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Large Graphs with Given Degree and Diameter. II

J. C. BERMOND, C. DELORME, AND G. FARHI

Laboratoire de Recherche en Informatique, ERA CNRS 452, Bât. 490, Université Paris Sud, 91405 Orsay, France

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The following problem arises in the study of interconnection networks: find graphs of given maximum degree and diameter having the maximum number of vertices. Constructions based on a new product of graphs, which enable us to construct graphs of given maximum degree and diameter, having a great number of vertices from smaller ones are given; therefore the best values known before are improved considerably.

INTRODUCTION

We are interested in the (Δ, D) graph problem, a problem which arises in telecommunications networks (or microprocessor networks).

Let $G = (X, E)$ be an undirected graph with vertex set X and edge set E. The distance between two vertices x and y, denoted $\delta(x, y)$ is the length of a shortest path between x and y. The *diameter* D of the graph G is defined as: $D = \max_{(x,y) \in X^2} \delta(x, y)$. The degree $d(x)$ of a vertex x is the number of vertices adjacent to x and Δ denotes the maximum degree of G.

The (Δ, D) graph problem consists of finding the maximum number of vertices $n(\Delta, D)$ of a graph with given maximum degree Δ and diameter D.

This problem arises quite naturally in the study of interconnection networks: the vertices represent the stations (or processors); the degree of a vertex is the number of links incident at this vertex and the diameter represents the maximum number of links to be used to transmit a message. The problem seems to have been set first in the literature by Elspas [8]. Different contributions have been made in the seventies and the known results were summarized in Storwick's article [12]. These results have been recently improved by Memmi and Raillard [10] and Quisquater [11].

A theoretical bound on $n(\Delta, D)$ is given by Moore (see [3, 4]) $n(2, D) \leqslant 2D + 1$ and for $\Delta > 2$, $n(\Delta, D) \leqslant (\Delta(\Delta - 1)^D - 2)/(\Delta - 2)$. The graphs satisfying the equality are called Moore graphs. It has been proved by different authors (see Biggs [3, Chap. 23]) that Moore graphs can exist only

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if $\Delta = 2$ (the graphs being the $(2D+1)$ cycles) or if $D = 2$ and $\Delta = 3, 7, 57$ (for $\Delta = 3$ and $\Delta = 7$ there exists a unique Moore graph, respectively, Petersen's graph on 10 vertices, and Hoffman and Singleton's graph on 50 vertices; for $\Delta = 57$ the answer is not known).

The aim of this article is to give constructions based on some new products of graphs, which are interesting in themselves, thus enabling us to construct large graphs having a given maximum degree Δ and diameter D from smaller ones. These constructions give graphs having more vertices than those known before (see Storwick's table [12] or Memmi and Raillard's table [10] in particular when Δ and D are small. Two other kinds of constructions will appear in other papers written by some of us, one giving infinite families of graphs with Δ and D given [7] and another based on cages and generalized quadrangles [6]. At the end of the paper we give the table of the best values we know now. For other extremal problem on diameters, in particular the minimum number of edges of graphs of given diameter and maximum degree, see [1, 4].

1. Definition of the Product $*$

Let $G = (X, E)$ and $G' = (X', E')$ be two graphs. Let us be given an arbitrary orientation of the edges of G and let U be the set of arcs. Finally let for each arc (x, y) of $U, f_{(x,y)}$ be a one to one mapping from X' to X'.

We define the product $G * G'$ as follows:

The vertex set of $G * G'$ is the cartesian product $X \times X'$. The vertex (x, x') is joined to (y, y') in $G * G'$ if and only if either

$$\{x = y \text{ and } \{x', y'\} \in E'\}$$

or

$$\{(x, y) \in U \text{ and } y' = f_{(x,y)}(x')\}.$$

Remarks. $G * G'$ can be viewed as formed by $|X|$ copies of G' where two copies generated by the vertices x and y are joined if (x, y) is an arc and in that case they are joined by a perfect matching depending on (x, y).

If G is K_2, we have a so-called permutation graph on G' introduced by Chartrand and Harary [5] (see also Harary's book [9, p. 175]).

Example 1.1. Let $G = K_2$, $(1, 2)$ being the oriented arc and let $G' = C_5$. In Fig. 1a we have represented $K_2 * C_5$, where $f_{(1,2)}(x') = x'$ and in Fig. 1b $K_2 * C_5$, where $f_{(1,2)}(x') = 2x'$ (mod 5). Note that the diameter of the graph of Fig. 1a is 3, but the diameter of the Petersen graph (Fig. 1b) is 2.

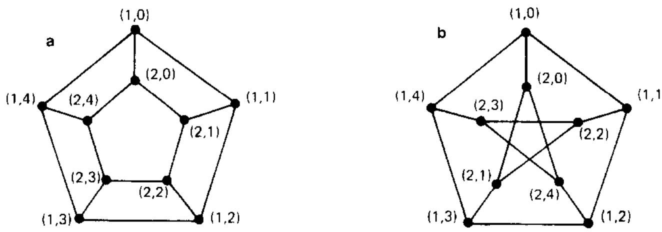

FIGURE 1

EXAMPLE 1.2. If we choose $f_{(x,y)}(x') = x'$ for any arc (x, y) then $G * G'$ is nothing other than the cartesian sum (called also cartesian product) of G and G' and in this case the diameter of $G + G'$ is the sum of the diameters of G and G'.

Degree and Diameter of $G * G'$

If G is of maximum degree Δ and G' of maximum degree Δ', then $G * G'$ has as maximum degree $\Delta + \Delta'$. If G is of diameter D and G' of diameter D', then the diameter of $G * G'$ is less than or equal to $D + D'$ and a clever choice of the functions $f_{(x,y)}$ can give a small diameter.

2. FIRST EXAMPLES

Let C_p denotes the cycle of length p.

LEMMA 2.1. *Let a be an integer* $\geqslant 1$. *In the graph* $K_2 * C_{2a^2+2a+1}$ *with orientation of* K_2 *the arc* $(1,2)$ *and with* $f_{(1,2)}(x') = (2a+1)x'$ $(\mathrm{mod}\ 2a^2 + 2a + 1)$, *the distance between the two vertices* $(1, x')$ *and* $(2, y')$ *is less than or equal to* $a + 1$.

Proof. It suffices to compute the distance between $(1,0)$ and $(2, y')$. Indeed the transformation $(1, y') \to (1, x' + y')$ and $(2, y') \to (2, y' + (2a+1)x')$ is an automorphism of the graph which acts transitively on the two cycles generated, respectively, by the vertices $(1, x')$ and $(2, x')$. Therefore we have to show that the distance between $(1,0)$ and any vertex $(2, x')$ is less than or equal to $a + 1$.

For p an integer, $0 \leqslant p \leqslant a$ let us define I_p as the set of vertices $(2, x')$ joined to $(1,0)$ by the following path: $(1,0)(1,1) \cdots (1,p)$, $(2, (2a+1)p)$ and then a path of length $\leqslant a - p$ on the cycle generated by the vertices $(2, x')$; then $I_p = [2ap + 2p - a, 2ap + a]$. Similarly let I_{-p} be the set of vertices $(2, x')$ joined to $(1,0)$ by the path $(1,0)(1,-1) \cdots (1,-p)(2, -(2a+1)p)$ and a path of length $\leqslant a - p$ on the cycle generated by the vertices $(2, x')$. Then $I_{-p} = [-2ap - a, -2ap - 2p + a]$. Note that for $0 \leqslant p \leqslant a$ all the vertices of I_p and I_{-p} are at most distance $a + 1$ from $(1, 0)$. Thus the set of vertices $(2, x')$ at most distance $a + 1$ from $(1, 0)$ contains $\bigcup_{0 \leqslant p \leqslant a} (I_p \cup I_{-p}) = I_{-a} \cup (\bigcup_{0 \leqslant p < a} (I_p \cup I_{p+1} \cup I_{p-a}))$. But $I_{p-a} = [-2a^2 + 2ap - a, -2a^2 + 2ap - a + 2p] \equiv [2ap + a + 1, 2ap + 2p + a + 1]$ (by adding $2a^2 + 2a + 1 \equiv 0 \pmod{2a^2 + 2a + 1}$) and $I_{p+1} = [2ap + 2p + a + 2, 2ap + 3a]$. Thus $I_p \cup I_{p+1} \cup I_{p-a} = [2ap + 2p - a, 2ap + 3a]$ and therefore $\bigcup_{0 \leqslant p \leqslant a} (I_p \cup I_{-p})$ contains all the elements of Z_{2a^2+2a+1}.

THEOREM 2.2. *The graph $K_2 * C_{2a^2+2a+1}$ defined as in Lemma 2.1 is a cubic graph of diameter 2 if $a = 1$, $a + 2$ if $a > 1$.*

Proof. It suffices to determine (in view of the automorphism described in the proof of Lemma 2.1) the distance between $(1, 0)$ and a vertex $(1, x')$. If $a = 1$ this distance is 2 by using a path of the cycle generated by the vertices $(1, x')$. If $a > 1$, every vertex $(1, x')$ is joined to a vertex $(2, (2a + 1)x')$, which is by Lemma 2.1 at most distance $a + 1$ from $(1, 0)$ and therefore $(1, x')$ is at most distance $a + 2$ from $(1, 0)$. Furthermore there is equality, because the distance between $(1, 0)$ and $(1, a + 2)$ is $a + 2$ if $a \geqslant 2$.

EXAMPLE 2.3. For $a = 1$, $K_2 * C_5$ is Petersen's graph as defined in Fig. 1b. For $a = 3$, $K_2 * C_{25}$ (with the definition of $f_{1,2}(x') = 7x' \pmod{25}$ is a cubic graph on 50 vertices with diameter 5. (The best example previously known had only 36 vertices.)

In fact the construction can be used with any graph G instead of K_2 and we have

THEOREM 2.4. *Let G be a graph of maximum degree Δ and diameter D, $D \geqslant 2$, with a given orientation U. Then $G * C_{2a^2+2a+1}$ with $f_{(x,y)}(x') = (2a + 1)x' \pmod{2a^2 + 2a + 1}$ for any arc (x, y) of U is a graph of maximum degree $\Delta + 2$ and of diameter $D + a$. The theorem is also true for $K_{\Delta+1} * C_5$ (case $D = 1$, $a = 1$).*

Proof. Let (x, x') and (y, y') be any two vertices of $G * C_{2a^2+2a+1}$ with $x \neq y$. By hypothesis there exists in G a path of length $\leqslant D$ between x and y, let z be the vertex before y on this path. Then there exists a vertex (z, z') at distance $\leqslant D - 1$ from (x, x'). By Lemma 2.1 (z, z') is at distance $\leqslant a + 1$ from any vertex (y, y') and therefore the distance between (x, x') and (y, y') is $\leqslant D + a$. Now let us consider two vertices (x, x') and (x, y'). By Theorem 2.2, these two vertices are at most distance $a + 2 \leqslant D + a$ if $D \geqslant 2$. If $D = 1$ and $a = 1$ two vertices (x, x') and (x, y') are at most distance

$2 \leqslant D+a$ and the theorem still works. (But if $a > 1$ there exist two vertices (x, x') and (x, y') at distance $a+2$ and the result is not true.)

To show that the equality is attained in the case $D \geqslant 2$, consider two vertices x and y at distance D in G. We will show that the distance between the vertices $(x, 0)$ and (y, a) is $D+a$. A shortest path between $(x, 0)$ and (y, a) consists of a path of length λ_0 in $x \times G'$, then one edge $(x, \alpha_0)(x_1, (2a+1)\alpha_0)$, a path of lengh λ_1 in $x_1 \times G'$, one edge $(x_1, \alpha_1)(x_2, (2a+1)\alpha_1) \cdots$, a path of length λ_i in $x_i \times G'$, one edge $(x_i, \alpha_i)((x_{i+1}, (2a+1)\alpha_i)$, and finally a path of length λ_k in $y \times G'$. The λ_i are positive or null. As it is a shortest path $x = x_0, x_1, ..., x_i, ..., x_k = y$ is a path in G of length at least D. Therefore the distance between $(x, 0)$ and (y, a) is at least $D + \sum_{0 \leqslant i \leqslant k} \lambda_i$. Furthermore we must have

$$a = (\pm\lambda_k \pm \lambda_{k-2} \pm \lambda_{k-4} \pm \cdots) + (2a+1)(\pm\lambda_{k-1} \pm \lambda_{k-3} \pm \cdots)$$
$$(\bmod 2a^2 + 2a + 1).$$

This last equality cannot be realized if $\sum \lambda_i < a$. Therefore $\sum \lambda_i \geqslant a$ and the distance is at least $D+a$. ∎

EXAMPLE 2.5. $K_3 * C_5$ is a graph of degree 4 and diameter 2 on 15 vertices (in fact, this graph is extremal). By using as graph G the graph on 24 vertices of degree 5 and diameter 2 (see [2]), $G * C_5$ is a graph of degree 7 and diameter 3 on 120 vertices (the best graph known before had only 80 vertices).

Using Theorem 2.4, we improved some values of [10], but now other constructions give better results.

3. GROUP THEORETICAL SETTING

In fact the product can be expressed in a group formulation. This formulation is not necessary to understand most of the constructions but it enables us to shorten the verifications and leads us to imagine other powerful examples.

Let Γ be a finite group (not necessarily abelian). Let S be a subset of Γ, which does not contain the unity of the group and such that if $s \in S$ then $s^{-1} \in S$. Then G' is constructed as follows: the vertices of G' are the elements of Γ, two vertices x' and y' being joined if there exists $s \in S$, such that $x' = sy'$. To define the product $G * G'$, we will choose for every arc (x, y) of G, a bijection $f_{(x,y)}$ defined by $f(x,y)(x') = \phi_{(x,y)}(x')\, \alpha_{(x,y)}$, where $\phi(x, y)$ is a given automorphism of Γ and $\alpha_{(x,y)}$ a given element of Γ.

In the examples of Section 2, we had $\Gamma = Z_{2a^2+2a+1}$ the cyclic group on

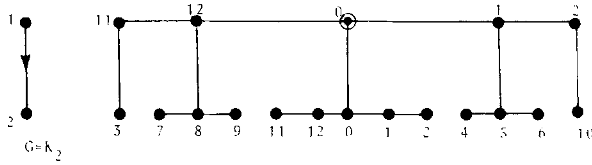

FIGURE 3

$2a^2+2a+1$ elements, written additively (integers modulo $2a^2+2a+1$). If we choose $S=\{-1,+1\}$ then G' is a C_{2a^2+2a+1}. If we choose the automorphism of Γ, $\phi_{(x,y)}$ defined by $\phi_{(x,y)}(x')=(2a+1)\,x'$ and $\alpha_{(x,y)}=0$, then we have $f_{(x,\,y)}(x')=(2a+1)\,x'$ for every arc (x,y) of G.

The interest of the formulation is that if T is a tree in G, then there exists a group acting on $T*G'$, transitively on each copy of G' and therefore it suffices to calculate the distance between any vertex of $y*G'$ to a given vertex of $x*G'$ to obtain all the distances to any vertex of $x*G'$. The group acting on $T*G'$ is the opposite group of Γ (defined with the law ba instead of ab): the automorphism is first defined on the subgraph $(x,y)*G'$ by $(x,x')\to(x,x'\psi(\gamma))$ and $(y,y')\to(y,y'\alpha_{(x,y)}^{-1}\phi_{x,y)}(\psi(\gamma))\,\alpha_{(x,y)})$, where ψ is some automorphism of Γ and γ any element of Γ; then we can define step by step the action of the opposite group of Γ on $T*G'$.

In what follows the calculation on the distances will be shortly explained on a figure indicating the vertices attained from the vertex $(x,0)$ with a tree rooted at $(x,0)$ the distances being the length of the path from $(x,0)$. We will draw the graph G on the left and we will dispose the vertices of the form $(x,\cdot)$ at the same level as the vertex x of G. The graph of Lemma 2.1 is pictured in Fig. 3 (here $a=2$, $f_{1,2}(x')=5x'$). The vertices of the form $(1,i)$ are drawn on the first level and the vertices $(2,i)$ on the second level.

It is clear that any vertex $(2,x')$ is at most distance 3 from $(1,0)$. For example, the path between $(1,0)$ and $(2,3)$ is $(1,0)$ $(1,12)$ $(1,11)$ $(2,3)$; the path between $(1,0)$ and $(2,4)$ is $(1,0)$ $(1,1)$ $(2,5)$ $(2,4)$, and so on.

4. PRODUCTS WITH CYCLES

Now we give examples using the cyclic group Z_p (of integers modulo p) written additively.

4.1 ($G*C_{19}$). *Let* $\Gamma=Z_{19}$, $S=\{-1,+1\}$ *then* G' *is isomorphic to* C_{19}. *Let* $\phi_{(x,y)}(x')=7x'$ *and* $\alpha_{(x,y)}=0$, *therefore* $f_{(x,y)}(x')=7x'$ (mod 19) *for every arc* (x,y) *of* G.

THEOREM 4.1. *Let G be a graph with diameter $D\geqslant 3$, maximum degree*

Δ and n vertices and U an orientation of G such that every pair of vertices at distance D is connected by a path of length D containing two consecutive arcs of U (that is, a directed path of length 2). *Then $G * C_{19}$ (as defined above with $f_{(x,y)}(x') = 7x'$) has diameter $D+2$, maximum degree $\Delta + 2$ and* 19*n vertices. Furthermore if the indegree and outdegree of every vertex of G are both* >0, *then the same conclusion holds with $D = 2$.*

Proof. To show that the diameter is at most $D+2$, it suffices to prove that:

(4.1a) If (x, y) and (y, z) are two consecutive arcs of U ((x, y, z) is a directed path of length two) then every vertex (z, z') is at most distance 4 from $(x, 0)$ (Fig. 4.1a).

(4.1b) If (x, y) is an arc of G, every vertex (y, y') is at most distance 4 from $(x, 0)$ (Fig. 4.1b).

(4.1c) If (y, x) and (x, z) are two arcs of U, then every vertex (x, x') is at most distance 4 from $(x, 0)$ (Fig. 4.1c).

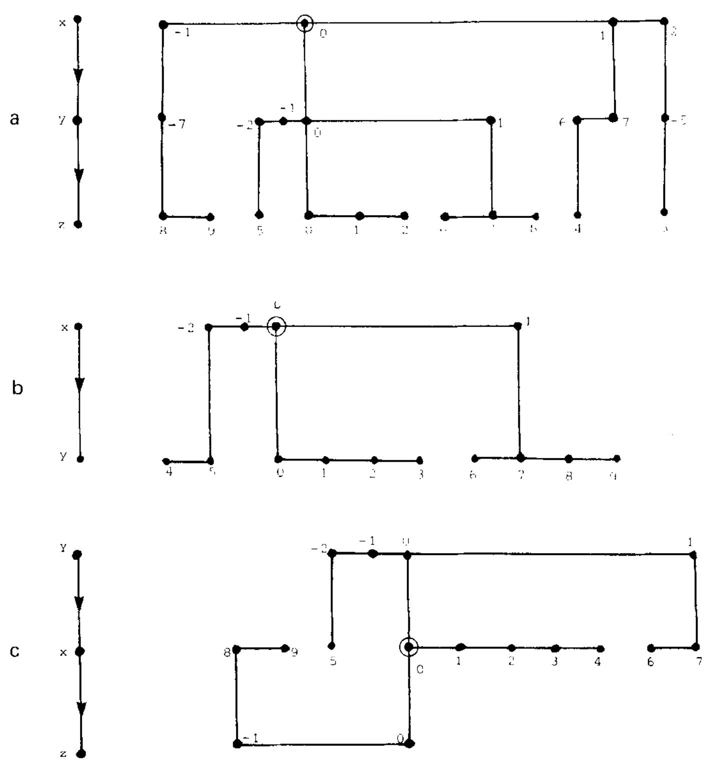

FIGURE 4.1

Indeed let (a, a') and (b, b') be any two vertices of $G * C_{19}$ with $a \neq b$. If the distance between a and b is d with $d < D$, let c be a vertex at distance $d - 1$ from a and adjacent to b. Then there exists a vertex (c, c') at most distance $d - 1$ from (a, a'), and by (4.1b) (c, c') and (b, b') are at most distance 4 (use the fact that by Section 3, (4.1b) implies that every vertex (z, z') is at most distance 4 from any vertex (x, x')); therefore (a, a') and (b, b') are at distance $d + 3 \leqslant D + 2$.

If a and b are at distance D, by the property of G there exists a path of length D between a and b containing a directed path of length two say (x, y, z). Let d_1 be the distance between a and x, and d_2 that between z and b; therefore $d_1 + d_2 + 2 = D$. There exist a vertex (x, x') at distance d_1 from (a, a') and a vertex (z, z') at distance d_2 from (b, b'). By (4.1a) the vertices (x, x') and (z, z') are at most distance 4. Then (a, a') and (b, b') are at distance at most $d_1 + d_2 + 4 = D + 2$.

Finally consider two vertices (x, x') and (x, x''); let y be adjacent to x, then there exists a vertex (y, y') adjacent to (x, x'') and by (4.1b) (y, y') is at most distance 4 from (x, x') therefore (x, x') and (x, x'') are at most distance $5 \leqslant D + 2$ if $D \geqslant 3$. If $D = 2$ and if there exist arcs (y, x) and (x, z); then the distance between (x, x') and (x, x'') is at most $4 = D + 2$ by (4.1c).

The proofs of (4.1a), (4.1b), (4.1c) are indicated on Figs. 4.1a,b,c; in fact, we give only one half of the tree by noticing that if there exists a path from $(x, 0)$ to (z, z') then there exists a path from $(x, 0)$ to $(z, 19 - z')$ by going in the opposite way on the cycles isomorphic to C_{19}. For example, from the path $(x, 0)$ $(x, 1)$ $(y, 7)$ $(y, 6)$ $(z, 4)$ from $(x, 0)$ to $(z, 4)$ in Fig. 4.1a we obtain immediately the path $(x, 0)$ $(x, 18)$ $(y, 12)$ $(y, 13)$ $(z, 15)$ from $(x, 0)$ to $(z, 15)$.

Finally the diameter is exactly $D + 2$; indeed if x and y are two vertices of G at distance D, then $(x, 0)$ is at distance at most $D + 1$ from (y, i) if and only if $i \equiv 0$ or $i \equiv \pm 7^P$ (mod 19), that is, $i \equiv 0, 1, 7, 8, 11, 12, 18$ (mod 19). Otherwise (y, i) is at distance at least $D + 2$ from $(x, 0)$.

Examples of Applications

Take G to be C_5 oriented as a directed circuit. We obtain a graph of diameter 4, degree 4 on 95 vertices. Take G as the graph on 11 vertices with arcs (x, y) if $y - x \equiv 1$ or 3 (mod 11). It can be checked easily that G satisfies the hypothesis and therefore $G * C_{19}$ is a graph of diameter 4, degree 6 on 209 vertices.

Another useful class of graphs was exhibited by Delorme [6] who proved that incidence graphs of generalized quadrangles or hexagons satisfy the hypothesis of the theorem. For example, we have such a graph of diameter 6, degree 5 on 2730 vertices; the product with C_{19} gives a graph of diameter 8, degree 7 on 51,870 vertices (compare with 3626 known before).

4.2 $(G * C_{17})$. *Let* $\Gamma = Z_{17}$; $S = \{-1, +1\}$, *then* G' *is isomorphic to* C_{17}.

THEOREM 4.2. *Let G be a graph with diameter* $D \geqslant 3$, *maximum degree* Δ *on n vertices. Let the edges of G be partitioned into two sets A and B and let U be an orientation such that every pair of vertices at distance D can be connected by a path of length D containing either two consecutive arcs of type A or two arcs of different type A and B. Then* $G * C_{17}$, *where* $f_{(x,y)}(x') = 3x' \pmod{17}$ *if* (x, y) *is of type A and* $f_{x,y}(x') = 4x' \pmod{17}$ *if* (x, y) *is of type B, is a graph of diameter at most* $D + 2$, *maximum degree* $\Delta + 2$ *on* $17n$ *vertices.*

Moreover the same conclusion holds for $D = 2$, *if furthermore every vertex of G is an end vertex of an edge of type B and the initial vertex of an arc of type A.*

Proof. As in the preceeding case (4.1) it suffices to prove that:

Every vertex (z, z') is at most distance 4 from the vertex $(x, 0)$ if x and z are at distance 2 in G with y the common neighbor of x and z with

$$(x, y) \in A, \quad (y, z) \in B \qquad \text{(Fig. 4.2a)},$$
$$(y, x) \in A, \quad (y, z) \in B \qquad \text{(Fig. 4.2b)},$$
$$(x, y) \in A, \quad (y, z) \in A \qquad \text{(Fig. 4.2c)}.$$

Note that the cases $(x, y) \in A$, $(z, y) \in B$ and $(y, x) \in A$, $(z, y) \in B$ can be deduced easily from cases (4.2a) and (4.2b) by noticing that $4^2 \equiv -1 \pmod{17}$ and therefore the orientations of the edges of B are not important.

The proofs are indicated in Figs. 4.2a–c where we have indicated only one half of the tree by noticing that if there exists a path from $(x, 0)$ to (z, z') there exists a path from $(x, 0)$ to $(z, 17 - z')$. If (x, y) is an arc of A (resp. B) then every vertex (y, y') is at most distance 4 from the vertex $(x, 0)$ (Fig. 4.2d) (resp. 4.2e). (The case (y, x) is an arc of A or B is dealt with similarly.) If (x, y) is an arc of A and (z, x) an arc of B (or (x, z) an arc of B) then every vertex (x, x') is at most distance 4 from the vertex $(x, 0)$ (Fig. 4.2f).

Remark. We do not have necessarily equality in Theorem 4.2. For example, if G is the directed cycle C_{15}, all the arcs belonging to A, $C_{15} \times C_{17}$ has only diameter 8 (and not 9).

APPLICATIONS. We can take for G the Petersen graph; A will be the set of edges of the two 5-cycles oriented to form directed circuits and B the other edges (see Fig. 6.3a): it satisfies the hypothesis and therefore we obtain a graph on 170 vertices, of diameter 4 and degree 5 different from the (5, 8) cage. More generally the theorem can in most cases be applied with G itself

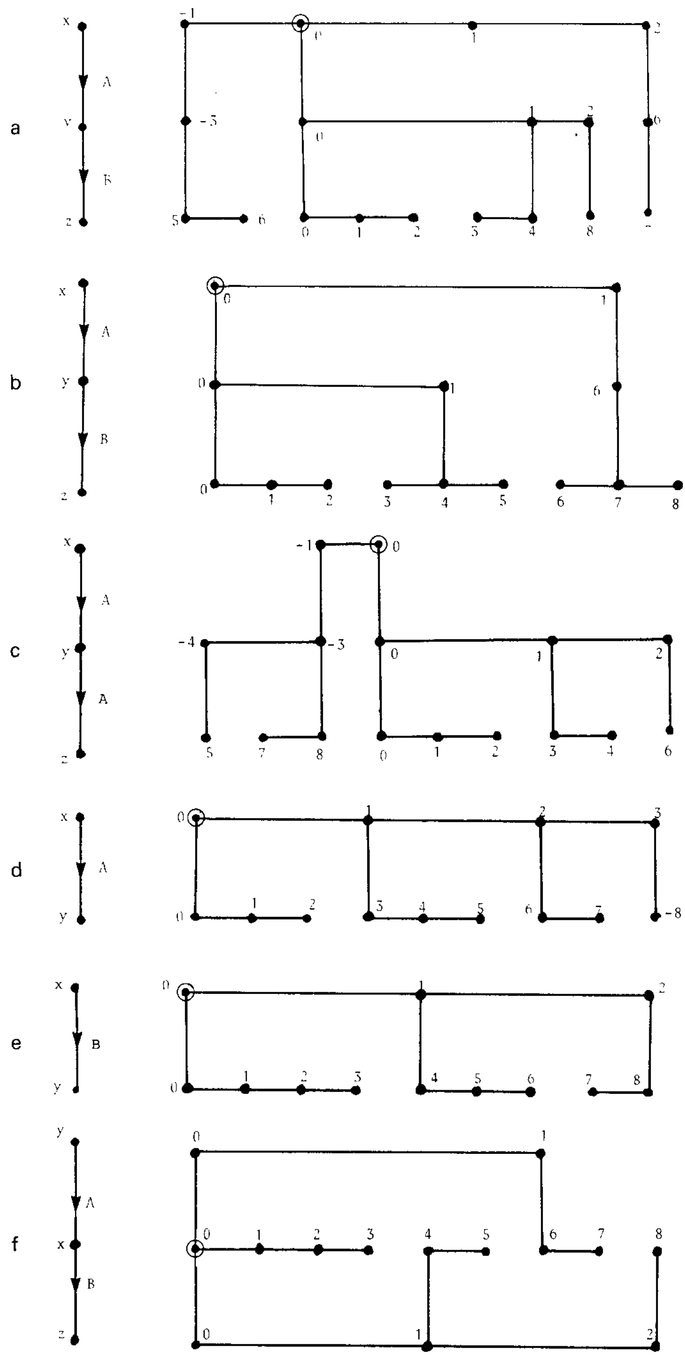

FIGURE 4.2

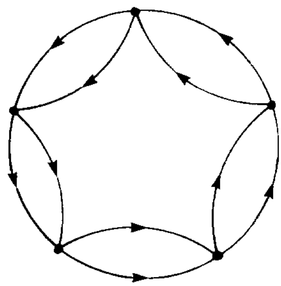

FIGURE 4.3.1

being a product, the edges of A being the edges of the copies of the second argument. For example, there exists a graph on 31248 vertices which is the product of the incidence graph of a generalized hexagon on 7812 vertices by F_4 (see Theorem 6.1); it can be multiplied by C_{17} and that gives a graph on 531216 vertices of diameter 9 and degree 9.

4.3 (An example of $G * C_{21}$). *Let $\Gamma = Z_{21}$, $S = \{-1, +1\}$ then G' is isomorphic to C_{21}. Let G_0 be the directed 5-cycle (Fig. 4.3.1) but with two arcs from any vertex to the following one. Let $f_{(x,y)}(x') = 4x' + 5 \pmod{21}$ if (x, y) is one of these two arcs, $f_{(x,y)}(x') = 4x' - 5 \pmod{21}$ if (x, y) is the other one.*

THEOREM. *The graph $G_0 * C_{21}$ as defined above, has* 105 *vertices, diameter* 3 *and degree* 6.

Proof. The fact that the diameter is 3 is shown by Fig. 4.3.2.

4.4 ($C_p * C_q$). *Let $\Gamma = Z_q$ with $S = \{-1, +1\}$ therefore G' is isomorphic to C_q. Let G be the directed cycle of length p: we consider the vertices of G*

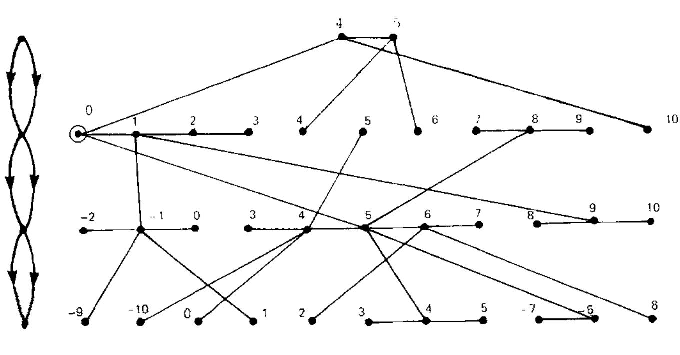

FIGURE 4.3.2

as elements of Z_p *and denote the arcs* $(i, i+1)$. *Let* $f_{(i,i+1)}(x') = kx'$ *for* $i = 0, 1, \dots, p-2$ *and* $f_{(p-1,0)}(x') = kx' + 1$. *Then a clever choice of* p, q, k *gives interesting graphs of degree* 4.

For example, with $p = 4$, $q = 9$, $k = 4$ one obtains a graph on 36 vertices of diameter 3 and degree 4. Similarly with $p = 10$, $q = 123$, $k = 16$ one obtains a graph on 1230 vertices of diameter 8 and degree 4. With $p = 12$, $q = 161$, $k = 13$ one obtains a graph on 1932 vertices of diameter 9 and degree 4. The verifications are left to the reader (they can be shortened by congruence considerations).

Remark 4.5. One can consider other products with G' a cycle. In many cases they do not give better results than those obtained by other methods. However, Delorme [6] used successfully products with C_{35} with graphs G having the same properties as those in Theorem 4.2 and in particular for the incidence graphs of generalized hexagons. (See in the table the graph of diameter 9, degree 8 on 273,420 vertices obtained by a product ith C_{35}.)

5. Product with S_3

Here Γ will be the non abelian group S_3 of the permutations of a set of 3 elements. Let t be one of the 3 transpositions and s be one of the two cyclic permutations and e the identity. Then the six elements of Γ are: e, t, s, ts, s^2, ts^2. Let $S = \{t\}$. Therefore G' is the graph consisting of the 3 edges $\{e, t\}\{s, ts\}$ and $\{s^2, ts^2\}$.

For every arc (x, y) of G let $f_{x,y}(x') = sx's^2$.

Theorem 5. *Let G be a graph with diameter* $D \geqslant 3$, *maximum degree* Δ, *on* n *vertices and* U *an orientation of* G *such that every pair of vertices at distance* D *is connected by a path of length* D *containing two consecutive arcs of* U. *Then* $G * S_3$ (*as defined above*) *has diameter* $D + 2$, *maximum degree* $\Delta + 1$ *and* $6n$ *vertices*.

Proof. In view of Section 3 it suffices to prove that:

(i) if (x, y, z) is a directed path of length two then every vertex (z, z') is at most distance 4 from (x, e) (Fig. 5),

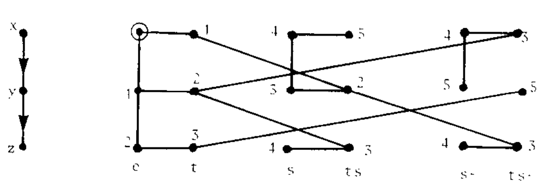

Fig. 5. The distances from (x, e) are indicated.

(ii) if (x, y) is an arc of G, then every vertex (y, y') or (x, x') is at most distance 5 from (x, e) (see also Fig. 5).

The equality is attained; indeed one can check that if x and y are two vertices of G at distance D, then (x, e) and (y, s) are at distance $D + 2$.

Examples of Applications

We have already seen (Section 4.1) that there are many graphs which have the property of the theorem (e.g., the incidence graphs of generalized hexagons giving therefore good graphs of diameter 8). For example, with the graph of diameter 6, degree 4 on 728 vertices as graph G, we obtain a graph of diameter 8, degree 5 on 4368 vertices.

6. Product with Finite Fields

Let Γ be the additive group of a finite field F of characteristic two. Let $S = \{1\}$; therefore G' is a perfect matching. For every arc (x, y) of G, let $f_{(x,y)}(x') = Xx'$, where X is some generator of F on F_2 (otherwise the product will not be connected).

6.1 $(G * F_4)$. *First let us consider* $F_4 = F_2(X)/X^2 + X + 1$. *The elements of* F_4 *(i.e., of* G'*) will be* 0, 1, X, $X + 1 = X^2$.

Theorem 6.1. *Let G be a graph of diameter $D \geqslant 3$, maximum degree Δ on n vertices, and U an orientation of G such that any pair of vertices at distance D is connected by a path of length D containing two consecutive arcs of U. Then $G * F_4$ (as defined above) has diameter $D + 1$, maximum degree $\Delta + 1$ and $4n$ vertices. Moreover, if the indegree and outdegree of every vertex of G are both >0, the same conclusion holds for $D = 2$.*

Proof. It suffices to prove that

(i) if (x, y) is an arc of U, every vertex (y, y') is at most distance 3 from $(x, 0)$ (Fig. 6.1a),

(ii) if (x, y, z) is a directed path of length two every vertex (z, z') is at most distance 3 from $(x, 0)$ (Fig. 6.1a),

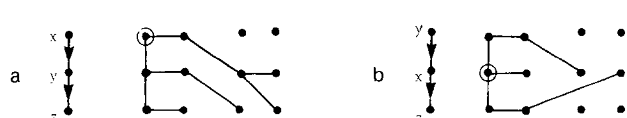

Figure 6.1

(iii) for the case $D=2$ if (y,x) and (x,z) are two arcs of U every vertex (x,x') is at most distance 3 from $(x,0)$ (Fig. 6.1b).

Examples of Applications

As we have seen in Section 4.1 many graphs have the property of the theorem and can be used as graphs G. For example, with C_5, oriented to form a directed circuit, we obtained a graph of degree 3, diameter 3 on 20 vertices which is extremal (by the introduction: $n(\Delta, D) \leqslant 20$). Also we can use for graphs G the incidence graphs of generalized quadrangles or hexagons to obtain many "good graphs" of diameter 5 or 7. For example, with the graph G of diameter 6, degree 6 on 7812 vertices we obtain a graph of diameter 7, degree 7 on 31,248 vertices.

6.2 (Product with other fields). *One can use in the same manner* $F_{16}=F_2[X]/X^4+X^3+X^2+X+1$ *or* $F_{32}=F_2[X]/X^5+X^2+1$. *We have obtained at least three interesting cubic graphs by using for G directed cycles*

$C_{13} * F_{16}$ *graph of diameter* 8 *on* 208 *vertices,*

$C_{15} * F_{16}$ *graph of diameter* 9 *on* 240 *vertices,*

$C_{15} * F_{32}$ *graph of diameter* 10 *on* 480 *vertices.*

The proof is somewhat long and we will not give it here. In Delorme [5] examples of products with F_8 of the incidence graphs of generalized quadrangles or hexagons are given.

6.3 (Another product with F_4). *In some cases we can use a better choice of the function* $f_{(x,y)}$ *(as in Theorem* 4.2). *For example, consider Petersen's graph with its arcs partitioned into two classes A and B (Fig.* 6.3a).

If (x,y) is an arc of A let $f_{(x,y)}(x')=Xx'$ and if (x,y) is an arc of B let $f_{(x,y)}(x')=Xx'+X$. We will show that $P * F_4$ is of diameter 3 (therefore having a graph of degree 4 on 40 vertices). Note that the affine functions f have the same determinant X. Thus any vertex (x,x') is at most distance 3 from any vertex (x,y') (Fig. 6.3b), from any vertex (y,y'), where (x,y) is an arc of P (Fig. 6.3c) and from any vertex (z,z'), where (x,y,z) is a directed path of length two (Fig. 6.3d).

Now if x and z are at distance two with y the common neighbour: if $(x,y)\in A$ and $(z,y)\in B$ we use the fact that there exists also a directed path from z to x of length 3 (z,u,v,x) (Fig. 6.3e) with $(z,u)\in A$; $(u,v)\in B$; $(v,x)\in A$. If $(y,x)\in A$ and $(y,z)\in B$ we use the fact that there exists a directed path (x,u,v,z) with $(x,u)\in B$; $(u,v)\in A$; $(v,z)\in A$ (Fig. 6.3f).

One can use the same product with F_4 by using the graph $K_3 * C_5$ suitably oriented to obtain a graph of degree 5, diameter 3 on 60 vertices, and using the Hoffman–Singleton graph to obtain a graph of diameter 3, degree 8 on 200 vertices.

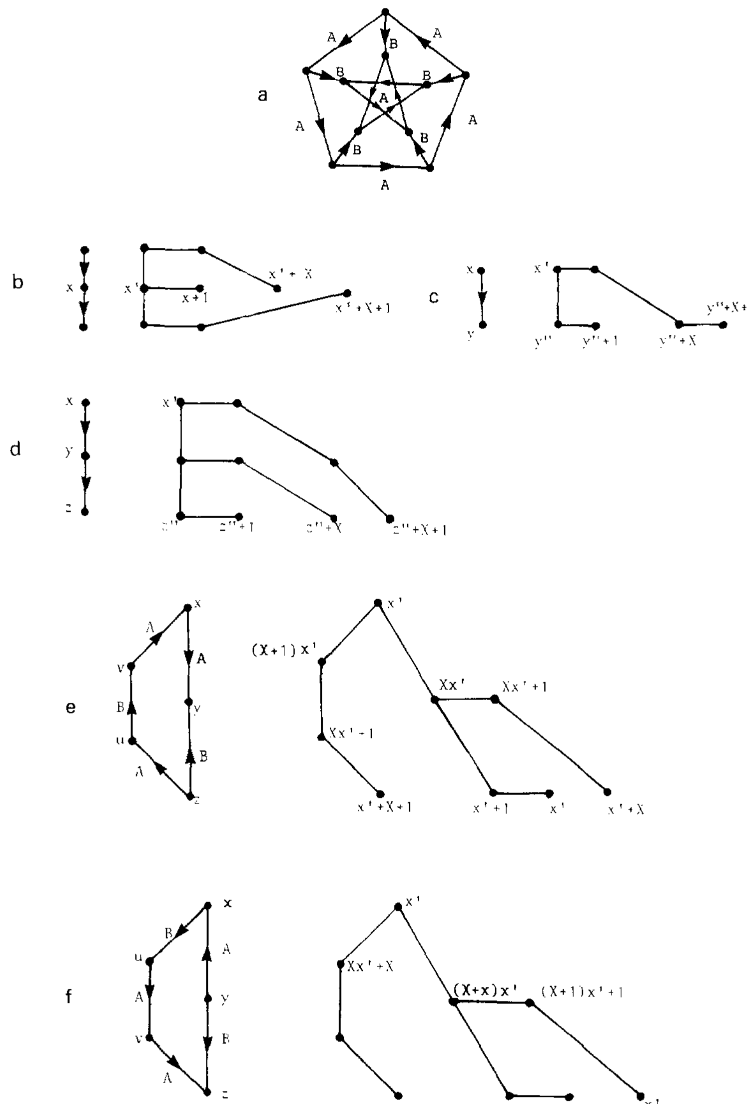

FIGURE 6.3

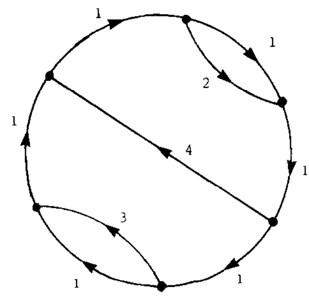

FIGURE 7.1

7. A LAST EXAMPLE: TUTTE–COXETER'S GRAPH

The famous Tutte–Coxeter's cubic graph on 30 vertices of diameter 4 can be considered as the product $G * G'$, where G is the graph of Fig. 7.1 and G' is the empty graph the vertex set (of Γ) being Z_5 and where $f_{(x,y)}(x') = x' + \alpha_{(x,y)} \pmod 5$ where $\alpha_{(x,y)}$ is indicated on Fig. 7.1. Tutte–Coxeter's graph is drawn in Fig. 7.2.

CONCLUSION

Other examples of applications appeared in another of the authors' papers [2] in which good asymptotic results were obtained. By using the product defined above and new results of Delorme [15], we obtain the best asymptotic results for diameters 4 and 6.

An earlier version of this paper contained a table of the best values known to us in 1981 and at that time many of them were obtained by the method of this paper. In many cases the values have been improved. A more recent

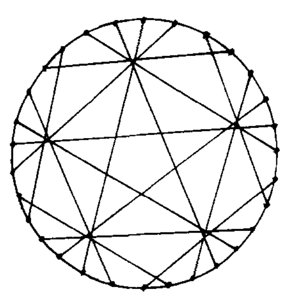

FIGURE 7.2

table appeared in [13] and an updated table can be requested of the authors. The reader can also find other results and references on the problem in the survey [14].

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