( b \)-type element and the restriction of \( \psi \) on \( \text{Ker}(\pi) \) is a group automorphism, the elements \( x_{j+1} a^{-s} \) and \( x_{j+1} \) are also of \( b \)-type. Therefore \( p \) maps an \( a \)-type element to a \( b \)-type element.

So far, we have shown that \( p \) maps an \( a \)-type element to a \( b \)-type one and vice versa. It implies that \( r \) must be even. Furthermore, since \( x_t \) is an \( a \)-type element, \( t \) must be odd. From 1 = \( \psi(a^t a^{-s}) = \psi(x_1 x_t) = \psi(x_1) \psi(x_t) = x_2 x_2 \), and \( x_2 \) is a \( b \)-type element, we have \( x_2 t = x_2^{-1} = x_2 \). It implies that \( 2t = r + 2 \). \( \square \)

By Lemma 4.1, if \( CM(D_n, X, p) \) is an \( r \)-valent regular \( t \)-balanced Cayley map with \( t > 1 \) then \( r = 4m \) and \( t = 2m + 1 \) for an integer \( m \).

**Lemma 4.2.** Let \( CM(D_n, X, p) \) be a 4\( m \)-valent regular \( (2m + 1) \)-balanced Cayley map on the dihedral group \( D_n \) with \( m \geq 1 \) and let \( \psi \) be the associated skew-morphism. Then \( \psi^2 \) is a group automorphism of \( D_n \).
Proof. Because $\psi$ is a bijection and $\psi(1) = 1$, it holds that $\psi^2 : D_n \to D_n$ is bijective and $\psi^2(1) = 1$. For any $g, h \in D_n$, we have
\[
\psi^2(gh) = \begin{cases} 
\psi(\psi(g)\psi(h)) = \psi^2(g)\psi^2(h) & \text{if } g \in \text{Ker}(\pi), \\
\psi(\psi(g)\psi^{2m+1}(h)) = \psi^2(g)\psi^{4m+2}(h) & \text{otherwise}.
\end{cases}
\]
Hence $\psi^2$ is a group automorphism of $D_n$. □

Proof of Theorem 3.1. For any even number $n = 2n'$ and for any $(n, \ell, k, m) \in \mathcal{S}$, $CM(n, \ell, k, m)$ is the Cayley map $CM(D_n, X, p)$, where
\[
X = \{a^{\ell}, a^{2k(\ell^i + \ell^{i-1} + \cdots + 1)}b \mid 0 \leq i \leq 2m - 1\}
\]
and $p$ is the permutation mapping $a^{\ell i}$ to $a^{2k(\ell^i + \ell^{i-1} + \cdots + 1)}b$ and mapping $a^{2k(\ell^i + \ell^{i-1} + \cdots + 1)}b$ to $a^{\ell^i+1}$ for any $i = 0, 1, \ldots, 2m - 1$.

Now define the functions $\psi : D_n \to D_n$ and $\pi : D_n \to \mathbb{Z}_{4m}$ by
\[
\psi(a^{2s}) = a^{-2ks(\ell^m + \ell^{m-1} + \cdots + \ell)}, \quad \psi(a^{2s+1}) = a^{-2ks(\ell^m + \ell^{m-1} + \cdots + \ell)+2k},
\]
\[
\psi(a^{2s}b) = a^{-2ks(\ell^m + \ell^{m-1} + \cdots + \ell)+1}, \quad \psi(a^{2s+1}b) = a^{-2ks(\ell^m + \ell^{m-1} + \cdots + \ell)+2k+1}
\]
and
\[
\pi(a^{2s}) = \pi(a^{2s+1}b) = 1, \quad \pi(a^{2s+1}) = \pi(a^{2s}b) = 2m + 1
\]
for any $s = 0, 1, \ldots, n' - 1$. Since $\gcd(n, \ell - 1) = 2$ and $2k^2(\ell^m + \ell^{m-1} + \cdots + \ell) \equiv 1 - \ell \pmod{n}$, we have $\gcd(n, 2k(\ell^m + \ell^{m-1} + \cdots + \ell)) = 2$. It follows that $\gcd(n', k(\ell^m + \ell^{m-1} + \cdots + \ell)) = 1$ and hence $\psi$ is a bijection. Moreover, since $\ell$ is odd, it holds that for any $i = 0, 1, \ldots, 2m - 1$,
\[
\psi(a^{\ell i}) = a^{-k(\ell^i - 1)(\ell^m + \ell^{m-1} + \cdots + \ell)+2k} = a^{-k(\ell^i - 1)\frac{\ell^{m+1} - \ell}{\ell - 1}+2k} = a^{-k(\ell^{i+1} - 1)(\ell^m + \ell^{m-1} + \cdots + \ell)+2k} = a^{2k(\ell^{i+1} + \ell^{i-2} + \cdots + 1)\ell+2k} = a^{2k(\ell^{i+1} + \ell^{i-2} + \cdots + 1)\ell+2k} = a^{2k(\ell^{i+1} + \ell^{i-2} + \cdots + 1)\ell+2k}
\]
and
\[
\psi(a^{2k(\ell^i + \ell^{i-1} + \cdots + 1)}b) = a^{-2k(\ell^i + \ell^{i-1} + \cdots + 1)(\ell^m + \ell^{m-1} + \cdots + \ell)+1} = a^{(\ell - 1)(\ell^{i+1} + \ell^{i-1} + \cdots + 1)+1} = a^{\ell^{i+1}}.
\]
That is, the restriction of $\psi$ on $X$ is $p$. By Theorem 2.1, the proof will be completed if we show that $\psi$ is a skew-morphism of $D_n$ with the power function $\pi$. For any $s = 0, 1, 2, \ldots, n' - 1$, we have
\[
\psi^2(a^{2s}) = \psi(a^{-2ks(\ell^m + \ell^{m-1} + \cdots + \ell)}) = a^{2k^2s(\ell^m + \ell^{m-1} + \cdots + \ell)^2} = a^{s(1 - \ell)(\ell^m + \ell^{m-1} + \cdots + \ell)} = a^{2s\ell},
\]
\[
\psi^2(a^{2s+1}) = \psi(a^{-2ks(\ell^m+\ell^{m-1}+\cdots+\ell)+2k})
\]
\[
= a^{2k(2s(\ell^m+\ell^{m-1}+\cdots+\ell)-1)(\ell^m+\ell^{m-1}+\cdots+\ell)+1}
\]
\[
= a^{(1-\ell)(s(\ell^m+\ell^{m-1}+\cdots+\ell)+1)}
= a^{2s\ell+\ell+1+1} = a^{(2s+1)\ell},
\]
\[
\psi^2(a^{2s}b) = \psi(a^{-2ks(\ell^m+\ell^{m-1}+\cdots+\ell)+1})
\]
\[
= a^{2k^2(\ell^m+\ell^{m-1}+\cdots+\ell)^2+2k} b
\]
\[
= a^{s(1-\ell)(\ell^m+\ell^{m-1}+\cdots+\ell)+2k} b = a^{2s\ell+2k} b, \quad \text{and}
\]
\[
\psi^2(a^{2s+1}b) = \psi(a^{-2ks(\ell^m+\ell^{m-1}+\cdots+\ell)+2k+1}) b
\]
\[
= a^{2k^2(\ell^m+\ell^{m-1}+\cdots+\ell)-2k^2(\ell^m+\ell^{m-2}+\cdots+1)(\ell^m+\ell^{m-1}+\cdots+\ell)+1}
\]
\[
= a^{2k^2(\ell^m+\ell^{m-1}+\cdots+\ell)+(1-\ell)(\ell^m+\ell^{m-2}+\cdots+1)+1}
\]
\[
= a^{2k^2(\ell^m+\ell^{m-1}+\cdots+\ell)-1}, \quad \text{and}
\]
\[
\psi^2(a^{2s+1}b) = \psi(a^{-2s-1+2k(\ell^m+\ell^{m-2}+\cdots+1)+1}) b
\]
\[
= a^{2k^2(s+1)(\ell^m+\ell^{m-1}+\cdots+\ell)-2k^2(\ell^m+\ell^{m-2}+\cdots+1)(\ell^m+\ell^{m-1}+\cdots+\ell)+2k+1} b
\]
\[
= a^{2k^2(s+1)(\ell^m+\ell^{m-1}+\cdots+\ell)+(1-\ell)(\ell^m+\ell^{m-2}+\cdots+1)+2k+1} b
\]
\[
= a^{2k^2(s+1)(\ell^m+\ell^{m-1}+\cdots+\ell)+2k-1} b.
\]

Therefore, for any \( r, s = 0, 1, 2, \ldots, n' - 1 \), we have
\[
\psi(a^{2r}) \psi(a^{2s}) = a^{-2kr(\ell^m+\ell^{m-1}+\cdots+\ell)} a^{-2ks(\ell^m+\ell^{m-1}+\cdots+\ell)}
\]
\[
= a^{-2k(r+s)(\ell^m+\ell^{m-1}+\cdots+\ell)} = \psi(a^{2r} a^{2s}),
\]
\[
\psi(a^{2r}) \psi(a^{2s+1}) = a^{-2kr(\ell^m+\ell^{m-1}+\cdots+\ell)} a^{-2ks(\ell^m+\ell^{m-1}+\cdots+\ell)+2k} b = \psi(a^{2r} a^{2s+1}),
\]
\[
\psi(a^{2r}) \psi(a^{2s} b) = a^{-2kr(\ell^m+\ell^{m-1}+\cdots+\ell)} a^{-2ks(\ell^m+\ell^{m-1}+\cdots+\ell)+1}
\]
\[
= \psi(a^{2r} a^{2s} b), \quad \text{and}
\]
\[
\psi(a^{2r}) \psi(a^{2s+1} b) = a^{-2kr(\ell^m+\ell^{m-1}+\cdots+\ell)} a^{-2ks(\ell^m+\ell^{m-1}+\cdots+\ell)+2k+1} b
\]
\[
= \psi(a^{2r} a^{2s+1} b).
\]

Similarly, a straightforward but tedious calculation gives
\[
\psi(a^{2r+1}) \psi^{2m+1}(a^{2s}) = \psi(a^{2r+1} a^{2s}),
\]
\[
\psi(a^{2r+1}) \psi^{2m+1}(a^{2s+1}) = \psi(a^{2r+1} a^{2s+1}),
\]
\[
\psi(a^{2r+1}) \psi^{2m+1}(a^{2s} b) = \psi(a^{2r+1} a^{2s} b),
\]
\[
\psi(a^{2r+1}) \psi^{2m+1}(a^{2s+1} b) = \psi(a^{2r+1} a^{2s+1} b),
\]
\[
\psi(a^{2r} b) \psi^{2m+1}(a^{2s}) = \psi(a^{2r} b a^{2s}).
\]
\[ \psi(a^{2r}b)\psi^{2m+1}(a^{2s+1}) = \psi(a^{2r}ba^{2s+1}), \]
\[ \psi(a^{2r}b)\psi^{2m+1}(a^{2s}b) = \psi(a^{2r}ba^{2s}b), \]
\[ \psi(a^{2r}b)\psi^{2m+1}(a^{2s+1}b) = \psi(a^{2r}ba^{2s+1}b), \]

and
\[ \psi(a^{2r+1}b)\psi(a^{2s}) = \psi(a^{2r+1}ba^{2s}), \]
\[ \psi(a^{2r+1}b)\psi(a^{2s+1}) = \psi(a^{2r+1}ba^{2s+1}), \]
\[ \psi(a^{2r+1}b)\psi(a^{2s}b) = \psi(a^{2r+1}ba^{2s}b), \]
\[ \psi(a^{2r+1}b)\psi(a^{2s+1}b) = \psi(a^{2r+1}a^{2s+1}b). \]

Hence \( \psi \) is a skew-morphism and \( CM(n, \ell, k, m) \) is a regular \((2m + 1)\)-balanced Cayley map. \( \square \)

Finally, we prove Theorem 3.2. By Lemma 4.1, we need only to consider \( 4m \)-valent regular \((2m + 1)\)-balanced Cayley maps on dihedral groups with \( m \geq 1 \).

**Proof of Theorem 3.2.** Suppose that \( \mathcal{M} = CM(D_n, X, p) \), where \( X = \{x_0, x_1, \ldots, x_{4m-1}\} \) and \( p = (x_0 \ x_1 \ \cdots \ x_{4m-1}) \), is a \( 4m \)-valent regular \((2m + 1)\)-balanced Cayley map for positive integers \( n = 2n' \) and \( m \). As in the proof of Lemma 4.1, one can assume that \( x_0 = b \) and \( x_1 = a^4 \) for some integer \( s \) up to isomorphism. Let \( \psi \) and \( \pi \) be the skew-morphism and the power function associated with \( \mathcal{M} \), respectively. Then \( \psi^2 \) is an automorphism of \( D_n \) by Lemma 4.2. Let \( \psi^2(a) = a^\ell \) and \( \psi^2(b) = a^w b \) for some integers \( w \) and \( \ell \) such that \( \gcd(\ell, n) = 1 \). Then \( x_2 = \psi^2(x_0) = a^w b \). Since \( \text{Ker} (\pi) \) is assumed to be \( \langle a^2, ab \rangle \) and \( x_2 \notin \text{Ker} (\pi) \), \( w \) must be even, say \( w = 2k \). Moreover, for any \( j = 0, 1, \ldots, 2m - 1 \), we have
\[ x_{2j} = \psi^{2j}(b) = a^{2k(\ell^j - 1)}b \quad \text{and} \quad x_{2j+1} = \psi^{2j}(a^s) = a^{sx^j}. \]
Clearly, \( a^{2k} = x_{2x_0} \in \text{Ker} (\pi) \). Thus
\[ \psi(a^{2k}) = \psi(x_2)\psi^{2m+1}(x_0) = x_3x_{2m+1} = a^{sx^0}\ell^m. \] (5)

The fact that \( D_n = \langle X \rangle = \langle b, a^s, a^{2k}b \rangle \) implies that \( \gcd(s, 2k, n) = 1 \). Because the restriction of \( \psi \) on \( \text{Ker} (\pi) \) is a group automorphism, we have \( \gcd(s, n) = \gcd(s, 2k, n) = 1 \). Hence one can also assume that \( s = 1 \) up to isomorphism. So
\[ x_{2j} = a^{2k(\ell^j - 1)}b \quad \text{and} \quad x_{2j+1} = a^{\ell^j} \]
for any \( j = 0, 1, \ldots, 2m - 1 \). Since \( a^{\ell^m} = x_{2m+1} = x_1^{-1} = a^{-1} \), we obtain \( \ell^m \equiv -1 \pmod{n} \). Moreover,
\[ \psi(a^2) = \psi(x_1x_1) = x_2x_{2m+2} = a^{2k}ba^{2k(\ell^m + \ell^{m-1} + \cdots + 1)} = a^{-2k(\ell^m + \ell^{m-1} + \cdots + \ell)}. \]

Since \( a^2 \in \text{Ker} (\pi) \) and the restriction of \( \psi \) on \( \text{Ker} (\pi) \) is a group automorphism, we have
\[ \psi(a^{2k}) = (\psi(a^2))^k = a^{-2k^2(\ell^m + \ell^{m-1} + \cdots + \ell)}. \]
Comparing it with Eq. (5) and noting that \( s = 1 \) and \( \ell^m \equiv -1 \pmod{n} \), we have
\[ 2k^2(\ell^m + \ell^{m-1} + \cdots + \ell) + (\ell - 1) \equiv 0 \pmod{n}. \]
Therefore, the $4m$-valent regular $(2m + 1)$-balanced Cayley map $M = CM(D_n, X, p)$ is isomorphic to $CM(n, \ell, k, m)$ for some $(n, \ell, k, m) \in \mathcal{S}$.

For any two quadruples $(n, \ell_1, k_1, m), (n, \ell_2, k_2, m) \in \mathcal{S}$, suppose that the two $4m$-valent regular $(2m + 1)$-balanced Cayley maps $CM(n, \ell_1, k_1, m) = CM(D_n, X, p)$ and $CM(n, \ell_2, k_2, m) = CM(D_n, Y, q)$ are isomorphic. Note that for $a$-type elements $x_i \in X$ and $y_i \in Y$, we have $x_i^{-1} = p^{2m}(x_i)$ and $y_i^{-1} = q^{2m}(y_i)$. Since $a$-type elements and $b$-type elements appear alternately in both $p$ and $q$ by Lemma $4.1$, $CM(n, \ell_1, k_1, m)$ and $CM(n, \ell_2, k_2, m)$ have the same rotation type. By Lemma $2.4$, there exists a group automorphism $\phi : D_n \to D_n$ such that $\phi(b) = b$ and $\phi p = q\phi$. From $p(b) = q(b) = a$, we have $\phi(a) = a$. Thus $\phi$ should be the trivial group automorphism, implying $\ell_1 = \ell_2$ and $k_1 = k_2$. □

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References