Computers and Mathematics with Applications 56 (2008) 3029–3037

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The normal parameter reduction of soft sets and its algorithm

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ARTICLE INFO

Article history:
Received 14 November 2007
Received in revised form 22 June 2008
Accepted 10 July 2008

Keywords:
Soft set
Fuzzy soft set
Normal parameter reduction
Decision partition
Parameter important degree

ABSTRACT

This paper is concerned with the reduction of soft sets and fuzzy soft sets. Firstly, the problems of suboptimal choice and added parameter set of soft sets are analyzed. Then, we introduce the definition of normal parameter reduction in soft sets to overcome these problems. In addition, a heuristic algorithm of normal parameter reduction is presented. Two new definitions, parameter important degree and decision partition, are proposed for analyzing the algorithm of normal parameter reduction. Furthermore, the normal parameter reduction is also investigated in fuzzy soft sets.

1. Introduction

Researchers in economics, engineering, environmental science, sociology, medical science, and many other fields deal with the complexity of uncertain data. The nature of the uncertainty data appearing in these domains can be very different. While probability theory, fuzzy sets [1], rough sets [2], and other mathematical tools are well-known and often useful approaches to describing uncertainty, each of these theories has its inherent difficulties, which are pointed out by Molodtsov [3]. Molodtsov proposed a completely new approach for modeling vagueness and uncertainty-soft set theory. The soft set theory is free from the difficulties affecting other existing methods.

A soft set is a parameterized family of the subsets of a universal set. It can be said that soft sets are neighborhood systems, and that they are a special case of context-dependent fuzzy sets. In soft set theory, the problem of setting the membership function simply does not arise. This makes the theory convenient and easy to apply in practice. Soft set theory has potential applications in various fields including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory, and measurement theory. Most of these applications have already been demonstrated by Molodtsov [4].

In recent years, research on soft set theory has been active, and great progress has been achieved. Aktas et al. [5] introduce the basic version of soft group theory, which extends the notion of group to include the algebraic structures of soft sets. Maji et al. [6] introduce the definition of reduct-soft-set and describe the application of soft set theory to a decision-making problem using rough sets. The same authors have also published a detailed theoretical study on soft sets [7]. Chen et al. [8] present a new definition of parameterization reduction in soft sets, and compare this definition to the related concept of attributes reduction in rough set theory.

Up to the present, all methods about reduction of soft sets only considered the optimal choice, the suboptimal choice is not referred. After the optimal choice is selected, the data of optimal object are deleted from the data set (such as sold products, repaired equipments). If we make next decision of the soft set in which the data of optimal choice are deleted, usually we need to make a new reduction of the soft set again. Obviously, much time is wasted on the reduction of the soft

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2008.07.013

set. Furthermore, the added parameter set is not considered too. If new parameters are added to the parameter set, new reduction of soft sets needs to be made again when we make the decision.

In order to overcome the above problems, a new definition of normal parameter reduction is proposed in this paper and a heuristic algorithm is presented to make normal parameter reduction. Two definitions of decision partition, based on the choice value and parameter important degree, are also introduced to analyze the algorithm. The algorithm excludes unnecessary parameters and searches suitable ones in the feasible parameter sets. Thus this algorithm performs efficiently.

The remainder of this paper is organized as follows. In Section 2, we review the basic concepts of soft set theory. In Section 3, parameterization reduction of soft sets [8] is described and the problems of suboptimal choice and added parameter set in soft sets are analyzed. The definition of normal parameter reduction is introduced to overcome these problems. Furthermore, two definitions of decision partition and parameter important degree are proposed, and some properties are discussed. The algorithm of normal parameter reduction is presented. In Section 4, normal parameter reduction of fuzzy soft sets is investigated. Finally, conclusions are given in Section 5.

2. Preliminaries

In this section, we review the notion of soft sets in [4] and some definitions of rough sets. Let U be an initial universe set and let E be a set of parameters.

Definition 2.1 (*See [4]*). A pair (F, E) is called a soft set (over U) if and only if F is a mapping of E into the set of all subsets of the set U, i.e., $F : E \to P(U)$, where $P(U)$ is the power set of U.

The soft set is a parameterized family of subsets of the set U. Every set $F(\varepsilon)$, $\varepsilon \in E$, from this family may be considered as the set of ε-elements of the soft set (F, E), or as the ε-approximate elements of the soft set. As an illustration, some examples such as fuzzy sets and topological spaces were listed in [4]. The way of setting (or describing) any object in soft set theory differs in principle from the way it is used in classical mathematics. In classical mathematics, a mathematical model of an object is usually constructed which is too complicated to find the exact solution. Therefore the notion of approximate solution has been introduced in soft set theory, whose approach is opposite the classical mathematics.

Definition 2.2 (*See [4]*). Assume that we have a binary operation, denoted by $*$, for subsets of the set U. Let (F, A) and (G, B) be soft sets over U. Then, the operation $*$ for soft sets is defined in the following way: $(F, A) * (G, B) = (H, A \times B)$, where, $H(\alpha, \beta) = F(\alpha) * G(\beta)$, $\alpha \in A$, $\beta \in B$, and $A \times B$ is the Cartesian product of the sets A and B.

The definition takes into account the individual nature of any soft set.

Definition 2.3 (*See [9]*). A knowledge representation system can be formulated as a pair $S = (U, A)$, where U is a nonempty finite set of objects and A is a nonempty finite set of attributes, such that $a : U \to V_a$ for any $a \in A$, where V_a is called the value set of a.

Definition 2.4 (*See [9]*). Each subset of attribute $B \subseteq A$ determines a binary indiscernibility relation $IND(B)$, as follows:

$$IND(B) = \{(x, y) \in U \times U | \forall a \in B, a(x) = a(y)\}.$$

The relation $IND(B)$, $B \subseteq A$, constitutes a partition of U, which we will denote by $U/IND(B)$. Obviously, $IND(B)$ is an equivalence relation and $IND(B) = \bigcap_{a \in B} IND(a)$.

3. Normal parameter reduction of soft set

In this section, we discuss the parameterization reduction and the normal parameter reduction of soft sets.

3.1. Analysis of parameterization reduction of soft set in [8]

Chen et al. [8] presented a parameterization reduction of soft sets and its applications in a decision making problem, which can be briefly described as follows.

Suppose $U = \{h_1, h_2, \ldots, h_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$ and (F, E) is a soft set with tabular representation. Let $f_E(h_i) = \sum_j h_{ij}$ where h_{ij} are the entries in the table of (F, E). Further we use M_E to denote the collection of objects in U which takes the max value of f_E. For every $A \subset E$, if $M_{E-A} = M_E$, then A is called a dispensable set in E, otherwise A is called an indispensable set in E. Roughly speaking, $A \subset E$ is dispensable means that the difference among all objects according to the parameters in A does not influence the final decision. The parameter set E is called independent if every proper subset of E is indispensable, otherwise E is dependent. $B \subseteq E$ is called a reduction of E if B is independent and $M_B = M_E$, i.e., B is the minimal subset of E that keeps the optimal choice objects invariant.

The problems of suboptimal choice and added new parameter set can be illustrated by the following example.

Example 3.1. Suppose we have a soft set (F, E) with the tabular representation displayed in Table 1.

Clearly $f_E(h_2) = 5$ is the maximum choice value, thus h_2 is the optimal choice object. The soft set in Table 1 has a parameterization reduction $\{e_3, e_6\}$ (not all) of E according to Chen's method in [8] (see Table 2). If the object h_2 is deleted from the Table 1, the suboptimal choice is h_1 or h_6 in Table 1. However, from the Table 2, we can see the maximum choice value is 1 except $f_{e_3,e_6}(h_2) = 2$, namely, the suboptimal choice is any one in $\{h_1, h_3, h_4, h_5, h_6\}$. Obviously, the suboptimal choice is incorrect in parameterization reduction tables.

If the character of objects can not be completely embodied by parameter set $E = \{e_1, e_2, \ldots, e_7\}$, new parameters need be added. Without loss of generality, let $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ be added parameter set. The soft set $(F, \{\bar{e}_1, \bar{e}_2, \bar{e}_3\})$ with the tabular representation is displayed in Table 3.

Combing original soft sets Table 1 with the added parameters Table 3 into a new Table 4. We can see h_1 is the optimal choice. While combing Table 2 with Table 3 into a new Table 5, we note that h_3 is the optimal choice. Thus the two optimal choices are inconsistent.

In order to overcome the above problems, we introduce the definition of normal parameter reduction in soft sets.

Table 1
Original table

U	e_1	e_2	e_3	e_4	e_5	e_6	e_7	$f(\cdot)$
h_1	1	0	1	1	1	0	0	4
h_2	0	0	1	1	1	1	1	5
h_3	0	0	0	0	0	1	1	2
h_4	1	0	1	0	0	0	0	2
h_5	1	0	1	0	0	0	0	2
h_6	0	1	1	1	0	1	0	4

Table 2
Parameterization reduction table of original table (Table 1)

U	e_3	e_6	$f(\cdot)$
h_1	1	0	1
h_2	1	1	2
h_3	0	1	1
h_4	1	0	1
h_5	1	0	1
h_6	1	0	1

Table 3
Added parameters table

U	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$
h_1	1	0	1
h_2	0	0	0
h_3	1	1	1
h_4	0	0	1
h_5	1	1	0
h_6	1	0	0

Table 4
Combing original table (Table 1) and added parameter table (Table 3)

U	e_1	e_2	e_3	e_4	e_5	e_6	e_7	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$f(\cdot)$
h_1	1	0	1	1	1	0	0	1	0	1	6
h_2	0	0	1	1	1	1	1	0	0	0	5
h_3	0	0	0	0	0	1	1	1	1	1	5
h_4	1	0	1	0	0	0	0	0	0	1	3
h_5	1	0	1	0	0	0	0	1	1	0	4
h_6	0	1	1	1	0	1	0	1	0	0	5

Table 5
Combing parameterization reduction table (Table 2) and added parameter table (Table 3)

U	e_3	e_6	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$f(\cdot)$
h_1	1	0	1	0	1	3
h_2	1	1	0	0	0	2
h_3	0	1	1	1	1	4
h_4	1	0	0	0	1	2
h_5	1	0	1	1	0	3
h_6	1	0	1	0	0	2

3.2. Normal parameter reduction of soft sets

Suppose $U = \{h_1, h_2, \ldots, h_n\}, E = \{e_1, e_2, \ldots, e_m\}, (F, E)$ is a soft set with tabular representation. Define $f_E(h_i) = \sum_j h_{ij}$, where h_{ij} are the entries in the table of (F, E).

Definition 3.1. With every subset of parameters $B \subseteq A$, a indiscernibility relation $IND(B)$ is defined by

$$IND(B) = \{(h_i, h_j) \in U \times U : f_B(h_i) = f_B(h_j)\}.$$

For soft set $\{F, E\}$, $U = \{h_1, h_2, \ldots, h_n\}$, denote $C_E = \{\{h_1, h_2, \ldots, h_i\}_{f_1}, \{h_{i+1}, \ldots, h_j\}_{f_2}, \ldots, \{h_k, \ldots, h_n\}_{f_s}\}$ as a partition of objects in U which partitions and ranks the objects according to value of $f_E(\cdot)$ based on the indiscernibility relation, and $C_E = \{\{h_1, h_2, \ldots, h_i\}_{f_1}, \{h_{i+1}, \ldots, h_j\}_{f_2}, \ldots, \{h_k, \ldots, h_n\}_{f_s}\}$ is called decision partition, where for subclass $\{h_v, h_{v+1}, \ldots, h_{v+w}\}_{f_i}, f_E(h_v) = f_E(h_{v+1}) = \cdots = f_E(h_{v+w}) = f_i$, and $f_1 \geq f_2 \geq \cdots \geq f_s$, s is the number of subclasses. Objects with the same value of $f_E(\cdot)$ are partitioned into a same subclass. In Table 1, $C_E = \{\{h_2\}_{f_1}, \{h_1, h_6\}_{f_2}, \{h_3, h_4, h_5\}_{f_3}\}$, where $f_1 = 5, f_2 = 4, f_3 = 2$.

Definition 3.2. For soft set $\{F, E\}$, $E = \{e_1, e_2, \ldots, e_m\}$, if there exists a subset $A = \{e'_1, e'_2, \ldots, e'_p\} \subset E$ satisfying $f_A(h_1) = f_A(h_2) = \cdots = f_A(h_n)$, then A is dispensable, otherwise, A is indispensable. $B \subset E$ is a normal parameter reduction of E if B is indispensable and $f_{E-B}(h_1) = f_{E-B}(h_2) = \cdots = f_{E-B}(h_n)$, that is to say $E - B$ is the maximal subset of E that the value $f_{E-B}(\cdot)$ keeps constant.

In Definition 3.2, $f_A(h_1) = f_A(h_2) = \cdots = f_A(h_n)$ implies $C_E = C_{E-A}$. $E - A$ is the minimal subset of E that keeps the classification ability invariant. Clearly, after the normal reduction of the parameter set E, we have fewer parameters, and the partitions of objects have not been changed. In the following discussion the pseudo parameter reduction is proposed.

Definition 3.3. For soft set $\{F, E\}$, $G = \{e''_1, e''_2, \ldots, e''_m\} \subset E$, if $C_{E-G} = C_E$, $E - G$ is called a pseudo reduction of E.

Note that in both Definitions 3.2 and 3.3, two equalities $C_E = C_{E-A}$ are all satisfied. However, in Definition 3.2, there exists a subset $A = \{e'_1, e'_2, \ldots, e'_p\} \subset E, f_A(h_1) = f_A(h_2) = \cdots = f_A(h_n)$. While in Definition 3.3, generally, the condition $f_A(h_1) = f_A(h_2) = \cdots = f_A(h_n)$ may not hold.

In order to better understand the two Definitions 3.2 and 3.3, let us consider a simple example. In Table 1 $f_{e_1,e_2,e_7}(h_1) = f_{e_1,e_2,e_7}(h_2) = \cdots = f_{e_1,e_2,e_7}(h_6) = 1$, Table 1 has a normal parameter reduction $\{e_3, e_4, e_5, e_6\}$ as displayed in Table 6. Table 1 shows $f_E(h_2) = 5, f_E(h_1) = f_E(h_6) = 4, f_E(h_3) = f_E(h_4) = f_E(h_5) = 2$, so $C_E = \{\{h_2\}_5, \{h_1, h_6\}_4, \{h_3, h_4, h_5\}_2, \}$, which means that h_2 is the optimal choice, h_1 or h_6 is the suboptimal choice and h_3, h_4 or h_5 is the inferior choice. $C_{E-\{e_1,e_2,e_7\}} = \{\{h_2\}_4, \{h_1, h_6\}_3, \{h_3, h_4, h_5\}_1, \}$, the results are same with decisions in Table 6. If e_4 is deleted from $\{e_3, e_4, e_5, e_6\}$, then the partition $C_{E-\{e_1,e_2,e_4,e_7\}} = \{\{h_2\}_3, \{h_1, h_6\}_2, \{h_3, h_4, h_5\}_1, \}$. $C_E = C_{E-\{e_1,e_2,e_4,e_7\}}$, so the partition of objects can not be changed. Thus we have the pseudo parameter reduction $\{e_3, e_5, e_6\}$ in Table 7.

Considering the normal parameter reduction Table 6, clearly, h_2 is the optimal object, h_1 or h_6 is the suboptimal object, and so on. Combing the Table 6 with Table 3 into a new table Table 8, the optimal choice is h_1, which is identical to the optimal choice by combining Table 1 with Table 3 (see Table 4). Therefore, the normal parameter reduction overcomes the problems of the suboptimal choice and updated parameter set.

From the definitions of normal parameter reduction and pseudo parameter reduction, we know normal parameter reduction is the special case of pseudo parameter reduction.

Table 6
Normal parameter reduction table of original table (Table 1)

U	e_3	e_4	e_5	e_6	$f(\cdot)$
h_1	1	1	1	0	3
h_2	1	1	1	1	4
h_3	0	0	0	1	1
h_4	1	0	0	0	1
h_5	1	0	0	0	1
h_6	1	1	0	1	3

Table 7
Pseudo parameter reduction of original table (Table 1)

U	e_3	e_5	e_6	$f(\cdot)$
h_1	1	1	0	2
h_2	1	1	1	3
h_3	0	0	1	1
h_4	1	0	0	1
h_5	1	0	0	1
h_6	1	0	1	2

Table 8
Combing added parameter table (Table 3) and normal parameter reduction table (Table 6)

U	e_3	e_4	e_5	e_6	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$f(\cdot)$
h_1	1	1	1	0	1	0	1	5
h_2	1	1	1	1	0	0	0	4
h_3	0	0	0	1	1	1	1	4
h_4	1	0	0	0	0	0	1	2
h_5	1	0	0	0	1	1	0	3
h_6	1	1	0	1	1	0	0	4

3.3. Algorithm of normal parameter reduction

For soft set $\{F, E\}$, $E = \{e_1, e_2, \ldots, e_m\}$ is the parameter set, $U = \{h_1, h_2, \ldots, h_n\}$ is the object set, $C_E = \{\{h_1, h_2, \ldots, h_i\}_{f_1}, \{h_{i+1} \ldots, h_j\}_{f_2}, \ldots, \{h_k, \ldots, h_n\}_{f_s}\}$ as a decision partition of objects in U. If the parameter e_i is deleted from set E, then the decision partition is changed and it can be denoted as $C_{E-e_i} = \{\{h_{1'}, h_{2'}, \ldots, h_{i'}\}_{f_{1'}}, \{h_{i+1'}, \ldots, h_{j'}\}_{f_{2'}}, \ldots, \{h_{k'}, \ldots, h_{n'}\}_{f_{s'}}\}$. For the sake of convenience we use C_E and C_{E-e_i} to denote $C_E = \{E_{f_1}, E_{f_2}, \ldots, E_{f_s}\}$ and $C_{E-e_i} = \{\overline{E-e}_{if_{1'}}, \overline{E-e}_{if_{2'}}, \ldots, \overline{E-e}_{if_{s'}}\}$, respectively. Where $E_{f_1} = \{h_1, h_2, \ldots, h_i\}_{f_1}$, $E_{f_2} = \{h_{i+1}, \ldots, h_j\}_{f_2}, \ldots, E_{f_s}\{h_k, \ldots, h_n\}_{f_s}$; $\overline{E-e}_{if_{1'}} = \{h_{1'}, h_{2'}, \ldots, h_{i'}\}_{f_{1'}}$, $\overline{E-e}_{if_{2'}} = \{h_{i+1'}, \ldots, h_{j'}\}_{f_{2'}}, \ldots, \overline{E-e}_{if_{s'}} = \{h_{k'}, \ldots, h_{n'}\}_{f_{s'}}$.

Definition 3.4. For soft set $\{F, E\}$, with parameter set $E = \{e_1, e_2, \ldots, e_m\}$, and object set $U = \{h_1, h_2, \ldots, h_n\}$. Decision partition and decision partition deleted e_i are $C_E = \{E_{f_1}, E_{f_2}, \ldots, E_{f_s}\}$ and $C_{E-e_i} = \{\overline{E-e}_{if_{1'}}, \overline{E-e}_{if_{2'}}, \ldots, \overline{E-e}_{if_{s'}}\}$, respectively. The importance degree of e_i for the decision partition is defined by

$$r_{e_i} = \frac{1}{|U|}(\alpha_{1,e_i} + \alpha_{2,e_i} + \cdots + \alpha_{s,e_i})$$

where $|\cdot|$ denotes the cardinality of set,

$$\alpha_{k,e_i} = \begin{cases} |E_{f_k} - \overline{E-e}_{if_{z'}}|, & \text{if there exist } z' \text{ such that } f_k = f_{z'} \quad 1 \le z' \le s', \quad 1 \le k \le s. \\ |E_{f_k}|, & \text{otherwise.} \end{cases}$$

Definition 3.5. For soft set $\{F, E\}$, with parameter set $E = \{e_1, e_2, \ldots, e_m\}$, $A = \{e'_1, e'_2, \ldots, e'_p\} \subset E$ and object set $U = \{h_1, h_2, \ldots, h_n\}$. Decision partition and decision partition deleted A are $C_E = \{E_{f_1}, E_{f_2}, \ldots, E_{f_s}\}$ and $C_{E-A} = \{\overline{E-A}_{f_{1'}}, \overline{E-A}_{f_{2'}}, \ldots, \overline{E-A}_{f_{s'}}\}$, respectively. The importance degree of A for the decision partition is defined by

$$r_A = \frac{1}{|U|}(\alpha_{1,A} + \alpha_{2,A} + \cdots + \alpha_{s,A})$$

where

$$\alpha_{k,A} = \begin{cases} |E_{f_k} - \overline{E-A}_{f_{z'}}|, & \text{if there exist } z' \text{ such that } f_k = f_{z'} \quad 1 \le z' \le s', \quad 1 \le k \le s. \\ |E_{f_k}|, & \text{otherwise.} \end{cases}$$

In order to understand the Definition 3.4 better, the following example is shown. We have a soft set (F, E) with the tabular representation displayed in Table 1, $C_E = \{\{h_2\}_5, \{h_1, h_6\}_4, \{h_3, h_4, h_5\}_2\}$. $s = 3$. $C_{E-e_i} = \{\{h_2\}_5, \{h_6\}_4, \{h_1\}_3, \{h_3\}_2, \{h_4, h_5\}_1\}$. $\alpha_{1,e_1} = |\{h_2\} - \{h_2\}| = 0$, $\alpha_{2,e_1} = |\{h_1, h_6\} - \{h_6\}| = |\{h_1\}| = 1$. $\alpha_{3,e_1} = |\{h_3, h_4, h_5\} - \{h_3\}| = |\{h_4, h_5\}| = 2$. Therefore $r_{e_1} = \frac{1}{6}(0 + 1 + 2) = 0.5$.

The important degree of parameter has the following properties.

Property 3.1. *For soft set $\{F, E\}$, parameter set $E = \{e_1, e_2, \ldots, e_m\}$, $0 \le r_{e_i} \le 1$.*

Proof. Let $r_{e_i} = \frac{1}{|U|}(\alpha_{1,e_i} + \alpha_{2,e_i} + \cdots + \alpha_{s,e_i})$, we have $\alpha_{k,e_i} = |E_{f_k} - \overline{E-e}_{if_{z'}}| \le |E_{f_k}|$ (if there exist z' such that $f_k = f_{z'}$, $1 \le z' \le s'$, $1 \le k \le s$). Note that $|E_{f_1}| + |E_{f_2}| + \cdots + |E_{f_s}| = |U|$, therefore

$$\begin{aligned} r_{e_i} &= \frac{1}{|U|}(\alpha_{1,e_i} + \alpha_{2,e_i} + \cdots + \alpha_{s,e_i}) \\ &\le \frac{1}{|U|}(|E_{f_1}| + |E_{f_2}| + \cdots + |E_{f_s}|) \\ &= 1. \end{aligned}$$

It is easy to obtain $0 \le r_{e_i}$. Therefore we have $0 \le r_{e_i} \le 1$. $\square$

Property 3.2. *$r_{e_i} = 0$ if and only if value $f_{e_i}(\cdot) = 0$; $r_{e_i} = 1$ if and only if value $f_{e_i}(\cdot) = 1$.*

Proof. Let $r_{e_i} = \frac{1}{|U|}(\alpha_{1,e_i} + \alpha_{2,e_i} + \cdots + \alpha_{s,e_i})$, decision partition $C_E = \{E_{f_1}, E_{f_2}, \ldots, E_{f_s}\}$, decision partition deleted $e_i C_{E-e_i} = \{\overline{E-e_i}_{f_{1'}}, \overline{E-e_i}_{f_{2'}}, \ldots, \overline{E-e_i}_{f_{s'}}\}$.

Let $r_{e_i} = 0$, it means that $\alpha_{k,e_i} = 0$ $(1 \le k \le s)$, namely, $|E_{f_k} - \overline{E-e_i}_{f_{z'}}| = 0$ (if there exist z' satisfying $f_k = f_{z'}$, $1 \le z' \le s'$, $1 \le k \le s$). It is that $E_{f_k} - \overline{E-e_i}_{f_{z'}} = \emptyset$, obviously, $E_{f_k} = \overline{E-e_i}_{f_{z'}}$. It implies that the choice value of each object is not changed after deleting parameter e_i. Hence $f_{e_i}(\cdot) = 0$. While if $f_{e_i}(\cdot) = 0$, for any $h_i \in E_k$, then $h_i \in \overline{E-e_i}$. So $E_{f_k} - \overline{E-e_i}_{f_{z'}} = \emptyset$, $|E_{f_k} - \overline{E-e_i}_{f_{z'}}| = 0$ $(f_k = f_{z'}, 1 \le k \le s, 1 \le z' \le s')$. So we have $\alpha_{k,e_i} = 0$, $(1 \le k \le s)$. Therefore $r_{e_i} = 0$.

Similarly, if $r_{e_i} = 1$, we have $|E_{f_k} - \overline{E-e_i}_{f_{z'}}| = |E_{f_k}|(f_k = f_{z'}, 1 \le k \le s, 1 \le z' \le s')$. Obviously, the choice value of each object is changed after deleting parameter e_i. Hence $f_{e_i}(\cdot) = 1$. For $f_{e_i}(\cdot) = 1$, we have $|E_{f_k} - \overline{E-e_i}_{f_{z'}}| = |E_{f_k}|$, $(f_k = f_{z'}, 1 \le k \le s, 1 \le z' \le s')$, so $r_{e_i} = 1$. $\square$

Property 3.3. *If the number of value $f_{e_i}(\cdot) = 1$ on the column e_i labeled by the parameter e_i is larger than the number of value $f_{e_j}(\cdot) = 1$, $r_{e_i} > r_{e_j}$.*

Proof. For $|E_{f_k} - \overline{E-e_i}_{f_z}|$, $|E_{f_k} - \overline{E-e_j}_{f_{z'}}|$, $(f_k = f_z, 1 \le k \le s, 1 \le z \le s', 1 \le z \le s'')$. According to the Definition 3.4, the more the number of $f_{e_i}(\cdot) = 1$ we have, the more the number of changed choice value we have. So the sum of $|E_{f_k} - \overline{E-e_i}_{f_z}|$ is larger than that of $|E_{f_k} - \overline{E-e_j}_{f_{z'}}|$, $1 \le k \le s$. Thus $r_{e_i} > r_{e_j}$. $\square$

Definition 3.6. For soft set $\{F, E\}$, with parameter set $E = \{e_1, e_2, \ldots, e_m\}$, and object set $U = \{h_1, h_2, \ldots, h_n\}$. Decision partition and decision partition deleted e_i and e_j are $C_E = \{E_{f_1}, E_{f_2}, \ldots, E_{f_s}\}$ and $C_{E-e_i} = \{\overline{E-e_i}_{f_{1'}}, \overline{E-e_i}_{f_{2'}}, \ldots, \overline{E-e_i}_{f_{s'}}\}$, $C_{E-e_j} = \{\overline{E-e_j}_{f_{1''}}, \overline{E-e_j}_{f_{2''}}, \ldots, \overline{E-e_j}_{f_{s''}}\}$ respectively. The importance degree of $e_i \vee e_j$ and $e_i \wedge e_j$ are defined by

$$r_{e_i \vee e_j} = \frac{1}{|U|}(\alpha_{1,e_i\vee e_j} + \alpha_{2,e_i\vee e_j} + \cdots + \alpha_{s,e_i\vee e_j}),$$
$$r_{e_i \wedge e_j} = \frac{1}{|U|}(\alpha_{1,e_i\wedge e_j} + \alpha_{2,e_i\wedge e_j} + \cdots + \alpha_{s,e_i\wedge e_j}),$$

where $|\cdot|$ denotes the cardinality of set,

$$\alpha_{k,e_i\vee e_j} = |(E_{f_k} - \overline{E-e_i}_{f_{z'}}) \cup (E_{f_k} - \overline{E-e_j}_{f_{v'}})|,$$
$$\alpha_{k,e_i\wedge e_j} = |(E_{f_k} - \overline{E-e_i}_{f_{z'}}) \cap (E_{f_k} - \overline{E-e_j}_{f_{v'}})|,$$

if there exist $z'v'$ such that $f_k = f_{z'} = f_{v'}$ $1 \le z' \le s'$, $1 \le v' \le s''$, $1 \le k \le s$.

Property 3.4. $r_{e_i\vee e_j} \ge \max\{r_{e_i}, r_{e_j}\}$; $r_{e_i\wedge e_j} \le \min\{r_{e_i}, r_{e_j}\}$.

Proof. From the Definition 3.6, we can easily get

$$|(E_{f_k} - \overline{E-e_i}_{f_{z'}}) \cup (E_{f_k} - \overline{E-e_j}_{f_{v'}})| \ge |(E_{f_k} - \overline{E-e_i}_{f_{z'}})|,$$

and

$$|(E_{f_k} - \overline{E-e_i}_{f_{z'}}) \cup (E_{f_k} - \overline{E-e_j}_{f_{v'}})| \ge |(E_{f_k} - \overline{E-e_j}_{f_{v'}})|.$$

Obviously, $\alpha_{k,e_i\vee e_j} = |(E_{f_k} - \overline{E-e_i}_{f_{z'}}) \cup (E_{f_k} - \overline{E-e_j}_{f_{v'}})|$, $\alpha_{k,e_i} = |(E_{f_k} - \overline{E-e_i}_{f_{z'}})|$, $\alpha_{k,e_j} = |(E_{f_k} - \overline{E-e_j}_{f_{v'}})|$. So we have $\alpha_{k,e_i\vee e_j} \ge \max\{\alpha_{k,e_i}, \alpha_{k,e_j}\}$. Therefore

$$\begin{aligned} r_{e_i\vee e_j} &= \frac{1}{|U|}(\alpha_{1,e_i\vee e_j} + \alpha_{2,e_i\vee e_j} + \cdots + \alpha_{s,e_i\vee e_j}) \\ &\ge \frac{1}{|U|}(\max\{\alpha_{1,e_i}, \alpha_{1,e_j}\} + \cdots + \max\{\alpha_{s,e_i}, \alpha_{s,e_j}\}) \\ &\ge \max\{r_{e_i}, r_{e_j}\}. \end{aligned}$$

Similarly, we can obtain $r_{e_i\wedge e_j} \le \min\{r_{e_i}, r_{e_j}\}$. $\square$

Theorem 3.1. *For soft set $\{F, E\}$, $E = \{e_1, e_2, \ldots, e_m\}$, $U = \{h_1, h_2, \ldots, h_n\}$, if there exists a subset $A = \{e'_1, e'_2, \ldots, e'_p\} \subset E$, such that $E - A$ is the normal parameter reduction of E, then $r_A = 1$ or $r_A = 0$ and $r_{e'_1} + r_{e'_2} + \cdots + r_{e'_p} = f_A(\cdot)$.*

Proof. $E-A$ is the normal parameter reduction of E, so we get $f_A(h_1)=f_A(h_2)=\cdots=f_A(h_n)$. Obviously, $f_A(\cdot)=$ *natural number* or $f_A(\cdot)=0$.

Let $f_A(\cdot)=0$, then $f_{e'_1}(\cdot)=f_{e'_2}(\cdot)=\cdots=f_{e'_p}(\cdot)=0$. So for any $E_{f_k}\subseteq C_E$, $\overline{E-A}_{f_{z'}}\subseteq C_{E-A}$, $E_{f_k}=\overline{E-A}_{f_{z'}}$ $(f_k=f'_z)$, it is that $\alpha_{k,A}=0$. Therefore $r_A=0$. From Property 3.2 we get $r_{e'_1}=r_{e'_2}=\cdots=r_{e'_p}=0$. Therefore $r_{e'_1}+r_{e'_2}+\cdots+r_{e'_p}=0$. Thus $r_{e'_1}+r_{e'_2}+\cdots+r_{e'_p}=f_A(\cdot)$.

Let $f_A(\cdot)=$ *natural number*, $C_E=\{E_{f_1},E_{f_2},\ldots,E_{f_s}\}$, and $C_{E-A}=\{\overline{E-A}_{f_{1'}},\overline{E-A}_{f_{2'}},\ldots,\overline{E-A}_{f_{s'}}\}$. Since $f_A(\cdot)=$ *natural number*, we can easily get $s=s'$ and $f_1=f_{1'}+$ *natural number*, $\ldots$, $f_s=f_{s'}+$ *natural number*, subclass $E_{f_1}=\overline{E-A}_{f_{1'}},\ldots,E_{f_s}=\overline{E-A}_{f_{s'}}$. Therefore $\alpha_{i,A}=|E_{f_i}-\overline{E-A}_{f_{z'}}|=|E_{f_i}|$ $(1\le i\le s$, $1'\le z'\le s'$, if there exists z' such that $f_i=f_{z'})$. Thus $r_A=1$.

Let $E_k=\{h_{i_1},h_{i_2},\ldots,h_{i_v}\}$, $i_1,i_2,\ldots,i_v<n$. Since $E-A$ is normal parameter reduction, $f_A(h_j)=$ *natural number*. For each $h_j\in E_k$, there exist *natural number* parameters in A to make $f_{e_l}(h_j)=1$, $e_l\in\bar{A}$, where $\bar{A}$ is the subset of A, $|\bar{A}|=$ *natural number*, and $f_{\bar{A}}(\cdot)=f_A(\cdot)=$ *natural number*. For different object, the subset $\bar{A}$ is different. So $\alpha_{k,e'_1}+\alpha_{k,e'_2}+\cdots+\alpha_{k,e'_p}=$ *natural number* $\times\,|E_k|$. Therefore,

$$
\begin{aligned}
r_{e'_1}+r_{e'_2}+\cdots+r_{e'_p} &= \frac{1}{|U|}\left(\sum_{i=1}^{s}\alpha_{i,e'_1}+\sum_{i=1}^{s}\alpha_{i,e'_2}+\cdots+\sum_{i=1}^{s}\alpha_{i,e'_p}\right)\\
&= \frac{1}{|U|}\left(\sum_{j=1}^{p'}\alpha_{1,e'_j}+\sum_{j=1}^{p'}\alpha_{2,e'_j}+\cdots+\sum_{j=1}^{p'}\alpha_{s,e'_j}\right)\\
&= \frac{1}{|U|}\times f_A(\cdot)\times(|E_1|+|E_2|+\cdots+|E_s|)\\
&= f_A(\cdot).
\end{aligned}
$$

This completes the proof. □

Example 3.2. Suppose we have a soft set (F,E) with the tabular representation displayed in Table 9, $E=\{e_1,e_2,\ldots,e_{10}\}$, $U=\{h_1,h_2,\ldots,h_6\}$. $\{e_3,e_4,e_7,e_{10}\}$ and $\{e_1,e_2,e_3,e_4,e_5,e_8,e_{10}\}$, $\{e_2,e_3,e_5,e_7,e_{10}\}$, are the normal parameter reductions of soft set. Namely, $A=\{e_1,e_2,e_5,e_6,e_8,e_9\}$ or $A=\{e_6,e_7,e_9\}$, $A=\{e_1,e_4,e_6,e_8,e_9\}$.

$C_E=\{\{h_1\}_6,\{h_5\}_5,\{h_2,h_4,h_6\}_4,\{h_3\}_3\}$, $C_{E-e_1}=\{\{h_1\}_5,\{h_2,h_5,h_6\}_4,\{h_3,h_4\}_3\}$, $C_{E-e_2}=\{\{h_1\}_6,\{h_5\}_5,\{h_2,h_4\}_4,\{h_3,h_6\}_3\}$, $C_{E-e_3}=\{\{h_1\}_5,\{h_5\}_4,\{h_2,h_3,h_4,h_6\}_3\}$, $C_{E-e_4}=\{\{h_1,h_5\}_5,\{h_4\}_4,\{h_2,h_3,h_6\}_3\}$, $C_{E-e_5}=\{\{h_1,h_5\}_5,\{h_4,h_6\}_4,\{h_2,h_3\}_3\}$, $C_{E-e_6}=\{\{h_1\}_6,\{h_5\}_5,\{h_4\}_4,\{h_2,h_6\}_3,\{h_3\}_2\}$, $C_{E-e_7}=\{\{h_1\}_5,\{h_2,h_4,h_5,h_6\}_4,\{h_3\}_3\}$, $C_{E-e_8}=\{\{h_1\}_6,\{h_2,h_4,h_5,h_6\}_4,\{h_3\}_2\}$, $C_{E-e_9}=\{\{h_1\}_6,\{h_5\}_5,\{h_2,h_6\}_4,\{h_3,h_4\}_3\}$, $C_{E-e_{10}}=\{\{h_1\}_5,\{h_2,h_5,h_6\}_4,\{h_4\}_3,\{h_3\}_2\}$.

$r_{e_1}=\frac{1}{6}\times(1+1+1+0)=\frac{3}{6}$, $r_{e_2}=\frac{1}{6}\times(0+0+1+0)=\frac{1}{6}$, $r_{e_3}=\frac{1}{6}\times(1+1+3+0)=\frac{5}{6}$, $r_{e_4}=\frac{1}{6}\times(1+0+2+0)=\frac{3}{6}$, $r_{e_5}=\frac{1}{6}\times(1+0+1+0)=\frac{2}{6}$, $r_{e_6}=\frac{1}{6}\times(0+0+2+1)=\frac{3}{6}$, $r_{e_7}=\frac{1}{6}\times(1+1+0+0)=\frac{2}{6}$, $r_{e_8}=\frac{1}{6}\times(0+1+0+1)=\frac{2}{6}$, $r_{e_9}=\frac{1}{6}\times(0+0+1+0)=\frac{1}{6}$, $r_{e_{10}}=\frac{1}{6}\times(1+0+1+1)=\frac{3}{6}$.

From the Table 9, $\{e_3,e_4,e_7,e_{10}\}$ and $\{e_1,e_2,e_3,e_4,e_5,e_8,e_{10}\}$, $\{e_2,e_3,e_5,e_7,e_{10}\}$ are the normal parameter reductions of soft set, we can calculate $r_{e_1}+r_{e_2}+r_{e_5}+r_{e_6}+r_{e_8}+r_{e_9}=2$, $r_{e_6}+r_{e_7}+r_{e_9}=1$, $r_{e_1}+r_{e_4}+r_{e_6}+r_{e_8}+r_{e_9}=2$, which satisfied the Theorem 3.1. Note that $r_{e_4}+r_{e_6}=1$, but $f_{e_4,e_6}(h_1)\neq f_{e_4,e_6}(h_2)$, so $\{e_1,e_2,e_3,e_5,e_7,e_8,e_9,e_{10}\}$ is not the normal parameter reduction of soft set (F,E). Thus the converse proposition of thesis does not hold. Theorem 3.1 implied that for parameter subset $A=\{e'_1,e'_2,\ldots,e'_p\}\subset E$, if the value $r_{e'_1}+r_{e'_2}+\cdots+r_{e'_p}$ are nonnegative integers, $E-A$ may be the normal parameter reduction; otherwise, $E-A$ is certainly not the normal parameter reduction. Therefore, constructing a feasible parameter reduction set and excluding redundant parameters, the algorithm of normal parameter reduction based on Theorem 3.1 can be proposed as follows:

Table 9
Soft set table

U	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	$f(\cdot)$
h_1	1	0	1	1	1	0	1	0	0	1	6
h_2	0	0	1	1	1	1	0	0	0	0	4
h_3	0	0	0	0	0	1	0	1	0	1	3
h_4	1	0	1	0	0	0	0	0	1	1	4
h_5	1	0	1	0	0	0	1	1	0	1	5
h_6	0	1	1	1	0	1	0	0	0	0	4

1. Input the soft set (F, E);
2. Input the parameter set E;
3. Compute parameter importance degree r_{e_i} $(1 \le i \le m)$;
4. Select maximal subset $A = \{e'_1, e'_2, \dots, e'_p\}$ in E which satisfying that sum of $r_{e'_i}$ $(1 \le i \le p)$ is nonnegative integer, then put the A into a feasible parameter reduction set;
5. Check A, if $f_A(h_1) = f_A(h_2) = \cdots = f_A(h_n)$, then $E - A$ is the normal parameter reduction and A is saved in the feasible parameter reduction set, otherwise A is deleted from the feasible parameter reduction set.
6. Find the maximum cardinality of A in feasible parameter reduction set.
7. Compute $E - A$ as the optimal normal parameter reduction.

4. Normal parameter reduction of fuzzy soft sets

In real life, many problems are imprecise in nature. The classical mathematical tools are not capable of successfully dealing with such problems. Fuzzy set theory has been used quite extensively to deal with such imprecision [1,10]. Most results of fuzzy soft sets may be found in [11]. Roy et al. [12] presented a method of object recognition from an imprecise multiobserver data. And Maji [6] et al. considered the reduction of weighted soft sets. In this section, the reduction of fuzzy soft sets is discussed.

Definition 4.1 (*See [12]*). Let $\Psi(U)$ denote the set of all fuzzy sets of U. Let $A_i \subset E$. A pair (F_i, A_i) is called a fuzzy soft set over U, where F_i is a mapping given by $F_i : A_i \to \Psi(U)$.

Definition 4.2 (*See [12]*). For two fuzzy soft sets (F, A) and (G, B) over a common universe U, (F, A) is a fuzzy soft set subset of (G, B) if (i) $A \subset B$, and (ii) $\forall \varepsilon \in A$, $F(\varepsilon)$ is a fuzzy subset of $G(\varepsilon)$. We write $(F, A)\tilde{\subseteq}(G, B)$. (F, A) is said to be a fuzzy soft super set of (G, B), if (G, B) is a fuzzy soft subset of (F, A). We denote it by $(F, A)\tilde{\supseteq}(G, B)$.

The tabular representation of fuzzy soft set is given in Table 10. h_{ij} is the membership degree of h_i in a parameter set $\{e_j\}$, $h_{ij} \in [0, 1]$.

Suppose $U = \{h_1, h_2, \dots, h_n\}$, $E = \{e_1, e_2, \dots, e_m\}$, (F, E) is a fuzzy soft set with tabular representation. Define $f_E(h_i) = \sum_j h_{ij}$ where h_{ij} are the entries in the fuzzy soft set table of (F, E).

Definition 4.3. For fuzzy soft set $\{F, E\}$, $E = \{e_1, e_2, \dots, e_m\}$, if there exists a subset $A = \{e'_1, e'_2, \dots, e'_p\} \subset E$ satisfying $\sum_{e_k \in A} h_{1k} = \sum_{e_k \in A} h_{2k} = \cdots = \sum_{e_k \in A} h_{nk}$, A is dispensable, otherwise, A is indispensable. $B \subset E$ is a normal parameter reduction of E if B is indispensable and $\sum_{e_k \in E-B} h_{1k} = \sum_{e_k \in E-B} h_{2k} = \cdots = \sum_{e_k \in E-B} h_{nk}$, that is to say $E - B$ is the maximal subset of E that the value $f_{E-B}(\cdot)$ keeps constant.

Example 4.1. Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$, be the set of objects having different colors, size and surface texture features. The parameter set, E = {blackish, dark brown, yellowish, reddish, large, small, very small, average, very large, course, moderately course, fine, extra fine} $= \{e_1, e_2, \dots, e_{14}\}$.

The fuzzy soft set (F, E) is defined as follows, which is illustrated by a tabular representation in Table 11.

We can obtain $A = \{e_1, e_2, e_5, e_7, e_8, e_9, e_{12}\}$, $\sum_{e_k \in A} h_{1k} = \sum_{e_k \in A} h_{2k} = \sum_{e_k \in A} h_{3k} = \sum_{e_k \in A} h_{4k} = \sum_{e_k \in A} h_{5k} = \sum_{e_k \in A} h_{6k} = 2.7$. Thus $\{e_3, e_4, e_6, e_{10}, e_{11}, e_{13}, e_{14}\}$ is the normal reduction of fuzzy soft set (F, E).

Table 10
Fuzzy soft set table

U	e_1	$\cdots$	e_m
h_1	h_{11}	$\cdots$	h_{1m}
$\cdots$	$\cdots$	$\cdots$	$\cdots$
h_n	h_{n1}	$\cdots$	h_{nm}

Table 11
Example of fuzzy soft set table

U	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}
h_1	0.1	0.2	0.3	0.1	0.5	0.4	0.2	0.8	0.3	0.1	0.4	0.6	0.1	0.5
h_2	0.3	0.2	0.3	0.3	0.7	0.5	0.3	0.2	0.1	0.3	0.1	0.9	0.1	0.5
h_3	0.2	0.3	0.4	0.3	0.2	0.5	0.9	0.4	0.5	0.3	0.1	0.3	0.1	0.6
h_4	0.7	0.1	0.7	0.4	0.2	0.2	0.5	0.5	0.1	0.2	0.1	0.5	0.1	0.3
h_5	0.5	0.3	0.2	0.5	0.6	0.2	0.1	0.7	0.4	0.5	0.3	0.1	0.5	0.4
h_6	0.4	0.4	0.3	0.5	0.3	0.2	0.2	0.3	0.5	0.4	0.2	0.6	0.3	0.3

5. Conclusions

In this paper, the problems of suboptimal choice and added parameter set are discussed in the reduction of soft sets. A new definition of normal parameter reduction is introduced. The data of optimal objects can be deleted directly from the normal parameter reduction, and the next optimal choice can be obtained exactly from the normal parameter reduction in which the data of optimal objects are deleted. Furthermore, by adding new parameters to the parameter set of the normal parameter reduction, the exact optimal choice can be obtained. In addition, the heuristic algorithm of normal parameter reduction of soft sets is presented. The algorithm excludes unnecessary parameters and searches for suitable parameters in the feasible sets using the decision partition and parameter importance degree. Thus this algorithm performs more efficiently.

Acknowledgments

This work was supported by the National Research Foundation for the Doctoral Program of Higher Education of China No. 20020145007, and National Science Foundation of PR China under Grants 60274099.

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