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Bounded decomposition in the Brieskorn lattice and Pfaffian Picard–Fuchs systems for Abelian integrals ☆

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Abstract

We suggest an algorithm for derivation of the Picard–Puchs system of Pfaffian equations for Abelian integrals corresponding to semiquasihomogeneous Hamiltonians. It is based on an effective decomposition of polynomial forms in the Brieskorn lattice. The construction allows for an explicit upper bound on the norms of the polynomial coefficients, an important ingredient in studying zeros of these integrals.

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1. Introduction

Given a polynomial in two variables $f \in \mathbb{R}[x, y]$ and a polynomial 1-form ω on $\mathbb{R}^2$, how many isolated ovals δ on the level curves $f = \text{const}$ may satisfy the condition $\oint_\delta \omega = 0$? This is the long-standing *infinitesimal Hilbert problem*, see [1]. The answer is to be given in terms of the degrees of f and ω.

A recent approach to this problem, suggested in [15,16,18] is based on the fact that periods of polynomial 1-forms restricted on level curves of polynomials, satisfy a system of differential equations with rational coefficients, called the *Picard–Fuchs system*. Under certain restrictions on the monodromy group, the number of zeros of solutions of such

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systems can be estimated from above in terms of the magnitude of coefficients of this system, more precisely, the norms of its matrix residues. Thus it becomes important to derive the Picard–Fuchs system for Abelian integrals so explicitly as to allow for the required estimates for the residues.

In [16] a Fuchsian system was derived in the *hypergeometric* form

$$(t\cdot \mathbf{1}+A)\dot I = BI, \qquad \dot I=\frac{d}{dt}I(t), \tag{1.1}$$

where $I(t)=(I_1(t),\dots,I_l(t))$ is a collection of integrals of some monomial forms over any oval of the level curve $\{f=t\}$, and A, B are two constant $(l\times l)$-matrices of explicitly bounded norms, depending on f ($\mathbf{1}$ always stands for the identity matrix of the appropriate size). The rational matrix function $R(t)=(t\cdot\mathbf{1}+A)^{-1}B$ has only simple poles and the norm of its matrix residues can be explicitly majorized provided that the eigenvalues of A remain well apart. This allows to solve the infinitesimal Hilbert problem for all polynomials f whose critical values (after a suitable normalization) are sufficiently distant from each other. What remains is to study the case of confluent critical values (including those at infinity).

In a general hypergeometric system (1.1), the residues may or may not blow up as some of the singular points tend to each other. The particular feature of the Picard–Fuchs system is its *isomonodromy*: the monodromy group remains the same under deformations of f (at least for sufficiently generic f). This implies that even if the explosion of residues occurs, it cannot be caused by the explosion of the eigenvalues. In order to find out what indeed happens with the residues, the first step is to write down as explicitly as possible the Picard–Fuchs system as a flat meromorphic connexion with singularities in the holomorphic bundle over the variety of all polynomials f of a given degree.

This problem is solved in the paper for polynomials with a fixed principal (quasi) homogeneous part having an isolated critical point at the origin.

As an auxiliary first step, we need to describe explicitly the structure of the relative cohomology module. While the subject is fairly classic and sufficiently well understood, the existing tools do not allow for the quantitative analysis. We suggest an alternative, completely elementary construction that immediately yields all necessary bounds. This construction, exposed in Section 2 is based on "division by f", a lemma distilled from the paper [8] by J.-P. Françoise. The Pfaffian form of the Picard–Fuchs system is derived in Section 4. In the last section we mention some simple properties of the derived system and formulate a conjecture that it has only logarithmic singularities in the affine part.

2. Relative cohomology revisited

2.1. Relative cohomology, Brieskorn and Petrov modules

Denote by $\boldsymbol{\Lambda}^k$, $k=0,1,\dots,n$, the module of polynomial k-forms on the complex affine space $\mathbb{C}^n$ for a fixed $n\geqslant 1$. If $f\in\mathbb{C}[x_1,\dots,x_n]\simeq\boldsymbol{\Lambda}^0$ is a polynomial, then the collection $df\wedge\boldsymbol{\Lambda}^{k-1}$ of k-forms *divisible* by $df\in\boldsymbol{\Lambda}^1$, is a $\mathbb{C}$-linear subspace in $\boldsymbol{\Lambda}^k$, and the quotient

$$\boldsymbol{\Lambda}^k_f=\boldsymbol{\Lambda}^k/df\wedge\boldsymbol{\Lambda}^{k-1},\quad k=1,\dots,n, \tag{2.1}$$

is called the space of *relative k-forms*. Since the exterior derivative d preserves divisibility by df, the *relative de Rham complex* $\boldsymbol{\Lambda}_f^\bullet$,

$$0 \to \boldsymbol{\Lambda}_f^1 \xrightarrow{d} \boldsymbol{\Lambda}_f^2 \cdots \xrightarrow{d} \boldsymbol{\Lambda}_f^{n-1} \xrightarrow{d} \boldsymbol{\Lambda}_f^n \xrightarrow{d} 0, \tag{2.2}$$

naturally appears. A form $\omega \in \boldsymbol{\Lambda}^k$ is called *relatively closed* if $d\omega = df \wedge \eta$ and *relatively exact* if $\omega = df \wedge \xi + d\theta$ for appropriate $\eta \in \boldsymbol{\Lambda}^k$ and $\xi, \theta \in \boldsymbol{\Lambda}^{k-1}$. The *relative cohomology groups* $\boldsymbol{H}_f^k = H^k(\boldsymbol{\Lambda}_f^\bullet)$, relatively closed k-forms modulo relatively exact ones, are important characteristics of the polynomial f.

Together with the natural $\mathbb{C}$-linear structure, the relative cohomology groups $\boldsymbol{H}_f^k$ possess the structure of a module over the ring $\mathbb{C}[f] = f^*\mathbb{C}[x_1, \ldots, x_n]$. This follows from the identity

$$f \cdot (df \wedge \eta + d\theta) = df \wedge (f\eta - \theta) + d(f\theta) \tag{2.3}$$

meaning that relatively exact forms are preserved by multiplication by f.

As is well-known, the highest module $\boldsymbol{H}_f^n$, as well as all $\boldsymbol{H}_f^k$ with $0 < k < n-1$, is zero. Instead, we consider another important module, called *Brieskorn module* (lattice) [4, 6,7], defined as the quotient

$$\boldsymbol{B}_f = \boldsymbol{\Lambda}^n / df \wedge d\boldsymbol{\Lambda}^{n-2}, \tag{2.4}$$

and the $\mathbb{C}[f]$-module $\boldsymbol{P}_f$, the quotient of *all* $(n-1)$-forms by the *closed*, $(n-1)$-forms,

$$\boldsymbol{P}_f = \boldsymbol{\Lambda}^{n-1} / \bigl(df \wedge \boldsymbol{\Lambda}^{n-2} + d\boldsymbol{\Lambda}^{n-2}\bigr) \supseteq \boldsymbol{H}_f^{n-1}. \tag{2.5}$$

The latter is an extension of $\boldsymbol{H}_f^{n-1}$: the quotient $\boldsymbol{P}_f / \boldsymbol{H}_f^{n-1}$ is naturally isomorphic to the finite-dimensional $\mathbb{C}$-space $\boldsymbol{\Lambda}_f^n = \boldsymbol{\Lambda}^n / df \wedge \boldsymbol{\Lambda}^{n-1}$. In several sources, $\boldsymbol{P}_f$ is referred to as the *Petrov module*. The exterior differential naturally projects as a *bijective* map d: $\boldsymbol{P}_f \to \boldsymbol{B}_f$ which obviously *is not* a $\mathbb{C}[f]$-module homomorphism.

Clearly, a relatively exact (closed) form is exact (resp., closed) after being restricted on any nonsingular level set $f^{-1}(t) \subset \mathbb{C}^n$, $t \in \mathbb{C}$ since df vanishes on all such sets.

The inverse inclusion is considerably more delicate. Gavrilov studied the case $n = 2$ and proved that for a 1-form with exact restrictions on all level curves $f^{-1}(t) \subset \mathbb{C}^2$ to be relatively exact, it is sufficient to require that the polynomial f has only isolated singularities and all level curves $f^{-1}(t)$ be connected [9,10]. This result generalizes the earlier theorem by Ilyashenko [13]. A multidimensional generalization in the same spirit was obtained by I. Pushkar' [17]. The affirmative answer depends on the topology of a generic level set $f^{-1}(t)$ (its connectedness for $n = 2$ or vanishing of the Betti numbers b_k for k between 0 and $n-2$, see [3,5]).

Both the isolatedness and connectedness assumptions can be derived from a single assumption that the principal (quasi)homogeneous part $\hat{f}$ of the polynomial f has an isolated critical point at the origin: such polynomials are called *semiquasihomogeneous* [2]. For two variables with equal weights it suffices to require that $\hat{f}$ factors as a product of pairwise different linear homogeneous terms.

2.2. Computation of relative cohomology

Besides the above question on the relationship between the algebraically defined cohomology of the relative de Rham complex and analytically defined cohomology of (generic) fibers, the natural problem of computing $\boldsymbol{H}_f^\bullet$ arises.

This problem was addressed in the papers [3,5–7,9,10] mentioned above. Using analytic tools or theory of perverse sheaves and D-modules, their authors prove that under certain genericity-type assumptions on f, the highest relative cohomology module $\boldsymbol{H}_f^{n-1}$ and the Petrov module $\boldsymbol{P}_f$ are finitely generated over the ring $\mathbb{C}[f]$. For semiquasihomogeneous polynomials one can describe explicitly the collection of generators for $\boldsymbol{B}_f$, the polynomial forms $\omega_1, \dots, \omega_l \in \boldsymbol{\Lambda}^{n-1}$ such that any other form $\omega \in \boldsymbol{\Lambda}^{n-1}$ can be represented as

$$\begin{aligned} &\omega = \sum_{i=1}^{l} p_i \omega_i + df \wedge \eta + d\xi, \\ &p_i = p_i(f) \in \mathbb{C}[f], \ \eta, \xi \in \boldsymbol{\Lambda}^{n-2}, \end{aligned} \tag{2.6}$$

with appropriate polynomial coefficients p_i that are uniquely defined.

The proofs of this and related results, obtained in either analytic or algebraic way, are sufficiently involved. In particular, it is very difficult if possible at all to get an information on (i) how the decomposition (2.6) depends on parameters, in particular, if f itself depends on parameters, and (ii) how to place explicit *quantitative bounds* on the coefficients $p_i(f)$ in terms of the magnitude of coefficients of the form ω. For example, to extract such bounds from the more transparent analytic proof by Gavrilov, one should place a *lower* bound on the determinant of the period matrix of the forms ω_i over a system of vanishing cycles on the level curves $f^{-1}(t)$. The mere nonvanishing of this determinant is a delicate assertion whose proof in [9] is incomplete (a simple elementary proof was supplied by Novikov [14]). The explicit computation of this determinant for a specific choice of the generators ω_i was achieved by A. Glutsuk [11], but the answer is given by a very cumbersome expression.

In the next section we suggest an elementary derivation of the formula (2.6) under the assumption that the polynomial f is semiquasihomogeneous. This derivation:

(1) gives an independent elementary demonstration of the Gavrilov–Bonnet–Dimca theorem for the most important particular case of semiquasihomogeneous polynomials;
(2) proves that the polynomial coefficients p_i and the forms η, ξ from the decomposition (2.6) depend polynomially on the coefficients of the nonprincipal part of f, provided that the principal quasihomogeneous part of f remains fixed;
(3) yields the collection of the coefficients $(p_1, \dots, p_l)$ of (2.6) as a result of application of a certain *linear operator* to the form ω. The norm of this operator can be explicitly bounded in terms of f (and the chosen set of generators $\{\omega_i\}$) and the degree $\deg \omega$.

3. Bounded decomposition in the Brieskorn and Petrov modules

3.1. Degrees, weights, norms

In this section we first consider quasihomogeneous polynomials from the ring $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ with rational positive weights $w_i = \deg x_i$ normalized by the condition $w_1 + \cdots + w_n = n$ to simplify the treatment of the most important *symmetric* case when $w_i = 1$. The symbol $\deg f$ always means the quasihomogeneous degree.

Remark 1. Later on we will introduce additional variables $\lambda = (\lambda_1, \dots, \lambda_m)$ considered as *parameters*, assign them appropriate weights and work in the extended ring $\mathbb{C}[x, \lambda] = \mathbb{C}[x_1, \dots, x_n, \lambda_1, \dots, \lambda_m]$. Even in the symmetric case the weights of the parameters will in general be different from 1.

The *Euler field* associated with the weights $w_1, \dots, w_n$ is the derivation $X = \sum w_i x_i \partial/\partial x_i$ of $\mathbb{C}[x]$. By construction, $Xf = rf$, $r = \deg f \in \mathbb{Q}$, for any quasihomogeneous polynomial f (the Euler identity).

We put $\deg dx_i = \deg x_i = w_i$. This extends the quasihomogeneous grading on all k-forms: in the symmetric case, the degree of a polynomial k-form will be k plus the maximal degree of its coefficients. Obviously, $\deg \omega = \deg d\omega$ for any form, provided that $d\omega \neq 0$. The Lie derivative $X\omega$ of a quasihomogeneous form ω of degree r by the Euler identity is $r\omega$. Note that $\deg \omega > 0$ for all k-forms with $k \geqslant 1$.

The *norm* of a polynomial in one or several variables is defined as the sum of absolute values of its (real or complex) coefficients. This norm is multiplicative. The norm of a k-form by definition is the sum of the norms of its polynomial coefficients; it satisfies the inequality $\|\omega \wedge \eta\| \leqslant \|\omega\| \cdot \|\eta\|$ for any two forms ω, η.

The exterior derivative operator is bounded in the sense of this norm if the degree is restricted: $\|d\omega\| \leqslant (\max_i w_i) \deg \omega \cdot \|\omega\|$. In particular, in the symmetric case $\|d\omega\| \leqslant r\|\omega\|$, $r = \deg \omega$. Conversely, a primitive of an n-from μ can be always chosen bounded by the same norm $\|\mu\|$.

Unless explicitly stated differently, a monomial (monomial form, etc.) has always the unit coefficient.

3.2. Parameters

We will systematically treat the case when all objects (forms, functions etc.) depend polynomially on finitely many additional parameters $\lambda = (\lambda_1, \dots, \lambda_m)$. We will denote by $\boldsymbol{\Lambda}^k[\lambda]$, $k = 0, \dots, n$, the collection of k-forms whose coefficients polynomially depend on λ. For instance, the notation $\eta \in \boldsymbol{\Lambda}^{n-1}[\lambda]$ means that

$$\eta = \sum_{i=1}^{n} a_i(x, \lambda)\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

with polynomial coefficients $a_i \in \mathbb{C}[x, \lambda]$.

In such case the norm of forms, functions etc. will be always considered relative to the ring $\mathbb{C}[x, \lambda]$, that is, as the $\sum_i \|a_i\|$ of absolute values of coefficients a_i of the complete expansion in x, λ. If the parameters λ_s are assigned weights, we take them into account when defining the degree of the form. To stress the fact that the norm is computed relative to the ring $\mathbb{C}[x,\lambda]$ and not to $\mathbb{C}[x]$ (i.e., that the situation is parametric), we will sometimes denote the norm by $\|\cdot\|_\lambda$. For an instance, $\|2\lambda_1 x_1\| = 2|\lambda_1| \neq 2 = \|2\lambda_1 x_1\|_\lambda$.

3.3. *Division by a quasihomogeneous differential df. The division modulus*

If $f \in \mathbb{C}[x_1,\dots,x_n]$ is a quasihomogeneous polynomial having an *isolated* singularity at the origin, then the multiplicity l of this singularity can be easily found by Bézout theorem, since no roots of the system of algebraic equations $\partial f/\partial x_i = 0$, $i = 1,\dots,n$, can escape to infinity. In the symmetric case $l = (\deg f - 1)^n$. Choose any monomial basis $\varphi_1,\dots,\varphi_l$ of the local algebra $\mathbb{C}[[x_1,\dots,x_n]]/\langle\partial f\rangle$, $\langle\partial f\rangle = \langle\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\rangle$. Then the monomial n-forms $\mu_i = \varphi_i\, dx_1 \wedge\cdots\wedge dx_n$ form a basis of $\boldsymbol{\Lambda}^n_f = \boldsymbol{\Lambda}^n/df \wedge \boldsymbol{\Lambda}^{n-1}$ over $\mathbb{C}$: any n-form μ can be divided out as

$$\mu = \sum_{i=1}^{l} c_i\mu_i + df \wedge \eta, \quad c_i \in \mathbb{C},\ \eta \in \boldsymbol{\Lambda}^{n-1}, \tag{3.1}$$

with appropriate constants $c_1,\dots,c_l \in \mathbb{C}$ (coefficients of the "remainder" $\sum c_i\mu_i$) and a polynomial form $\eta \in \boldsymbol{\Lambda}^{n-1}$ (the "incomplete ratio"). Moreover, if μ is quasihomogeneous, then the decomposition (3.1) contains only terms with $\deg\mu_i = \deg\mu$ and $\deg\eta = \deg\mu - \deg f$. This immediately follows from quasihomogeneity and the uniqueness of the coefficients c_i. From this observation we also conclude that all monomial forms of degree $< \deg f$ must be among μ_i, and, moreover, any monomial form of degree greater than $\max_i \deg\mu_i$, is divisible without remainder by df.

The choice of the monomial forms μ_i spanning the quotient, is not unique, though the distribution of their degrees is. Denote by $\rho = \rho(f)$ the maximal difference

$$\rho(f) = \max_i \deg\mu_i - \min_i \deg\mu_i = \max_i \deg\varphi_i - \min_i \deg\varphi_i. \tag{3.2}$$

The following results are well-known.

Proposition 1. 1. *In the symmetric case* $\rho(f) < l = (r-1)^n$ [2, §5.5].

2. *In the bivariate case $n = 2$ the inequality $\rho(f) < r = \deg f$ holds if and only if f is a "simple singularity" of one of the following types,*

$$\begin{aligned}
&A_k\colon\ f = x_1^{k+1} + x_2^2, \quad k \geqslant 2,\\
&D_k\colon\ f = x_1^2x_2 + x_2^{k-1}, \quad k \geqslant 4,\\
&E_6\colon\ f = x_1^3 + x_2^4,\\
&E_7\colon\ f = x_1^3 + x_1x_2^3,\\
&E_8\colon\ f = x_1^3 + x_2^5,
\end{aligned}$$

see e.g., [2, §13, Theorem 2].

From these observations it can be immediately seen that the division with remainder (3.1) is a bounded linear operation in the space of all n-forms of restricted degrees.

Lemma 1. *Assume that $f \in \boldsymbol{\Lambda}^0$ is a quasihomogeneous polynomial having an isolated critical point of multiplicity l at the origin, and the monomial n-forms $\mu_1, \dots, \mu_l \in \boldsymbol{\Lambda}^n$ form the basis of $\boldsymbol{\Lambda}^n_f$.*

Then there exists a finite constant $M < +\infty$ depending only on f and the choice of the basis $\{\mu_i\}$, such that any n-form $\mu \in \boldsymbol{\Lambda}^n$ can be divided with remainder by df as in (3.1) *subject to the follouring constraints,*

$$\deg \eta \leqslant \deg \mu - \deg f, \qquad \|\eta\| + \sum |c_i| \leqslant M\|\mu\|. \tag{3.3}$$

If the form μ is quasihomogeneous, then $\deg \eta = \deg \mu - \deg f$ and c_i can be nonzero only if $\deg \mu_i = \deg \mu$.

The constant M depends on the choice of the monomial basis $\{\mu_i\}$. The optimal choice of such basis (out of finitely many possibilities) results in the smallest value $M = M(f)$ that depends only on f. We will always assume that the basis $\{\mu_i\}$ is chosen optimal in this sense.

Definition 1. The minimal constant $M(f)$ corresponding to an optimal choice of the monomial basis of the quotient $\boldsymbol{\Lambda}^n_f$ is called the *division modulus* of the quasihomogeneous polynomial $f \in \boldsymbol{\Lambda}^0$.

Corollary 1. *Assume that $\mu \in \boldsymbol{\Lambda}^n[\lambda]$ depends polynomially on additional parameters λ. Then μ can be divided with remainder by df so that the remainder and the incomplete ratio depend polynomially on λ with the same division modulus,*

$$\begin{aligned} &c_i = c_i(\lambda) \in \mathbb{C}[\lambda], \quad i = 1, \dots, n, \ \eta \in \boldsymbol{\Lambda}^{n-1}[\lambda], \\ &\|\eta\| + \sum \|c_i\| \leqslant M(f)\|\mu\|, \quad \|\cdot\| = \|\cdot\|_\lambda. \end{aligned}$$

Proof. Every monomial from the expansion of μ in x, λ can be divided out separately by df which is independent of λ. □

Proof of Lemma 1. Let M be the best constant such that (3.3) holds for all monomial n-forms with $\deg \mu \leqslant l$. It is finite since there are only finitely many such forms. In particular, since any form of degree l is divisible by df by Proposition 1, the respective fraction η will be of the norm at most $M\|\mu\|$.

Writing an arbitrary monomial n-form of degree $> l$ as a product of a monomial form of degree l times a monic monomial function $x^\alpha \in \mathbb{C}[x]$, $\alpha \in \mathbb{Z}^n_+$, we construct the explicit division formulas (without remainders) for all monomial forms of higher degrees. The division constant will be given by the same number M, since multiplication by a monic monomial preserves the norms of both $\|\mu\|$ and $\|\eta\|$.

All the other assertions of the Lemma are well-known [2]. □

3.4. Computability of the division modulus

Despite its general nature, the above proof is constructive, at least in the low dimensional cases $n = 1, 2$, allowing for an explicit computation of the division modulus in these cases.

The one-dimensional case is trivial: for the monomial $f(x) = x^r$ the division modulus $M(f)$ is equal to r and it can be obviously recalculated for any other principal homogeneous part. The "special case" of a multivariate polynomial $f(x) = x_1^r + \cdots + x_n^r$, see [12], is reducible to the one-dimensional situation. In this case $l = (r-1)^n$ monomial forms $x^\alpha dx_1 \wedge \cdots \wedge dx_n$ with $0 \leqslant \alpha_i \leqslant r-1$ form the basis, and the corresponding division modulus is again equal to r. This example admits an obvious generalization for quasihomogeneous "special polynomials" with different weights.

For a bivariate truly homogeneous polynomial f (i.e., in the symmetric case, the most important for applications), the division modulus M for all higher degree forms ($\deg \mu \geqslant 2 \deg f$) can be explicitly computed as the norm of the inverse Sylvester matrix for the partial derivatives $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ [16]. The "*quasimonic*" polynomials, introduced in that paper, are defined by the condition $M(f) = 1$, which in many respects is a natural normalizing condition for multivariate polynomials.

The choice of the basic forms even in the symmetric bivariate case depends on f: while it is generically possible to choose them as $x_1^{\alpha_1} x_2^{\alpha_2} dx_1 \wedge dx_2$ with $0 \leqslant \alpha_{1,2} \leqslant r-1$, for a badly chosen f some of these forms of degree greater than $r = \deg f$ can become linear dependent modulo df, requiring a different choice. In order to avoid making this choice, one may allow a *redundant* (i.e., linear dependent) collection of generating forms μ_i. Choosing all monomial forms of degree $\leqslant 2r$ makes the corresponding division for low degree forms trivial, so that the division modulus $M(f)$ is determined only by division of forms of higher degree. Details and accurate estimates in the bivariate symmetric case can be found in [16].

To describe the division modulus $M(f)$ in the case of $n \geqslant 3$ variables is a considerably more difficult problem, though it still can be reduced to analysis of finitely many monomial divisions. One can (at least, theoretically) express $M(f)$ via lower bounds for minors of certain explicitly formed matrices.

Remark 2. It is worth mentioning that the division modulus $M(f)$ is not directly related to the norm $\|f\|$, even in the symmetric bivariate case. If $\deg \mu \geqslant l$ and $\mu = df \wedge \eta$, then $\|\mu\| \leqslant \|df\|\,\|\eta\|$. On the other hand, $\|\mu\| \geqslant M^{-1}\|\eta\|$ by the definition of $M(f)$. Therefore

$$M(f) \geqslant \|df\|^{-1} \geqslant r^{-1}\|f\|^{-1}, \qquad r = \deg f,$$

that is, the division modulus for a polynomial f with the small norm must be large. The inverse is not true: a polynomial with a small division modulus can have a very large norm. Simple examples can be constructed in the form $f(x) = c\prod_i (x_1 - \lambda_i x_2)$ with sufficiently close values of the parameters $\lambda_i \in [0, 1]$ and a suitably chosen normalizing constant $c \in \mathbb{C}$.

3.5. Division by f

We begin by establishing an analog of the Euler identity in the Brieskorn module. It plays the central role for explicitly constructing the decomposition (2.6).

Lemma 2. *Assume that $f \in \boldsymbol{\Lambda}^0$ is a quasihomogeneous polynomial of degree r. Then any polynomial n-form divisible by df in $\boldsymbol{\Lambda}^n$, can itself be divided by f in the Brieskorn module $\boldsymbol{B}_f$. It also admits a polynomial primitive divisible by f.*

In other words, for any form $\eta \in \boldsymbol{\Lambda}^{n-1}$ there exist four forms $\mu \in \boldsymbol{\Lambda}^n$, $\omega \in \boldsymbol{\Lambda}^{n-1}$ and $\xi, \xi' \in \boldsymbol{\Lambda}^{n-2}$ such that

$$df \wedge \eta = f\mu + df \wedge d\xi \tag{3.4}$$

$$= d(f\omega) + df \wedge d\xi'. \tag{3.5}$$

The degrees of all forms μ, ω, ξ, ξ' are all equal to $\deg \eta$ in case the latter is quasihomogeneous.

The division operation is always well-posed in the sense that the decomposition (3.5) *can be always chosen to meet the inequality*

$$\|\omega\| + \|\xi'\| \leqslant (n+3)\deg\eta \cdot \|\eta\| \tag{3.6}$$

(*a similar inequality can be proved also for the first decomposition* (3.4)).

Proof. Note that for any n-form $\mu \in \boldsymbol{\Lambda}^n$ and any vector field X on $\mathbb{C}^n$,

$$(Xf)\mu = (\mathrm{i}_X df)\mu = df \wedge \mathrm{i}_X\mu,$$

where i_X is the inner antiderivative, since $df \wedge \mu = 0$. We will need this formula for the case when X is the Euler vector field.

To prove the first divisibility assertion (3.4), we have to show that the identity

$$df \wedge \eta = f\mu + df \wedge d\xi \tag{3.7}$$

can be always resolved as a linear equation with respect to μ and ξ for any choice of η. Using the Euler identity for functions and the above remark, we represent $f\mu$ as a form divisible by df,

$$f\mu = r^{-1}(Xf)\mu = r^{-1}(\mathrm{i}_X df)\mu = df \wedge r^{-1}\mathrm{i}_X\mu. \tag{3.8}$$

Eq. (3.7) will obviously be satisfied if

$$\eta = r^{-1}\mathrm{i}_X\mu + d\xi,$$

that is, when η is cohomologous to $r^{-1}\mathrm{i}_X\mu$. This last condition is equivalent to the equality between the exterior derivatives

$$d\eta = r^{-1}d\mathrm{i}_X\mu = r^{-1}X\mu,$$

since by the homotopy formula, $d\mathrm{i}_X\mu = X\mu - \mathrm{i}_X d\mu = X\mu$. Thus resolving Eq. (3.7) is reduced to inverting the Lie derivative X on the linear space of n-forms.

We claim that the linear map $\mu \mapsto X\mu$ of $\boldsymbol{\Lambda}^n$ to itself, is surjective (and obviously degree-preserving), guaranteeing thus solvability of the last equation for any choice of η.

Indeed, any monomial n-form $\mu_\alpha = x^\alpha dx_1 \wedge \cdots \wedge dx_n$ is an eigenvector of X with the strictly positive eigenvalue $\deg \mu_\alpha \geqslant n$ (recall that the weights w_i are normalized so that the volume form $dx_1 \wedge \cdots \wedge dx_n$ is of degree n). Thus X is surjective on $\boldsymbol{\Lambda}^n$ (actually, bijective) and one can choose $\mu = rX^{-1}(d\eta)$. The norm of the inverse operator X^{-1} does not exceed $(r/n)\deg\eta$ in the symmetric case. The proof of (3.4) is complete.

To prove the second assertion (3.5), we transform it using (3.8) as follows,

$$df \wedge \eta = f\,d\omega + df \wedge (\omega + d\xi') = r^{-1} df \wedge \mathrm{i}_X d\omega + df \wedge (\omega + d\xi'),$$

which will be obviously satisfied if

$$\eta = r^{-1}\mathrm{i}_X d\omega + \omega + d\xi'. \tag{3.9}$$

Taking the exterior derivative as before, we reduce this equation to the form

$$d\eta = r^{-1} d\mathrm{i}_X d\omega + d\omega = r^{-1}X\mu + \mu, \quad \mu = d\omega.$$

Solvability of this equation with respect to μ, (and hence to ω) for any left-hand side $d\eta$ follows from invertibility of the differential operator $r^{-1}X + \mathbf{1}$ on the linear space of polynomial n-forms ($\mathbf{1}$ stands for the identity operator). Exactly as in the previous situation, all monomial n-forms are eigenvectors for $(r^{-1}X + \mathbf{1})|_{\boldsymbol{\Lambda}^n}$ with the positive eigenvalues, all greater or equal to $r^{-1}n + 1$, hence $r^{-1}X + \mathbf{1}$ is invertible on $\boldsymbol{\Lambda}^n$ and ω can be chosen as a primitive of $(r^{-1}X + \mathbf{1})^{-1} d\eta$.

To prove the inequality between the norms, notice that $\mu = d\omega$ satisfies the inequality $\|\mu\| \leqslant \|d\eta\| \leqslant \deg\eta \|\eta\|$. A primitive ω can be always take of the norm $\|\omega\| \leqslant \|d\omega\|$. Together this yields $\|\omega\| \leqslant \deg\eta\|\eta\|$.

The norm $\|\xi'\|$ can be found from (3.9). Clearly, $\|\mathrm{i}_X\mu\| \leqslant n\|\mu\|$ because of the choice of the weights $\deg x_i$ which satisfy the condition $\sum w_i = n$. Substituting this inequality into (3.9), we obtain

$$\|\xi'\| \leqslant \|d\xi'\| \leqslant \|\eta\| + n\|d\omega\| + \|\omega\| \leqslant (n+2)\deg\eta\|\eta\|,$$

since $\deg\omega = \deg\eta \geqslant 1$. $\square$

3.6. Generating Petrov and Brieskorn modules: the algorithm

Division by the gradient ideal together with the Euler identity as formulated in Lemma 2, allows for a constructive proof of the representation (2.6) for an arbitrary semi-quasihomogeneous polynomial F.

Let $F = f + h \in \mathbb{C}[x_1, \ldots, x_n]$ be a semiquasihomogeneous polynomial with the principal quasihomogeneous part f and the lower-degree part h. Denote as before by $\mu_1, \ldots, \mu_l \in \boldsymbol{\Lambda}^n$ the forms spanning $\boldsymbol{\Lambda}^n_f = \boldsymbol{\Lambda}^n / df \wedge \boldsymbol{\Lambda}^{n-1}$ (note that the quotient is computed using only the principal part f). We claim that:

(1) any n-form $\mu \in \boldsymbol{\Lambda}^n$ can be represented as

$$\mu = \sum_{i=1}^{l} q_i \mu_i + dF \wedge d\zeta, \quad q_i \in \mathbb{C}[F],\ \zeta \in \boldsymbol{\Lambda}^{n-2}, \tag{3.10}$$

(2) any $(n-1)$-form $\omega \in \boldsymbol{\Lambda}^{n-1}$ can be represented as

$$\omega = \sum_{i=1}^{l} p_i \omega_i + dF \wedge \xi + d\xi', \quad p_i \in \mathbb{C}[F],\ \xi, \xi' \in \boldsymbol{\Lambda}^{n-2}. \tag{3.11}$$

The construction of the decomposition (3.10) begins by division of μ by df as explained in Lemma 1:

$$\mu = \sum c_i \mu_i + df \wedge \eta, \quad c_i \in \mathbb{C},\ \eta \in \boldsymbol{\Lambda}^{n-1}.$$

If $\deg \mu < r = \deg f = \deg F$, then the incomplete ratio is in fact absent, $\eta = 0$, and we arrive to a particular case of (3.10) with $q_i = c_i$ of degree 0 (constants).

If $\deg \mu$ is higher than r, we transform the term $df \wedge \eta$ using Lemma 2 and then substitute $f = F - h$:

$$\mu - \sum c_i \mu_i = f\mu' + df \wedge d\zeta = F\mu' + dF \wedge d\zeta - \mu'', \quad \mu'' = h\mu' + dh \wedge d\zeta.$$

Obviously, both μ' and μ'' are of degree *strictly inferior* to $\deg \mu$, which allows to continue the process inductively. Assuming that the reprasentations (3.10) are known for both μ' and μ'', we substitute them into the last identity and after collecting terms arrive to a representation for μ. In the symmetric case the inductive process cannot take more than $\deg \mu - r$ steps. It is a direct analog of the process of division of univariate polynomials, see also [16].

To construct (3.11), we divide $d\omega$ by df. If $\deg \omega < r$, then the incomplete ratio is absent and we obtain a special kind of (3.11) exactly as before.

Otherwise in the division with remainder

$$d\omega = \sum_{i=1}^{l} c_i\, d\omega_i + df \wedge \eta, \quad c_i \in \mathbb{C},\ \eta \in \boldsymbol{\Lambda}^{n-1},$$

substitute $df \wedge \eta = d(f\omega') + df \wedge d\xi$ and pass to the primitives. We obtain

$$\begin{aligned} \omega - \sum c_i \omega_i &= f\omega' + df \wedge \xi + d\xi' \\ &= F\omega' + dF \wedge \xi + d\xi' - \omega'', \quad \omega'' = h\omega' + dh \wedge \xi. \end{aligned} \tag{3.12}$$

For the same reasons as before, the degrees of ω', ω'' are strictly smaller than $\deg \omega$, hence the process can be continued inductively.

Remark 3. In a somewhat surprising way, it turned out impossible to transform directly the decomposition (3.10) for the form $d\omega \in \boldsymbol{\Lambda}^n$ into (3.11) for ω.

3.7. Effective decomposition in the Petrov module

The construction above is so transparent that any qualitative as well as quantitative assertion concerning these expansions, can be immediately verified.

We will show that

(1) all terms of the decomposition (3.11) depend polynomially on the lower order terms of F, assuming that the principal part if fixed, and

(2) the well-posedness of the construction is determined solely by the division modulus $M(f)$ of the principal homogeneous part.

In order to formulate the result, consider a *general semiquasihomogeneous polynomial* with the prescribed principal quasihomogeneous part,

$$F(x,\lambda) = f(x) + h(x,\lambda), \quad h(x,\lambda) = \sum_{\deg f_s < \deg f} \lambda_s f_s(x), \tag{3.13}$$

where $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$ are all (monic) monomials of degree strictly inferior to $r = \deg f$, arbitrarily ordered. We treat the coefficients $\lambda_1, \ldots, \lambda_m$ as the parameters of the problem, assigning to them the weights so that

$$\deg \lambda_s + \deg f_s = \deg f = r \quad \text{for all } s.$$

This choice makes the entire polynomial F quasihomogeneous of the same degree r in the ring $\mathbb{C}[x,\lambda] = \mathbb{C}[x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m]$. Instead of the ring $\mathbb{C}[F]$, the coefficients p_i of the decomposition (3.11) will belong to the ring $\mathbb{C}[F,\lambda]$ and their quasihomogeneity will be understood in the sense that the formal variable F is assigned the weight $\deg F = r$.

Theorem 1. *If the quasihomogeneous polynomial $f \in \mathbb{C}[x]$ has an isolated critical point at the origin and $F \in \mathbb{C}[x,\lambda]$ is a general semiquasihomogeneous polynomial* (3.13), *then any polynomial quasihomogeneous $(n-1)$-form $\omega \in \boldsymbol{\Lambda}^{n-1}[\lambda]$ of degree k can be represented as*

$$\omega = \sum_{i=1}^{l} p_i \omega_i + dF \wedge \xi + d\xi'. \tag{3.14}$$

The coefficients $p_i \in \mathbb{C}[F,\lambda]$ and the $(n-2)$-forms $\xi, \xi' \in \boldsymbol{\Lambda}^{n-2}[\lambda]$ are all polynomial and quasihomogeneous jointly in F, λ (resp., in x, λ) of the degrees $k - \deg \omega_i$, $k - r$ and k respectively.

The norm of the coefficients relative to the ring $\mathbb{C}[F, \lambda_1, \ldots, \lambda_m]$ is explicitly bounded in terms of n, r, k and the division modulus $M(f)$. In particular, for the symmetric case when $\deg x_1 = \cdots = \deg x_n = 1$,

$$\sum_{i=1}^{l} \|p_i\| \leqslant k! r^{k(n+3)} M^k \|\omega\|, \quad k = \deg \omega, \ M = M(f), \ \|\cdot\| = \|\cdot\|_\lambda. \tag{3.15}$$

Remark 4. The fact that the form ω is quasihomogeneous, is not important: any polynomial form is the sum of quasihomogeneous parts, each of them being divisible separately.

Remark 5. Even in the symmetric case, the degrees of the parameters are different from 1: $\deg \lambda_s = r - \deg f_s$ will take all natural values from 1 to r.

Proof of Theorem 1. The first assertion of Theorem 1 (on polynomiality and quasihomogeneity) follows from direct inspection of the algorithm described above, since all transformations on each inductive step (exterior differentiation, division by df which is independent of λ, and the Euler identity in $\boldsymbol{P}_f$) respect the quasihomogeneous grading.

The only assertion that has to be proved is that on the norms. In order for a sequence of increasing with k real constants $C_k > 0$ to be upper bounds for the decomposition (3.14),

$$\sum_{i=1}^{l} \|p_i\| \leqslant C_k \|\omega\|, \quad \text{for all } \omega \text{ with } \deg \omega \leqslant k,$$

they should satisfy a certain recurrent inequality which we will instantly derive from the suggested algorithm.

Denote by $p_i \in \mathbb{C}[F, \lambda]$ (resp., by p'_i and p''_i) the polynomial coefficients of the decomposition of the forms ω (resp., ω' and ω'') from the identity (3.12): since the degrees of both ω', ω'' are less than k and the sequence C_k is increasing, we have

$$\sum_i \|p'_i\| \leqslant C_{k-1} \|\omega'\|, \qquad \sum_i \|p''_i\| \leqslant C_{k-1} \|\omega''\|.$$

Multiplication by F corresponds to a shift of coefficients in the decomposition of ω'. Thus from (3.12) follows the inequality

$$\sum_i \|p_i\| \leqslant \sum_i \|c_i\| + \sum_i \|p'_i\| + \sum_i \|p''_i\| \leqslant \sum_i \|c_i\| + C_{k-1}(\|\omega'\| + \|\omega''\|).$$

By Lemma 2, $\|\omega'\| \leqslant (n+3)k\|\eta\|$. The norm of the inferior part h is by definition equal to the number of terms, that is, the number of monomials in n variables of degree $\leqslant r-1$. Therefore $\|h\| \leqslant r^n$ and $\|dh\| \leqslant r^{n+1}$. This implies an upper bound for $\|\omega''\|$:

$$\|\omega''\| \leqslant \|h\| \|\omega'\| + \|dh\| \|\xi\| \leqslant (\|h\| + \|dh\|)(\|\omega'\| + \|\xi\|) \leqslant 2r^{n+1}(n+3)k\|\eta\|$$

by Lemma 2. Finally, $\|\eta\| + \sum \|c_i\| \leqslant M\|\omega\|$ by definition of the division modulus $M = M(f)$. Assembling all these bounds together, we conclude that

$$\sum \|p_i\| \leqslant M\|\omega\| + C_{k-1} \cdot 3r^{n+1}(n+3)k\|\omega\|.$$

Thus the increasing sequence $C_k \geqslant 1$ will form upper bounds for the norms of the coefficients of decomposition for polynomial forms of degree $\leqslant k$, provided that

$$C_k \geqslant AkC_{k-1}, \quad A \geqslant 4r^{n+1}(n+3)M \geqslant r^{n+3}M$$

(notice that $r \geqslant 2$), which can be immediately satisfied if we put

$$C_k = k!\, r^{k(n+3)} M^k.$$

This proves the inequality for the norms. □

Note that the bound established in this theorem, is polynomial in $M = M(f)$ and (for a fixed r) factorial in $k = \deg \omega$, that is, only slightly overtaking the exponential growth.

3.8. Nonhomogeneous division

By a completely similar procedure one can describe the result of division by a *nonhomogeneous* differential dF as a sequence of divisions by the principal homogeneous part df.

More precisely, if $\mu \in \boldsymbol{\Lambda}^n[\lambda]$ is a polynomial n-form polynomially depending on the parameters $\lambda_1, \dots, \lambda_m$ and $F = f + \sum \lambda_s f_s$ is as in (3.13), then there exists a representation

$$\mu = \sum_{i=1}^{l} = c_i(\lambda)\mu_i + dF \wedge \eta, \quad c_1, \dots, c_n \in \mathbb{C}[\lambda],\ \eta \in \boldsymbol{\Lambda}^{n-1}[\lambda], \tag{3.16}$$

polynomially depending on parameters. If μ is quasihomogeneous, then so are c_i, and η, with $\deg c_i = \deg \mu - \deg \mu_i$ and $\deg \eta = \deg \mu - \deg F$. Moreover, the ratio $(\|c_i\|_\lambda + \|\eta\|_\lambda)/\|\mu\|_\lambda$ bounded in terms of $\deg \mu$ and the division modulus $M(f)$ of f only.

Indeed, dividing μ by df yields

$$\mu = \sum c_i \mu_i + df \wedge \eta = \sum c_i \mu_i + dF \wedge \eta - \mu', \quad \mu' = dh \wedge \eta,$$

where $h = F - f$, hence $\deg h < \deg f = \deg F$ and therefore $\deg \mu' < \deg \mu$. This means that the process of division can be continued inductively. Since $\|\mu'\| \leqslant \|h\|\|\eta\| \operatorname{const}_{r,n} M(f)\|\mu\|$, the norms of the remainder and the incomplete ratio are bounded in terms of $M(f)$ and the degrees. In the symmetric case the bound looks especially simple.

Proposition 2. *In the symmetric case of all weights equal to* 1, *the division of a form of degree $k = \deg \mu$ is bounded as follows,*

$$\|\eta\|_\lambda + \sum_{i=1}^{l} \|c_i\|_\lambda \leqslant M_k(F) \cdot \|\mu\|, \quad M_k(F) = kr^{n(k-r)}\bigl(M(f)\bigr)^k.$$

Proof. In this case $\|h\| \leqslant r^n$, so that $\|\mu'\| \leqslant Mr^n\|\mu\|$, and finally $\|\eta\| + \sum \|c_i\| \leqslant M\|\mu\|(1 + K + \cdots + K^{\deg \mu - r})$, where $K = Mr^n$. Thus the norm of the nonhomogeneous division operator obviously does not exceed $M^k(kr^{n(k-r)})$. This expression is exponential in $k = \deg \mu$ and polynomial in $M = M(f)$. □

4. Picard–Fuchs system for Abelian integrals

Consider a quasihomogeneous polynomial $f \in \boldsymbol{\Lambda}^0$ of degree $r = \deg f$ having an isolated singularity of multiplicity l at the origin. As before, let $\mu_1, \dots, \mu_l$ be generators of $\boldsymbol{\Lambda}^n_f$ over $\mathbb{C}$ and $\omega_1, \dots, \omega_l$ their monomial primitives. Consider the general semiquasihomogeneous polynomial $F = f + \sum_1^m \lambda_s f_s \in \mathbb{C}[x, \lambda]$ as in (3.13) with the fixed principal part f, whose coefficients $\lambda_1, \dots, \lambda_m$ are the natural parameters. Consider in the parameter space $\mathbb{C}^m$ the locus Σ such that for $\lambda \in \mathbb{C}^m \setminus \Sigma$ the level set $\{x \in \mathbb{C}\colon F(x, \lambda) = 0\}$ is a nonsingular algebraic hypersurface. Denote by $\Gamma = \Gamma(\lambda),\ \lambda \notin \Sigma$, any continuous family of $(n-1)$-cycles on the zero level. The Abelian integrals

$$I_i(\lambda) = \int_{\Gamma(\lambda)} \omega_i, \quad i = 1, \dots, l, \tag{4.1}$$

are well defined multivalued analytic functions on $\mathbb{C}^m \setminus \Sigma$. In this section we will derive a Pfaffian system of linear equations satisfied by these integrals.

We will always assume that the weights of the parameters λ_s are chosen so that F becomes a quasihomogeneous polynomial in x, λ of degree r: $\deg \lambda_s = r - \deg f_s$. The enumeration of the monomials f_s begins with the free term $f_1 \equiv 1$ of degree 0 so that the respective coefficient λ_1 is necessarily of degree r. Recall that $\rho(f)$ is the maximal difference (3.2) between the degrees of the forms μ_i.

Theorem 2. *There exist* $(l \times l)$*-matrix polynomials* $C_0(\lambda), C_1(\lambda), \dots, C_m(\lambda)$,

$$\begin{aligned} &C_0(\lambda) = \lambda_1 \cdot \mathbf{1} + C'(\lambda_2, \dots, \lambda_m), \\ &\deg C_0 \leqslant r + \rho(f), \quad \deg C_s \leqslant \deg f_s + \rho(f), \quad s = 1, \dots, m \end{aligned} \tag{4.2}$$

(*the degrees are quasihomogeneous*), *such that on* $\mathbb{C}^m \setminus \Sigma$

$$\frac{\partial}{\partial \lambda_s}\bigl(C_0(\lambda) I\bigr) = C_s(\lambda) I, \quad s = 1, \dots, m. \tag{4.3}$$

The norms $\|C_s\|_\lambda$ *are bounded by a power of the division modulus* $M(f)$.

In other words, the column vector function $I(\lambda)$ on the complement to Σ satisfies the matrix Pfaffian equation

$$\mathbf{d}I = \Omega I, \quad \Omega = C_0^{-1} \cdot \left(-\mathbf{d}C_0 + \sum_{s=1}^{m} C_s \, d\lambda_s\right), \tag{4.4}$$

with a rational matrix-valued 1-form Ω having the poles only on the locus $\Sigma' = \{\det C_0 = 0\} \subset \mathbb{C}^m$. Here $\mathbf{d}$ is the exterior derivation with respect to the variables λ_s only: for $c(\lambda) \in \mathbb{C}[\lambda]$, $\mathbf{d}c = \sum_s \frac{\partial c(\lambda)}{\partial \lambda_s} d\lambda_s$.

The proof is constructive. The description of the matrix polynomials $C_s(\lambda)$ is given below.

4.1. Gelfand–Leray derivative with respect to parameters

Lemma 3. *If* $\omega \in \boldsymbol{\Lambda}^{n-1}$ *is a polynomial form with constant* (*independent of* λ) *coefficients, and* $\eta_s \in \boldsymbol{\Lambda}^{n-1}[\lambda]$ *any form satisfying the identity*

$$f_s \, d\omega = -dF \wedge \eta_s, \tag{4.5}$$

(*recall that* $f_s = \frac{\partial F}{\partial \lambda_s}$), *then*

$$\frac{\partial}{\partial \lambda_s} \int_{\Gamma(\lambda)} \omega = \int_{\Gamma(\lambda)} \eta_s.$$

Proof. To derive this formal identity, we express $\lambda_s = H(x)$ from the equation $F(x, \lambda_s) = 0$, assuming all other parameters fixed, and apply the Gelfand–Leray formula to H: for (4.5) to hold, it would be sufficient if $\eta = \eta_s$ satisfies

$$d\omega = dH \wedge \eta.$$

It remains to observe that by the implicit function theorem and the definition of the parameters,

$$dF + \frac{\partial F}{\partial \lambda_s} dH = 0, \qquad \frac{\partial F}{\partial \lambda_s} = f_s.$$

Here and above d stands for the exterior derivative with respect to the "spatial" variables $x_1, \ldots, x_n$. $\square$

The standard Gelfand–Leray derivative appears for the parameter occurring before the constant term $f_1 \equiv 1$ (modulo the sign).

4.2. *Derivation of the system: beginning of the proof of Theorem 2*

Divide each of the forms $F\mu_i \in \boldsymbol{\Lambda}^n[\lambda]$, $\mu_i = d\omega_i$, by dF with with the remainder coefficients and the incomplete ratios polynomially depending on λ as in Proposition 2:

$$F\mu_i = dF \wedge \eta_i + \sum_{j=1}^{l} c_{ij}\mu_j, \qquad c_{ij} = c_{ij}(\lambda). \tag{4.6}$$

Clearly, the quasihomogeneous degree $\deg c_{ij}$ in $\mathbb{C}[\lambda]$ is equal to $r + \deg \mu_i - \deg \mu_j \leqslant \rho(f) + r$ ($c_{ij} \equiv 0$ if the difference is negative).

Let $C_0 = C_0(\lambda)$ be the $(l \times l)$-matrix polynomial with the entries $c_{ij}(\lambda)$. Since dF does not depend on λ_1 (the free term of F), while the only term depending on λ_1 in $F\mu_i$ is $\lambda_1\mu_i$, the dependence of C_0 on λ_1 can be immediately described: the corresponding remainder coefficients $c_{ij}(\lambda_1)$ for the division of $\lambda_1\mu_i$ by dF form the scalar matrix $\lambda_1 \cdot \mathbf{1}$ (the incomplete ratio is absent).

Since c_{ij} do not depend on x (being "constants depending on the parameters"), the identity (4.6) implies that

$$d\biggl(F\omega_i - \sum_j c_{ij}\omega_j\biggr) = -dF \wedge (-\omega_i - \eta_i), \qquad i = 1, \ldots, l.$$

Let

$$\omega'_{i,s} = -f_s(\omega_i + \eta_i), \qquad i = 1, \ldots, l,\ s = 1, \ldots, m.$$

All these forms are polynomial and polynomially depending on parameters. Their degrees can be easily computed: $\deg \eta_i = \deg \mu_i = \deg \omega_i$, $\deg \omega'_{i,s} = \deg f_s + \deg \mu_i$.

By the parametric Gelfand–Leray formula (Lemma 3), the partial derivatives of integrals of the forms $F\omega_i - \sum_j c_{ij}\omega_j$ over the cycle $\Gamma(\lambda) \subset \{F = 0\} \subset \mathbb{C}^n$ are equal to the integrals of the forms $\omega'_{i,s}$. Since the terms $F\omega_i$ vanish on $\Gamma(\lambda)$ for all values of λ, we have

$$\frac{\partial}{\partial \lambda_s}\biggl(\sum_j c_{ij}(\lambda) I_j(\lambda)\biggr) = I'_{i,s}(\lambda), \qquad I'_{i,s}(\lambda) = \oint_{\Gamma(\lambda)} \omega'_{i,s}.$$

The forms ω_i were chosen to generate the Petrov module $\boldsymbol{P}_F$ over $\mathbb{C}[F,\lambda]$, so each of the Abelian integrals $\oint \omega'_{i,s}$ can be expressed as a polynomial combination,

$$I'_{i,s} = \sum_{j=1}^{l} p_{ij,s} I_j, \quad p_{ij,s} \in \mathbb{C}[F,\lambda],$$

for all i, s. Denote by $C_s = C_s(\lambda)$ the polynomial $(l \times l)$-matrix function formed by the free terms of the polynomials $p_{ij,s}(\cdot\,,\lambda)$:

$$C_s(\lambda) = \left[p_{ij,s}(F,\lambda)|_{F=0}\right]_{i,j=1}^{l}, \quad s = 1,\dots,m.$$

All other terms, being divisible by F, disappear after integration over the cycle on the level surface $\{F=0\}$. Collecting the terms, we conclude that the partial derivatives of the column vector function $I(\lambda) = (I_1(\lambda),\dots,I_l(\lambda))$, $I_i = \oint \omega_i$, satisfy the system

$$\frac{\partial(C_0 I)}{\partial\lambda_s} = C_s I, \quad s = 1,\dots,m.$$

4.3. Bounds for the norms: end of the proof of Theorem 2

The construction described above, does not yet imply the assertion on the norms of the matrix polynomials $C_0,\dots,C_m$ for only one reason: multiplication by $F = f + h$, $h = \sum \lambda_s f_s$, is *not* a bounded operator. While multiplication by h increases the norm at most by $\|h\|_\lambda = \mathrm{const}_{n,r}$ (not exceeding $(r-1)^n$ in the symmetric case), the norm $\|f\|$ cannot be bounded in terms of $M(f)$, as required in the theorem (see Remark 2).

To correct this drawback, exactly as in [16], the division line (4.6) should be first prepared using (3.8) as follows,

$$\begin{aligned} &F\mu_i = (f+h)\mu_i = df \wedge \eta'_i + h\mu_i = dF \wedge \eta'_i + \mu'_i, \\ &\eta'_i = r^{-1}\mathrm{i}_X\mu_i, \quad \mu'_i = h\mu_i - dh \wedge \eta'_i, \end{aligned} \tag{4.7}$$

where (we again make all estimates for the symmetric case only),

$$\|\eta'_i\| \leqslant (n/r)\|\mu_i\|, \quad \|\mu'_i\| \leqslant \|h\|(1+r)(n/r)\|\mu_i\|.$$

Then forms μ'_i should be divided by dF with remainder: since their norms are bounded by a constant depending only on n, r (the norms of the monomial forms μ_i are equal to 1), the results of such division will be bounded by suitable powers of $M(f)$ by virtue of Proposition 2.

Collecting the terms, we conclude that the coefficients $c_{ij} \in \mathbb{C}[\lambda]$ of the corresponding remainders in (4.6) and the incomplete ratios $\eta_i \in \boldsymbol{\Lambda}^{n-1}[\lambda]$ will be bounded by expressions polynomial in $M(f)$.

The rest of the derivation remains unchanged and the estimates completely straightforward: the polynomial bounds for η_i imply those of the polynomial coefficients $p_{ij,s} \in \mathbb{C}[F,\lambda]$ by Theorem 1. This proves the last assertion of Theorem 2. $\square$

5. Observations. Discussion

The algorithm of derivation of the Picard–Fuchs system in the Pfaffian form is so transparent that many things become obvious.

5.1. Bounds

Though the matrix polynomials $C_s(\lambda)$ are not quasihomogeneous (their entries have different degrees), the determinant $\det C_0(\lambda)$ is a quasihomogeneous polynomial from $\mathbb{C}[\lambda]$. Its degree can be immediately computed as lr from the explicit representation (4.2). This same representation proves that this determinant, equal to $\lambda_1^n +$ polynomial in $(\lambda_2, \ldots, \lambda_m)$, does not vanish identically, so that the system (4.4) is indeed meromorphic.

Moreover, the norm of the inverse matrix C_0^{-1} can be explicitly majorized in terms of the distance to the critical locus. One possibility to do this is to consider the sections $\lambda_1 = 1$ and apply the Cartan inequality as in [16], using the quasihomogeneity.

5.2. Spectrum

The spectrum of $C_0(\lambda)$ can be also easily computed: it consists of all l critical values of the polynomial $F(x, \lambda)$, at least when $F(\cdot, \lambda)$ is a Morse polynomial. To see this, it is sufficient to evaluate both parts of (4.6) at any of l critical points $a_1, \ldots, a_l \in \mathbb{C}^n$. The column vectors $v_i = (\varphi_1(a_i), \ldots, \varphi_l(a_i))^{\mathrm{T}}$, $i = 1, \ldots, l$, are the corresponding eigenvectors (recall that $\mu_i = \varphi_i \, dx_1 \wedge \cdots \wedge dx_n$).

5.3. Hypergeometric form

Restricting the Pfaffian system (4.4) on the one-dimensional complex lines $\lambda_s =$ const, $s = 2, \ldots, m$, parameterized by the value of $t = \lambda_1$, one obtains a parameterized family of Picard–Fuchs systems of ordinary differential equations. In this case only the matrix C_1 is relevant.

By Theorem 2, it is quasihomogeneous of degree $\leqslant \rho(f)$ jointly in the variables $\lambda_1, \ldots, \lambda_m$. If $\rho(f) < r = \deg \lambda_1$, then C_1 cannot depend on λ_1 and hence the Picard–Fuchs system in this case will have the hypergeometric form (1.1). By Proposition 1, this happens only when f is a simple quasihomogeneous polynomial of one of the types listed there. For hyperelliptic polynomials (the singularity of the type A_k) this was well-known, see [16]. In turn, the hypergeometric form implies that all singular points of the Picard–Fuchs system are *Fuchsian* (with simple poles of the rational coefficients) when $F(\cdot, \lambda)$ is a Morse polynomial.

5.4. Logarithmic poles

For the full Pfaffian system (4.4) the polar locus, occurring where $\det C_0(\lambda)$ vanishes, is of multiplicity 1 (it is sufficient to produce just one value of the parameters λ such that $F(\cdot, \lambda)$ has simple critical points). Yet it is not the characteristic property.

A rational 1-form ω analytic outside a hypersurface $\Sigma' = \{g = 0\} \subset \mathbb{C}^m$, g being a polynomial without multiple factors, is said to have a *logarithmic singularity* on this hypersurface, if both $g\omega$ and $dg \wedge \omega$ extend as polynomial forms across Σ' on $\mathbb{C}^m$.

This is only one of several close but non-equivalent definitions, probably the strongest possible. It ensures that the restriction of ω on any holomorphic curve γ cutting Σ' at a point a, has a Fuchsian singularity with the residue independent on the choice of γ, depending only on the point a.

The basic question concerning the system (4.4) is whether this system itself or a suitable gauge transformation of this system with a rational matrix gauge function, are Fuchsian with bounded residues. If the answer is positive, this would mean a positive solution of the infinitesimal Hilbert problem.

Using symbolic computation for implementing the algorithm, we discovered that in the hyperelliptic case (singularity of the type A_k) the Picard–Fuchs system (4.4) indeed has only logarithmic poles until the degree $k = 6$ of the polynomial $f = x_1^k + x_2^2$. This naturally suggests the following conjecture.

Conjecture. *All singularities of the Picard–Fuchs system* (4.4) *are only logarithmic poles on* $\Sigma' = \{\det C_0 = 0\}$.

It would be interesting to verify this conjecture for other simple singularities listed in Proposition 1, perhaps first by symbolic computation.

The next step could be to study the behavior of residue of (4.4), the matrix function defined on the regular part of Σ', checking whether it is bounded near singular points of the discriminant.

5.5. Singular perturbations

The polynomial dependence of the matrices C_s on the lower degree coefficients of the polynomial $F = f + \cdots$ fails for the coefficients of the principal part. Though apparently rational, this dependence certainly must exhibit singularities when f degenerates into a quasihomogeneous form with nonisolated singularities. The Picard–Fuchs system in such cases may have singular points corresponding to *atypical values* of F. Their appearance must somehow be related to the fact that the division modulus explodes when such degeneracy occurs, thus creating a *singularly perturbed* system of linear differential equations. These phenomena seem to be worth of detailed study.

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