JOURNAL OF ALGEBRA 70, 261–269 (1981)

The Brauer Complex of a Chevalley Group

D. I. DERIZIOTIS

Department of Mathematics, University of Crete, Iraklion, Crete, Greece

Communicated by Jacques Tits

Received October 29, 1979

1. INTRODUCTION

Let G be a simple linear algebraic Chevalley group of universal type over K, the algebraic closure of the prime field $\mathbf{F}_p$, $p > 1$. In [7] J. Humphreys introduced a geometrical object called the Brauer complex of G which has been suggested for the solution of the "decomposition" problem of the modular representation theory of $G(q)$, the group of $\mathbf{F}_q$-rational points of G, where q is a power of p. The present paper provides some new properties of this complex which are summarized in Theorem 3.3. A consequence of this theorem (see [4]) is that one can relate the Brauer complex to the families of semisimple complex representations of $G^*(q)$ constructed by Deligne and Lusztig in [5], where G^* is an adjoint group being the dual of G (see [5]). In particular, by considering the semisimple classes in $G(q)$, we define certain points of the complex, the positions of which can determine what subgroups of $G(q)$ are the centralizers of semisimple elements in $G(q)$. This is shown in [6]. In [5] the degrees of the semisimple representations of $G^*(q)$ are given by a formula involving the orders of these subgroups. Hence, one can now determine in a simple way these degrees for all groups of adjoint type (see detailed tables in [6]). Also, in a similar context, we have computed in [6] the number of semisimple representations belonging to a given family for groups of rank $\leqslant 2$. This gives us an idea how the Brauer complex might be used to obtain this number for groups of higher rank than 2. Another immediate consequence of Theorem 3.3 is the well-known result of R. Steinberg [9] that $G(q)$ has q^l semisimple classes where l is the rank of G.

2. PRELIMINARIES

We follow the definitions and notations of [3] and [8], where the reader will also find further references as well as proofs of standard results. Let g be

0021-8693/81/050261–09$02.00/0

the simple complex Lie algebra of G and $H_{\mathbb{C}}$ a Cartan subalgebra of g. Consider the root system Φ of g relative to $H_{\mathbb{C}}$ and fix a fundamental basis $\{r_i;\ i=1,2,\dots,l\}$ in Φ. If r is a root, then h_r denotes the coroot associated to r. In particular, let h_i denote the coroot associated to r_i, $i=1,2,\dots,l$. Consider the $\mathbb{Z}$-lattice Y of the coroots. Then the real space $H = Y \otimes \mathbb{R}$ endowed with the Killing form of g is a Euclidean space.

Let W be the Weyl group generated by all the reflections w_r in the hyperplanes in H orthogonal to the coroots h_r, $r \in \Phi$. For each $r \in \Phi$ and $k \in \mathbb{Z}$, we put $w_{r,k} = d(kh_r)\, w_r$ to denote the affine reflection in the hyperplane $H_{r,k} = \{h \in H;\ r(h) = k\}$, where $d(kh_r)$ denotes the translation by the element kh_r. Now the set of all hyperplanes $H_{r,k}$, $(r \in \Phi,\ k \in \mathbb{Z})$, is stable under the semidirect product $D \cdot W = W_a$, where D is the group of translations by elements of Y. W_a is called the affine Weyl group of Φ. Any connected component C of the open subset $H - \bigcup_{r,k} H_{r,k}$ of H is an open simplex, called an alcove. The alcove $C_0 = \{h \in H;\ r_i(h) > 0,\ i = 1,2,\dots,l,\ r_0(h) < 1\}$ is called the fundamental alcove, where r_0 denotes the highest root in Φ. A fundamental region of the action of W_a on H is the closed simplex $\bar{C}_0$, the closure of C_0.

Following [8], the group G is constructed from a given finite dimensional faithful g-module V. Let $\mu_1, \mu_2, \dots, \mu_n$ be the weights of a fixed admissible basis of V. As G is assumed to be of universal type, the $\mathbb{Z}$-span of these weights is just the weight lattice $X = \sum_{i=1}^{l} \mathbb{Z}\lambda_i$, where the λ_i's are the fundamental weights, i.e., the linear functions on H defined by $\lambda_i(h_j) = \delta_{ij}$ (Kronecker delta) $i,j = 1,2,\dots,l$. In G there is a maximal torus T which splits over $\mathbf{F}_p$. The torus T is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(X, K^*)$, where K^* denotes the multiplicative group of K. In fact, T is generated by all $h(\mathscr{X}) = \mathrm{diag}(\mathscr{X}(\mu_1), \mathscr{X}(\mu_2), \dots, \mathscr{X}(\mu_n))$, $\mathscr{X} \in \mathrm{Hom}_Z(X, K^*)$. Moreover, each $h(\mathscr{X})$ is written uniquely in the form $\prod_1^l h(\mathscr{X}_{r_i,z_i})$, where $\mathscr{X}_{r_i,z_i}(\lambda) = z_i^{\lambda(h_i)}$, $\lambda \in X$ and $z_i = \mathscr{X}(\lambda_i)$, $i = 1,2,\dots,l$. Identifying the Weyl group W with the group generated by all the reflections in the space $X \otimes \mathbb{R}$ associated to the roots, W acts on T by $w(h(\mathscr{X})) = h(w(\mathscr{X}))$, where $w(\mathscr{X})(\lambda) = \mathscr{X}(w^{-1}(\lambda))$ for all $\lambda \in X$.

Consider now the additive group $\mathbb{Q}_{p'}$ of the rational numbers whose denominators are not divisible by p. We recall that the additive group $\mathbb{Q}_{p'}/\mathbb{Z}$ is isomorphic to K^* and that $Y \otimes (\mathbb{Q}_{p'}/\mathbb{Z})$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}_{p'}/\mathbb{Z})$. The following lemma is needed in the next section.

LEMMA 2.1. (a) *Let ρ be a homomorphism of $\mathbb{Q}_{p'}$ onto K^*. Then there exists a surjective homomorphism $Y \otimes \mathbb{Q}_{p'} \to \mathrm{Hom}_{\mathbb{Z}}(X, K^*)$ given by $\Sigma y_i \otimes \xi_i \to \Pi \mathscr{X}_{y_i \otimes \rho(\xi_i)}$, where $\mathscr{X}_{y_i \otimes \rho(\xi_i)}(\lambda) = \rho(\xi_i)^{\lambda(y_i)}$. Thus every element of T is of the form $h(\mathscr{X}_y)$ for some $y \in Y \otimes \mathbb{Q}_{p'}$.*

(b) *The semisimple conjugacy classes of G are in one-to-one correspondence with the points in $Y \otimes \mathbb{Q}_{p'} \cap \bar{C}_0$.*

Proof. (a) The map $Y \times \mathbb{Q}_{p'} \to \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}_{p'}/\mathbb{Z})$ defined by $(y, \xi) \to \mathscr{X}_{(y,\xi)}$, where $\mathscr{X}_{(y,\xi)}(\lambda) = \lambda(y)\, \xi \bmod Z$, induces a unique homomorphism $Y \otimes \mathbb{Q}_{p'} \to \mathrm{Hom}(X, \mathbb{Q}_{p'}/\mathbb{Z})$ given by $\Sigma y_i \otimes \xi_i \to \mathscr{X}_{\Sigma y_i \otimes \xi_i \bmod \mathbb{Z}} = \Pi \mathscr{X}_{y_i \otimes \xi_i \bmod \mathbb{Z}}$, where $\Pi \mathscr{X}_{y_i \otimes \xi_i \bmod \mathbb{Z}}: \lambda \to \Sigma \lambda(y_i)\, \xi_i \bmod \mathbb{Z}$ for all $\lambda \in X$. Since for all $\lambda \in X$ we have $\Sigma \lambda(y_i)\, \xi_i \in \mathbb{Z}$ if and only if $\Sigma y_i \otimes \xi_i \in Y$, the kernel of this homomorphism is the lattice Y. Now the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q}_{p'} \to \mathbb{Q}_{p'}/\mathbb{Z} \to 0$ gives the tensored sequence $0 \to Y \otimes \mathbb{Z} = Y \to Y \otimes \mathbb{Q}_{p'} \to Y \otimes (\mathbb{Q}_{p'}/\mathbb{Z}) \to 0$ which is also exact. Hence, $\mathrm{Hom}(X, \mathbb{Q}_{p'}/\mathbb{Z}) \cong Y \otimes (\mathbb{Q}_{p'}/\mathbb{Z}) \cong Y \otimes \mathbb{Q}_{p'}/Y$ and the above homomorphism is surjective. Therefore, under ρ we have the surjective homomorphism as the lemma states.

(b) We know (c.f. [1, 9]) that two elements $h(\mathscr{X}_y), h(\mathscr{X}_{y'})$, $y, y' \in Y \otimes \mathbb{Q}_{p'}$, of T are conjugate in G if and only if are conjugate under W, i.e., $w(\mathscr{X}_y) = \mathscr{X}_{y'}$ for some $w \in W$. If $y = \Sigma y_i \otimes \xi_i$, $y_i \in Y$, $\xi_i \in \mathbb{Q}_{p'}$, then by (a) we have $w(\mathscr{X}_y)(\lambda) = \Pi \rho(\xi_i)^{w^{-1}(\lambda)(y)} = \Pi \rho(\xi_i)^{\lambda(w(y_i))} = \mathscr{X}_{w(y)}(\lambda)$ for all $\lambda \in X$. Therefore, $h(\mathscr{X}_y)$ is conjugate to $h(\mathscr{X}_{y'})$ in G if and only if $\mathscr{X}_{w(y)} = \mathscr{X}_{y'}$, i.e., $\mathscr{X}_{w(y)-y'} = 1$. That is, $w(y) - y' \in Y$ which means that y and y' determine the same W_a-orbit in $Y \otimes \mathbb{Q}_{p'}$. Now since the W_a-orbits in $Y \otimes \mathbb{Q}_{p'}$ are represented by the elements of $Y \otimes \mathbb{Q}_{p'} \cap \bar{C}_0$ and since every semisimple element of G is conjugate to some element of T, we obtain the requirement.

Finally let $\sigma: G \to G$ be the Frobenius map which raises every matrix entry to its qth power. Then the group G_σ of the fixed points under σ is a finite Chevalley group. As G is assumed to be of universal type, G_σ is the group $G(q)$ of $\mathbf{F}_q$-rational points of G. σ acts on the semisimple classes of G. By the above lemma the σ-stable semisimple classes correspond to elements of $\bar{C}_0$. These elements of $\bar{C}_0$ will be called the σ-invariant points of $\bar{C}_0$.

3. The Brauer Complex

In this section we introduce the Brauer complex which will be used to find properties characterizing the σ-invariant points of $\bar{C}_0$.

For a positive integer n we consider the affine reflection group $W_{a,n} = D_n \cdot W$, where D_n denotes the group of translation $(1/q^n)\, D$. The groups $W_{a,n}$, $n \in \mathbb{Z}^+$, have properties similar to the affine Weyl group W_a. We state some of them needed for the present purposes.

(a) $W_{a,n}$ is a Coxeter group on generators $w_{r_1}, \ldots, w_{r_l}$ and $w_{r_0, 1/q^n}$.

(b) The simplex $\mathcal{O}_0^n = (1/q^n)\, \bar{C}_0 = \{h \in X;\ r_i(h) \geqslant 0,\quad i = 1, 2, \ldots, l,\ r_0(h) \leqslant 1/q^n\}$ is a fundamental region for the action of $W_{a,n}$ on H.

Let $d((1/q^n)\, y)\, w$, $(y \in Y, w \in W)$, be an element of $W_{a,n}$. Then we denote $\mathcal{O}_{w,y}^n$ the open simplex $\{h \in H;\ w(r_i)(h) > (1/q^n)\, w(r_i)(y),\ w(r_0)(h) <$

$(1/q^n)(w(r_0)(y)+1)$, $i=1,2,\ldots,l\}$. If $w' \in W$, $y' \in Y$ and $m \in \mathbb{N}$, then we call the element w' (resp. y') the relative orientation (resp. position) of $\mathcal{A}^{mn}_{w'w,q^m my+w(y')}$ with respect to $\mathcal{A}^n_{w,y}$. The simplices $\mathcal{A}^1_{w,y}$, $(w \in W, y \in Y)$, will be called the (w,y)-alcoves and denoted simply by $\mathcal{A}_{w,y}$. In particular, the simplex $\mathcal{A}^1_0$ will be denoted by $\mathcal{A}_0$.

Let Σ_n be the set of hyperplanes $H_{r,z/q^n}$, $(r \in \Phi, z \in Z)$.

(c) $\Sigma_n = \{w(H_i),\ i=1,2,\ldots,l,\ w(H_{r_0}, 1/q^n);\ w \in W_{a,n}\}$, where $H_i = H_{r_i,0}$ and $H_0 = H_{r_0,1}$. Moreover, $\Sigma_n \subset \Sigma_{n+1}$ and $W_{a,n} \subset W_{a,n+1}$, $n \in \mathbb{N}$.

Let $I_0 = \{0, 1, 2,\ldots, l\}$ and for $J \subsetneq I_0$ let F_J denote the set of elements $h \in H$ satisfying $r_j(h)=0$, for $0 \neq j \in J$, $r_i(h) > 0$, for $0 \neq i \notin J$ and $r_0(h)=1$ if $0 \in J$, $r_0(h) < 1$ if $0 \notin J$. Then F_J is an open simplex in the affine space $H_J = \bigcap_{j \in J} H_j$. The simplices $F^n_J = (1/q^n) F_J$, $J \subsetneq I_0$, are the faces of $\mathcal{A}^n_0$. We say that the face F^n_J is of type J.

(d) For an element $w \in W_{a,n}$, the closed simplex $w(\mathcal{A}^n_0)$ can also be chosen as a fundamental region of $W_{a,n}$ with respect to which the Coxeter generators of $W_{a,n}$ are the elements $ww_{r_i}w^{-1}$, $i=1,2,\ldots,l$ and $ww_{r_0,1/q^n}w^{-1}$. Therefore, the type of the face $w(F^n_J)$ of $w(\mathcal{A}^n_0)$ can be defined as J. In other words, the action of $W_{a,n}$ on H does not affect the types of the faces of $\mathcal{A}^n_0$. The face of type J of $\mathcal{A}^n_{w,y}$ will be denoted by $F^n_{J,w,y}$. A good reference for proofs of the above statements is [2].

DEFINITION 3.1. The Brauer complex of G denoted by B_0 is the set of all faces of (w,y)-alcoves which are in $\bar{C}_0$.

Since $\bar{C}_0$ is an l-dimensional simplex we see (because of volume reasons) that it can be filled up by exactly $q^l (w,y)$-alcoves. These are the l-dimensional faces of B_0.

PROPOSITION 3.2. *Let x_0 be an element in $Y \otimes \mathbb{Q}_{p'} \cap \bar{C}_0$. Then x_0 is a σ-invariant point if and only if $x_0 \equiv qw(x_0) \bmod Y$, for some $w \in W$.*

Proof. Suppose that x_0 is a σ-invariant point. Let $t_0 = h(\mathscr{X}_{x_0})$ be the element of the torus T which represents the semisimple class corresponding to x_0. Thus also the element $\sigma(t_0) = t_0^q = h(\mathscr{X}_{qx_0})$ is in this class. By Lemma 2.1, the points x_0 and qx_0 are in the same W_a-orbit. Suppose that the point x_0 lies in the closure of $\mathcal{A}_{w,y}$ for some $w \in W$, $y \in Y$. Then the point qx_0 lies in the simplex $w(\bar{C}_0)+y$. Thus the point $qw^{-1}(x_0) - w^{-1}(y)$ is the unique point in $\bar{C}_0$ which determines the W_a-orbit of qx_0. But since x_0 lies in $\bar{C}_0$ and is equivalent to qx_0 under W_a, we must have $x_0 = qw^{-1}(x_0) - w^{-1}(y)$. This gives the result in one direction. Conversely, let $x_0 \in Y \otimes \mathbb{Q}_{p'} \cap \bar{C}_0$ such that $x_0 = qw(x_0)+y$, for some $w \in W$, $y \in Y$. Then we have $\mathscr{X}_{x_0} = \mathscr{X}_{qw(x_0)}$ as $\mathscr{X}_y = 1$. Therefore, $t_0 = h(\mathscr{X}_{x_0}) = h(\mathscr{X}_{qw(x_0)}) = w(t_0^q)$, which means that the

W-orbit of t_0 is fixed by σ. Hence (see [1; p. 197]), the conjugacy class of t_0 in G is a σ-stable class, as required.

Notice that if we work backward in the second part of the proof we obtain the first. But the above proof points out for what elements $w \in W$ and $y \in Y$ we have $x_0 = qw(x_0) + y$.

The question now is how the σ-invariant points are distributed in $\bar{C}_0$. To answer this we relate the σ-invariant points to the Brauer complex.

THEOREM 3.3. (*a*) *The closure of each l-dimensional face in the Brauer complex B_0 contains exactly one σ-invariant point.*

(b) *Let $\mathcal{O}_{w,y}$ be an alcove (i.e., an l-dimensional face) in B_0, and let F be a face of $\mathcal{O}_{w,y}$ which has the smallest dimension among the faces of $\mathcal{O}_{w,y}$ for which the type is the same as the type of the face of $\bar{C}_0$ on whch it lies. Then F is unique and contains the σ-invariant point lying in $\bar{\mathcal{O}}_{w,y}$.*

(c) *The closures of two distinct l-dimensional faces of B_0 contain distinct σ-invariant points.*

Proof. (a) Suppose that for some $w \in W$ and $y \in Y$ we have

$$\frac{1}{q}\, r_0(w(x)+y) \leqslant 1, \qquad \frac{1}{q}\, r_i(w(x)+y) \geqslant 0, \qquad i = 1, 2, \ldots, l, \qquad (*)$$

for all $x \in \bar{C}_0$. In other words the closed simplex $\bar{\mathcal{O}}_{w,y}$ lies in $\bar{C}_0$. Thus because of (*) we also have $\bar{\mathcal{O}}^k_{w,y} \subset \bar{\mathcal{O}}^{k-1}_0$, $k \geqslant 1$. Therefore, applying the affine transformation

$$d\left(\frac{1}{q}\, y + \frac{1}{q^2}\, w(y) + \cdots + \frac{1}{q^{k-1}}\, w^{k-2}(y)\right) w^{k-1}$$

on the last inclusion we get $\mathcal{S}_k \subset \mathcal{S}_{k-1}$, where

$$\mathcal{S}_k = \bar{\mathcal{O}}^k_{w^k, \sum_{i=0}^{k-1} q^{k-(i+1)} w^i(y)}.$$

Doing this for all $k \geqslant 1$, we obtain the infinite chain of simplices $\bar{C}_0 = \mathcal{S}_0 \supset \mathcal{S}_1 \supset \mathcal{S}_2 \supset \cdots \supset \mathcal{S}_k \supset \cdots$.

Let n be the order of w. As $k \to \infty$, the simplex $\bar{\mathcal{O}}^k_0$ tends to the origin and so the above chain tends to the point

$$\begin{aligned}
&\sum_{i=0}^{\infty} \frac{1}{q^{in+1}}\, y + \sum_{i=0}^{\infty} \frac{1}{q^{in+2}}\, w(y) + \cdots + \sum_{i=0}^{\infty} \frac{1}{q^{in+n}}\, w^{n-1}(y) \\
&\quad = \frac{1}{q}\, z + \frac{1}{q^{n+1}}\, z + \frac{1}{q^{2n+1}}\, z + \cdots = \sum_{j=0}^{\infty} \left(\frac{1}{q^n}\right)^j \left(\frac{1}{q}\, z\right) \\
&\quad = \frac{1/q}{1 - 1/q^n}\, z = \frac{q^{n-1}}{q^n - 1}\, z,
\end{aligned}$$

where

$$z = y + \frac{1}{q} w(y) + \cdots + \frac{1}{q^{n-1}} w^{n-1}(y).$$

Therefore, the point $x_0 = (q^{n-1}/q^n - 1) z$ lies in $Y \otimes \mathbb{Q}_{p'} \cap \bar{\mathcal{O}\!\ell}_{w,y}$. On the other hand, we have

$$\begin{aligned} &q w^{-1}(x_0) - w^{-1}(y) \\ &= \frac{q^n}{q^n - 1} \left(w^{n-1}(y) + \frac{1}{q} y + \cdots + \frac{1}{q^{n-1}} w^{n-2}(y) \right) \\ &= \frac{q^{n-1}}{q^n - 1} z = x_0. \end{aligned}$$

Thus by the previous proposition x_0 is a σ-invariant point. Moreover, x_0 is the unique σ-invariant point in the simplex $\bar{\mathcal{O}\!\ell}_{w,y}$. For, if x_0' were another point in $\bar{\mathcal{O}\!\ell}_{w,y}$ satisfying the condition $x_0' = q w^{-1}(x_0') - w^{-1}(y)$, then we would have $x_0 - x_0' = q w^{-1}(x_0 - x_0')$. This implies $x_0 = x_0'$ since the Killing form on H is W-invariant. This gives (a).

(b) We saw that the σ-invariant point x_0 which is in $\bar{\mathcal{O}\!\ell}_{w,y}$ lies in every simplex $\mathcal{S}_k$, for all $k \geqslant 1$ and has the form $x_0 = (q^{n-1}/q^n - 1) \sum_{i=0}^{n-1} (1/q^i) w^i(y)$, where n is the order of w. We may assume $x_0 \neq 0$ and so $x_0 \neq (1/q) y$ (the case $x_0 = 0$ is trivial). Now the bounding hyperplanes of $\bar{\mathcal{O}\!\ell}_{w,y}$ are of the form $H_{w(r_i),(1/q) w(r_i)(y)}$, $i = 1, 2, \ldots, l$, and $H_{w(r_0),(1/q)(1 + w(r_0) + y)}$. Let us suppose that x_0 lies on a face F_J of $\bar{C}_0$. We prove that x_0 lies also on the face $F^1_{J,w,y}$ of $\mathcal{O}\!\ell_{w,y}$ of type J. From the definition of the faces, this is equivalent to showing that

(i) $r_i(x_0) = 0$ if and only if

$$w(r_i)(x_0) = \frac{1}{q} w(r_i)(y) \qquad \text{for } i \neq 0,$$

(ii) $r_j(x_0) > 0$ if and only if

$$w(r_j)(x_0) > \frac{1}{q} w(r_j)(y) \qquad \text{for } j \neq 0,$$

(iii) $r_0(x_0) = 1$ if and only if

$$w(r_0)(x_0) = \frac{1}{q} (1 + w(r_0)(y)),$$

and

(iv) $r_0(x_0) < 1$ if and only if

$$w(r_0)(x_0) < \frac{1}{q}(1 + w(r_0)(y)).$$

Let us prove equivalence (i). Equivalence (ii) is proved in the same way. We have $r_i(x_0) = 0$ if and only if $(q^{n-1}/q^n - 1)\sum_{m=0}^{n-1}(1/q^m)\, r_i(w^m(y)) = 0$ and this is equivalent to $\sum_{m=0}^{n-1}(1/q^{m+1})\, w(r_i)\,(w^{m+1}(y)) = 0$. As the Killing form is W-invariant, the last equation is equivalent to $\sum_{m=0}^{n-2}(1/q^{m+1})$ $w(r_i)(w^{m+1}(y)) = -(1/q^n)\, w(r_i)(y)$ and adding in both sides of this equation the term $w(r_i)(y)$ we get $\sum_{m=0}^{n-1}(1/q^m)\, w(r_i)(w^m(y)) = (q^n - 1/q^n)\, w(r_i)(y)$ we obtain the equivalent equation $(q^{n-1}/q^n - 1)\sum_{m=0}^{n-1}(1/q^m)\, w(r_i)(w^m(y)) = (1/q)\, w(r_i)(y)$ as required. For equivalence (iii) we have $r_0(x_0) < 1$ if and only if $\sum_{m=0}^{n-1}(1/q^{m+1})\, w(r_0)(w^{m+1}(y)) < (q^n - 1/q^n)$. That is,

$$\sum_{m=0}^{n-2}\frac{1}{q^{m+1}}\, w(r_0)(w^{m+1}(y)) < \frac{q^n - 1 - w(r_0)(y)}{q^n}.$$

Adding the term $w(r_0)(y)$ in both sides of the last inequality we get

$$\sum_{m=0}^{n-1}\frac{1}{q^m}\, w(r_0)(w^m(y)) < \frac{(q^n - 1)(1 + w(r_0)(y))}{q^n}$$

or equivalently

$$\frac{q^{n-1}}{q^n - 1}\sum_{m=0}^{n-1}\frac{1}{q^m}\, w(r_0)(w^m(y)) < \frac{1}{q}(1 + w(r_0)(y))$$

as required.

Similarly one can prove equivalence (iv). Therefore the σ-invariant point x_0 lies on F_J if and only if it lies on $F^1_{J,w,y}$.

For the rest of the claim in (b) we look at the relative orientation and position of the simplex $\mathscr{S}_k$ wtih respect to $\mathscr{S}_{k-1}$. Suppose that a face $F^1_{J,w,y}$ of $\mathscr{S}_1$ lies on the face F_J of $\bar{C}_0$. We apply induction on k to show that the face of $\mathscr{S}_k$ of type J lies on F_J, for all $k \in \mathbb{N}$. Since the orientation and the position of $\mathscr{S}_2$ with respect to $\mathscr{S}_1$ is the same as that of $\mathscr{S}_1$ with respect to $\bar{C}_0$, the face of $\mathscr{S}_2$ of type J lies on $F^1_{J,w,y}$ and so on F_J. Now suppose that the face of $\mathscr{S}_k$ of type J lies on F_J. Then since the relative orientation and position of $\mathscr{S}_{k+1}$ with respect to $\mathscr{S}_k$ is the same as that of $\mathscr{S}_k$ with respect to $\mathscr{S}_{k+1}$ we see that the face of $\mathscr{S}_{k+1}$ of type J lies on the face of $\mathscr{S}_{k-1}$ of type J and so, by the induction hypothesis, on F_J. Taking the limit as $k \to \infty$ (as we did in (a)) we get a σ-invariant point on the closure of the face F_J. Doing this for all faces of $\mathscr{S}_1$ which lie on faces of $\bar{C}_0$ of the same type, we see that the intersection of their closures must be the face of $\mathscr{S}_1$ which contains the σ-invariant point lying in $\mathscr{S}_1$. From the first part of the proof of (b), this face must lie on a face of $\bar{C}_0$ of the same type. This completes the claim in (b).

(c) Since two distinct l-dimensional faces in B_0 obviously can not share a face which lies on a face of $\bar{C}_0$ of the same type, (c) is a consequence of (a) and (b).

Thus we have a one-to-one correspondence between σ-invariant points in $\bar{C}_0$ and (w, y)-alcoves in B_0. In other words, the faces of maximal dimension of the Brauer complex are in bijective correspondence with the σ-stable semisimple classes in G.

COROLLARY. (a) *The number of σ-stable semisimple conjugacy classes of $G(q)$ is q^l.*

(b) *Each σ-invariant point in $\bar{C}_0$ is of the form*

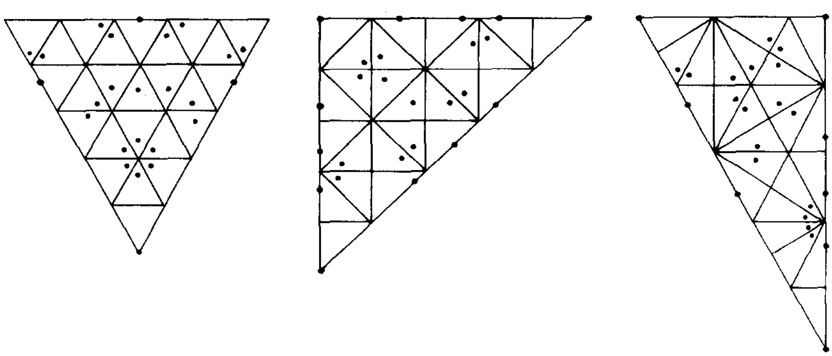
$$\frac{q^{n-1}}{q^n - 1}\left(y + \frac{1}{q}\, w(y) + \cdots + \frac{1}{q^{n-1}}\, w^{n-1}(y)\right)$$

for some $w \in W$, $y \in Y$, where n is the order of w.

Proof. (*a*) We know (see [1; p. 197]) that the semisimple classes in $G(q)$ are in bijective correspondence with the σ-stable semisimple classes in G which, by the above theorem, are q^l in number. This is the well known result obtained by Steinberg (see [9]);

(b) is in the proof of the theorem.

The figures below are examples for $q = 5$ showing how the σ-invariant points are distributed on the faces of the Brauer complex for types A_2, B_2 and G_2.



As we have mentioned in the introduction, in view of Theorem 3.3, the Brauer complex, in our version, has some interesting consequences to the representation theory of $G^*(q)$ (see [4, 6]) as well as to the structure of the centralizers of semisimple elements in $G(q)$ (see [6]). We hope to give a complete treatment of this in a forthcoming paper.

ACKNOWLEDGMENT

I take this opportunity to thank Professor Roger W. Carter for the encouragement he gave me during my studies at Warwick University.

REFERENCES

1. A. BOREL *et al.*, "Seminar on Algebraic Groups and Related Finite Groups," Lecture Notes in Mathematics No. 131, Springer, Berlin, 1970.
2. N. BOURBAKI, "Groups et algèbre de Lie," Chaps. IV–VI, Hermann, Paris, 1969.
3. R. W. CARTER, "Simple Groups of Lie Type," Wiley, London/New York, 1972.
4. R. W. CARTER, "Complex Representation Theory of Finite Groups of Lie Type," Research Symposium on Finite Simple Groups, Durham, 1978.
5. P. DELIGNE AND G. LUSZTIG, Representations of reductive groups over finite fields, *Ann. of Math.* **103** (1976).
6. D. DERIZIOTIS, Ph.D. thesis, University of Warwick, 1977.
7. J. E. HUMPHREYS, "Ordinary and Modular Representations of Chevalley Groups," Lecture Notes in Mathematics No. 528, Springer, Berlin, 1976.
8. R. REE, Construction of certain semisimple groups, *Canad. J. Math.* **16** (1964).
9. R. STEINBERG, Endomorphisms of linear algebraic groups, *Mem. Amer. Math. Soc.* **80** (1968).